Making Cornish–Fisher distributions fit

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Abstract

The truncated Cornish–Fisher inverse expansion is well known. It is used, for example, to approximate value-at-risk and conditional value-at-risk. It is known that this expansion gives a distribution for limited skewness and kurtosis and that the distribution may be a poor fit. drawing on Maillard (2012) we show how to find a unique corrected Cornish–Fisher distribution efficiently for a wide range of skewness and kurtosis. We show it has a unimodal density and a quantile function that is twice continuously differentiable as a function of mean, variance, skewness and kurtosis. We show how to obtain random variates efficiently and how to test goodness-of-fit. We apply the Cornish–Fisher distribution to fit hedge-fund returns and estimate conditional value-at risk. Finally, we investigate various generalisations of the Cornish–Fisher distributions and show they do not have the same desirable properties.

Keywords: Conditional value-at-risk; Goodness-of-fit; Kurtosis; Random variates; Skewness

Introduction

The Cornish–Fisher expansions are well known and allow us to express a random variable X with unknown distribution function F_X and known cumulants in terms of the normal distribution. In practice we must use only the first few cumulants, which leads to approximation errors. Although these can be serious, Cornish–Fisher expansions are widely used.

The purpose of this article is to develop a systematic correction to the expansion, with guaranteed convergence. We use hedge-fund data to illustrate the value of the method. For simplicity we ignore correlation and time-series effects, though, see for example Gabrielsen et al. (2015), these can easily be included as needed.

Cornish & Fisher (1938), Hill & Davis (1968) show that writing

$$F_X(x) = \Phi(u), \tag{1}$$

we can derive power series expansions for u and x:

$$u = \sum_{k=0}^{\infty} n_k x^k \quad \text{and} \quad x = \sum_{k=0}^{\infty} a_k u^k. \tag{2}$$

The first is called the *normalising expansion*, the second the *inverse expansion*. Both n_k and a_k are polynomials in the cumulants of X. The inverse expansion truncated to four terms is often used to approximate F_X, for example to approximate conditional value-at-risk (CVaR) (Bali et al. 2007, Liang & Park 2007). It is well known that the approximation is poor when X is not close to normal and that it gives a distribution function only for a small range of skewness and kurtosis. Extending Maillard (2012), we show how to overcome these problems. We define a family of distributions with four parameters: mean μ, variance σ^2, skewness κ_3 and (excess) kurtosis κ_4. We show it has properties desirable for optimisation, distribution fitting and random variate generation.

We write $C^2_{\mathbf{q}}$ for the set of twice continuously differentiable functions f of $\mathbf{q}$ for $\mathbf{q}$ in some region $\hat{R}$ of interest. We use the Cornish–Fisher inverse expansion to generate a family $\mathcal{F}(\mathbf{p})$, $\mathbf{p} = (\mu, \sigma, \kappa_3, \kappa_4)$ to approximate F_X. We show its quantile function $F^{-1}(u; \mathbf{p}) \in C^2_{\mathbf{p}}$ over a reasonable region $\hat{R}$. This is desirable when we optimise with smooth functions of $F^{-1}(u; \mathbf{p})$ such as conditional value-at-risk. There is much recent interest in coherent risk measures (Liang & Park 2007, Balbás et al. 2009), which often have this property.

We show the density function $f(x; \mathbf{p})$, is smooth, unimodal (that is, its density has one local maximum) and positive for $x \in \mathbb{R}$. That is, it is much like the normal density but can have nonzero skewness or kurtosis. This makes it attractive for distribution fitting.

If skewness and kurtosis are zero the normal family of distributions is a practical alternative to $\mathcal{F}(\mathbf{p})$. And if skewness is zero we may hope to use elliptical distributions. But these are not enough, for example, to model monthly returns from an investment, which may have significant skewness and kurtosis. The Box–Cox power exponential (BCPE) distribution (Rigby & Stasinopoulos 2004) appears a plausible alternative to $\mathcal{F}(\mathbf{p})$, because it is a four-parameter family of distributions. However, the BCPE quantile function does not in general have a continuous derivative and if the distribution is not symmetric it is truncated in one or other of its tails. The Cornish–Fisher distribution may be a reasonable alternative to BCPE in its main application to GAMLSS (Stasinopoulos & Rigby 2007).

Bowers (1966) shows, when $w(z)$ is a gamma density and $p_n(x)$ are the orthogonal polynomials with respect to $w(w)$, how to choose values of A_i so that

$$\hat{f}(x) = w(x) \sum_{n=0}^{N} A_n p_n(x) \tag{3}$$

is the density of a distribution with given first m moments. So it is reasonable to ask if we could do this for the normal density $\phi(x)$. We show we can but the function we obtain is only a density function for a range of skewness and kurtosis too small for practical use.

The Cornish–Fisher expansion

Suppose X has mean 0 and variance 1. Then, the Cornish–Fisher inverse expansion of equation (2) (right) gives

$$F_X^{-1}(u) = \sum_{k=0}^{\infty} a_k \left(\Phi^{-1}(u)\right)^k,$$

where each a_k is a polynomial in the cumulants of X and Φ is the standard normal distribution function. To use this in practice we need to truncate the series. For the approximation to be increasing the highest power of k must be odd. In practice the fourth order ($k = 3$) approximation is used. Cornish & Fisher (1938) give us $a_2 = -a_0 = s$, $a_1 = 1 + 5s^2 - 3k$ and $a_3 = k - 2s^2$ with $s = \kappa_3/6$ and $k = \kappa_4/24$. So the fourth order expansion is

$$x = \xi(u) = -s + (1 + 5s^2 - 3k)u + su^2 + (k - 2s^2)u^3 \tag{4}$$

giving quantile function

$$\tilde{F}^{-1}(u) = -s + (1 + 5s^2 - 3k)\Phi^{-1}(u) + s\left(\Phi^{-1}(u)\right)^2 + (k - 2s^2)\left(\Phi^{-1}(u)\right)^3. \tag{5}$$

Following Maillard (2012) we treat s and k as parameters because equations (7) and (8) show $\tilde{F}$ does not have skewness $s/6$ or kurtosis $k/24$ unless $s = k = 0$. We write $q = s^2$ to simplify expressions containing only even powers of s. $\tilde{F}$ can only be a distribution function if ξ is a strictly increasing function of u. It is straightforward to show (see, for example, Maillard (2012)) that ξ is strictly increasing in the region R given by $q < 3 - 2\sqrt{2}$ and

$$\frac{1 + 11q - \sqrt{q^2 - 6q + 1}}{6} < k < \frac{1 + 11q + \sqrt{q^2 - 6q + 1}}{6}. \tag{6}$$

Maillard (2012) computes the moments of the fourth-order Cornish–Fisher inverse expansion. These are (correcting a misprint and writing as functions of s and k):

$$\begin{aligned}
\mu_1(s,k) &= 0,\\
\mu_2(s,k) &= 1 + 6k^2 - 24s^2k + 25s^4,\\
\mu_3(s,k) &= 6s - 76s^3 + 510s^5 + 36sk - 468s^3k + 108sk^2,\\
\mu_4(s,k) &= 3 + 3348k^4 - 28080s^2k^3 + 1296k^3 - 6048s^2k^2\\
&\quad + 252k^2 - 123720s^6k + 8136s^4k - 504s^2k\\
&\quad + 24k + 64995s^8 - 2400s^6 - 42s^4 + 88380k^2s^4.
\end{aligned} \tag{7}$$

The skewness and kurtosis are given by

$$\hat{s}(s,k) = \frac{\mu_3(s,k)}{(\mu_2(s,k))^{3/2}} \quad \text{and} \quad \hat{k}(s,k) = \frac{\mu_4(s,k)}{(\mu_2(s,k))^2} - 3. \tag{8}$$

Equations (7) and (8) implicitly define a function

$$G(s,k) = \left(\hat{s}(s,k), \hat{k}(s,k)\right)^\top \tag{9}$$

Maillard (2012) suggests G might be invertible and tabulates some values. Figure 1 shows an empirical plot of R (top) and of $\hat{R} = \{G(s,k) : (s,k) \in R\}$ (bottom). If a distribution has skewness κ_3 and kurtosis κ_4 then we must have $\kappa_4 \geq \kappa_3^2 - 2$ and the grey regions show the areas that are excluded by this inequality. The points and dashed lines are discussed in Section 5. Both plots show lines on which s or k is constant.

G is not globally invertible. We show in Appendix A that we can write its Jacobian determinant as $|J(s,k)| = \mu_2^{9/2} S(q,k)/144$ where $S(q,k)$ is a polynomial in q and k, which has a root at approximately $(0, -0.139) \notin R$. G is still useful if we can establish two things. First, we need G to have a unique inverse for points in $\hat{R}$ so that $\tilde{F}^{-1}(u)$ is a twice continuously differentiable function of κ_3 and κ_4. Second, we need an efficient method to obtain this inverse: Newton's method is well known and converges quickly.

We show in Appendix A that $|J(s,k)| > 0$ for $(s,k) \in R$ and G has a unique inverse for $(\kappa_3, \kappa_4) \in \hat{R}$. To solve $G(s,k) = (\kappa_3, \kappa_4)$ starting from (s_0, k_0) by Newton's method we compute (s_j, k_j) iteratively using

$$J(s_j,k_j)\begin{pmatrix}\tilde{s}_j\\ \tilde{k}_j\end{pmatrix} = -\begin{pmatrix}\hat{s}(s_j,k_j) - \kappa_3/6\\ \hat{k}(s_j,k_j) - \kappa_4/24\end{pmatrix}, \tag{10}$$

$$\begin{pmatrix}s_{j+1}\\ k_{j+1}\end{pmatrix} = \begin{pmatrix}s_j\\ k_j\end{pmatrix} + \begin{pmatrix}\tilde{s}_j\\ \tilde{k}_j\end{pmatrix}, \tag{11}$$

where

$$J(s,k) = \begin{bmatrix} \frac{\partial \hat{s}}{\partial s} & \frac{\partial \hat{s}}{\partial k}\\ \frac{\partial \hat{k}}{\partial s} & \frac{\partial \hat{k}}{\partial k}\end{bmatrix}$$

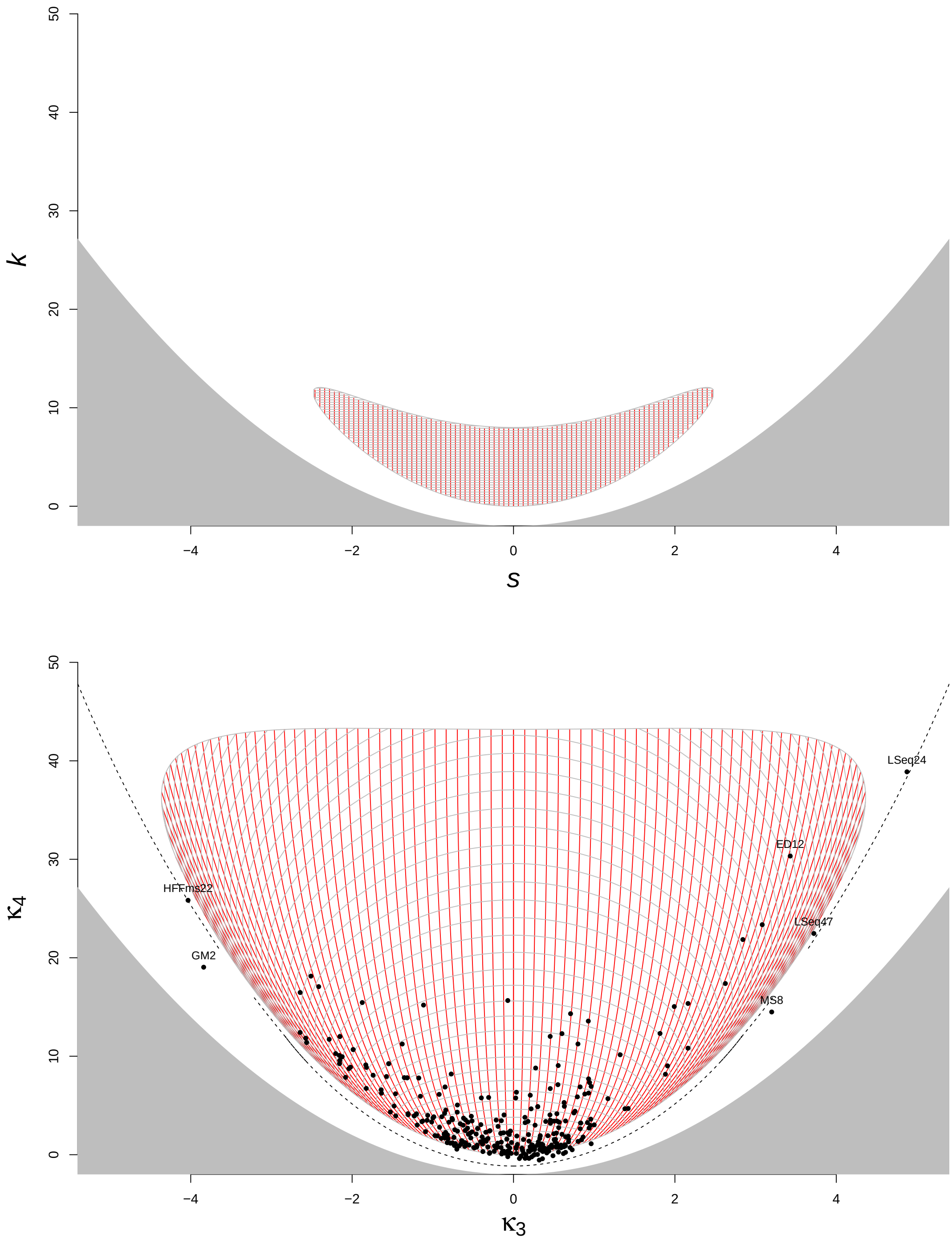

Figure 1: R and $\hat{R}$

is the Jacobian matrix of G. For Newton's method to converge we require that $\hat{s}(s,k)$ and $\hat{k}(s,k)$ be continuously differentiable and that $J(s,k)$ be nonsingular wherever we evaluate it. We note that

$$\mu_2(s,k) = 1 + 6k^2 - 24qk + 25q^2 \geq 1 + 6(k^2 - 4qk + 4q^2) \geq 1 + 6(k-2q)^2 \geq 1.$$

Hence $\hat{s}$ and $\hat{k}$ are twice continuously differentiable. Since $G(0,0) = (0,0)$ we can set $s_0 = k_0 = 0$. If $\kappa_3 \geq 0$ we must have $(s,k) \in S^+$ (see Appendix A) and so can replace equation (11) with

$$\begin{pmatrix} s_{j+1} \\ k_{j+1} \end{pmatrix} = \begin{pmatrix} s_j \\ k_j \end{pmatrix} + \alpha \begin{pmatrix} \tilde{s}_j \\ \tilde{k}_j \end{pmatrix} \tag{12}$$

where we can choose $\alpha \in (0,1]$ so that (s_{j+1}, k_{j+1}) is in some suitably chosen convex subset of S^+. In practice, we find that it is enough to restrict search to S^+. And if $\kappa_3 < 0$ we can choose $\alpha \in (0,1]$ in equation (12) so that $(s_{j+1}, k_{j+1}) \in S^-$. In either case $J(s_{j+1}, k_{j+1})$ is nonsingular and Newton's method improves at each step.

Suppose we want a distribution with mean μ, variance σ^2, skewness κ_3 and kurtosis κ_4. Suppose also μ and σ are finite and $(\kappa_3, \kappa_4) \in \hat{R}$. Then we can use Newton's method to evaluate $G^{-1}(\kappa_3, \kappa_4)$, giving us values for s and k and hence a_0, a_1, a_2 and a_3. Define

$$F^{-1}(u; \mathbf{p}) = \mu + \sigma \mu_2^{-1/2} \sum_{j=0}^{3} a_j \left(\Phi^{-1}(u)\right)^j, \tag{13}$$

where μ_2 is given by equation (7). $\tilde{F}$, defined by equation (5), has mean 0, variance μ_2 skewness κ_3 and kurtosis κ_4. Skewness and kurtosis are invariant under scaling and shifting. So F has mean μ, variance σ^2, skewness κ_3 and kurtosis κ_4. We call it the *Cornish–Fisher distribution with parameters* μ, σ, κ_3 *and* κ_4. Thus equation (13) defines a family of distributions for finite μ, finite $\sigma > 0$ and $(\kappa_3, \kappa_4) \in \hat{R}$. We write $Y \sim \mathcal{F}(\mathbf{p})$ to indicate Y is a distribution in this family. For reasons that become clearer at the end of Section 3 and in Section 5 we occasionally abuse notation by referring to F as a Cornish–Fisher distribution when $(\kappa_3, \kappa_4) \notin \hat{R}$.

The Cornish–Fisher distribution that approximates a distribution by putting $s = \kappa_3/6$ and $k = \kappa_4/24$ is widely used. We call it the *uncorrected Cornish–Fisher distribution* and show in Section 5 that it is often a poor approximation.

Properties of the Cornish–Fisher distribution

Suppose $Y \sim \mathcal{F}(\mathbf{p})$. Equations (4) and (13) give

$$F^{-1}(u; \mathbf{p}) = \mu + \sigma \mu_2^{-1/2} \xi \left(\Phi^{-1}(u)\right). \tag{14}$$

Proposition 1 F^{-1} *is twice continuously differentiable with respect to* $\mathbf{p} = (\mu, \sigma, \kappa_3, \kappa_4)$.

Proof This follows easily from equation (14) when we note that μ_2 and ξ are twice continuously differentiable. ∎

We can use the Cornish–Fisher distribution to estimate the α quantile, $F^{-1}(\alpha)$, and the (lower) α tail mean,

$$\textsc{tm}(X;\alpha) = \mathbb{E}\left[X : X \leq F^{-1}(\alpha)\right]$$

of X. These are often seen in finance as value-at-risk (Tee 2009), $\textsc{var}(X;\alpha) = -F^{-1}(\alpha)$, and conditional-value-at-risk, $\textsc{cvar}(X;\alpha) = -\textsc{tm}(X;\alpha)$. Acerbi (2002) shows

$$\textsc{tm}(X;\alpha) = \frac{1}{\alpha}\int_0^\alpha F^{-1}(u)\,\mathrm{d}u.$$

Put

$$\tau_r(\alpha) = \frac{1}{\alpha\sqrt{2\pi}}\int_{-\infty}^{\Phi^{-1}(\alpha)} z^r \exp\left(-\frac{1}{2}z^2\right)\mathrm{d}z.$$

Then, if Y has Cornish–Fisher distribution approximating X,

$$\begin{aligned}
\textsc{tm}(Y;\alpha) &= \frac{1}{\alpha}\int_0^\alpha \left(\mu + \sigma\mu_2^{-1/2}\left(\sum_{r=0}^{3} a_r(\Phi^{-1}(u))^r\right)\right)\mathrm{d}u \\
&= \mu + \sigma\mu_2^{-1/2}\sum_{r=0}^{3} a_r \frac{1}{\alpha}\int_0^\alpha (\Phi^{-1}(u))^r\,\mathrm{d}u \\
&= \mu + \sigma\mu_2^{-1/2}\sum_{r=0}^{3} a_r \frac{1}{\alpha\sqrt{2\pi}}\int_{-\infty}^{\Phi^{-1}(\alpha)} z^r \exp\left(-\frac{1}{2}z^2\right)\mathrm{d}z \\
&= \mu + \sigma\mu_2^{-1/2}\sum_{r=0}^{3} a_r\tau_r(\alpha).
\end{aligned}$$

This last expression is a twice continuously differentiable function of $\mathbf{p}$ and so, by Proposition 1, $\textsc{var}$, $\textsc{tm}$ and $\textsc{cvar}$ are also in $C^2_{\mathbf{p}}$. To use this result in practice we need expressions to compute the partial derivatives of F^{-1}. Appendix B derives such expressions.

The tail means and quantiles are only defined if $\xi(u)$ is strictly increasing on $\mathbb{R}$. Suppose, however, that $\xi(u)$ is increasing on (u_1, u_2) and that $\alpha \in (\alpha_1, \alpha_2)$ with $\alpha_1 = \Phi(u_1)$ and $\alpha_2 = \Phi(u_2)$. Then both $F^{-1}(Y;\alpha)$ and $\textsc{tm}(Y;\alpha)$ are sensibly defined and are the quantile and tail mean of a distribution given by

$$F'(x) = \begin{cases} F_1(x), & x \leq x_1, \\ F(x), & x_1 < x < x_2, \\ F_2(x), & x \geq x_2, \end{cases}$$

for $x_1 = F^{-1}(u_1)$ and $x_2 = F^{-1}(u_2)$ if we can find strictly increasing continuous functions F_1 and F_2 satisfying $F_1(x_1) = \alpha_1$, $F_2(x_2) = \alpha_2$,

$$\frac{1}{\alpha_1}\int_0^{\alpha_1} F_1^{-1}(u)\,\mathrm{d}u = \mu + \sigma\mu_2^{-1/2}\sum_{r=0}^{3} a_r\tau_r(\alpha_1), \tag{15}$$

$$\frac{1}{1-\alpha_2}\int_{\alpha_2}^{1} F_2^{-1}(u)\,\mathrm{d}u = \mu + \sigma\mu_2^{-1/2}\sum_{r=0}^{3} a_r\tau_r(1-\alpha_2).$$

Such functions are easy to find. For example, $F_1^{-1}(u) = mu + x_1 - m\alpha_1$ satisfies $F_1(x_1) = \alpha$ and

$$\frac{1}{\alpha_1}\int_0^{\alpha_1} F^{-1}(u)\,\mathrm{d}u = x_1 - \frac{1}{2}m\alpha_1,$$

and it is straightforward to show

$$\mu + \sigma\mu_2^{-1/2}\sum_{r=0}^{3} a_r\tau_r(\alpha) < x_1.$$

So if we choose m satisfying equation (15) holds then $m > 0$ and so F_1 is, as required, strictly increasing. It follows that if $(\kappa_3, \kappa_4) \notin \hat{R}$ but $\alpha \in (\alpha_1, \alpha_2)$ we can still use the Cornish–Fisher distribution to estimate quantiles and tail means. We return to this issue in Section 5.

We can rearrange equation (14) to get the distribution function

$$F(x) = u = \Phi\left(\xi^{-1}\left(\mu_2^{1/2}\frac{x-\mu}{\sigma}\right)\right). \tag{16}$$

In the R-package in the supplementary material we use Newton's method to evaluate ξ^{-1} rather than the formula for the roots of a cubic, because $\xi' > 0$ guarantees the root is unique and Newton's method converges.

Proposition 2 *Let $Y \sim \mathcal{F}(\mathbf{p})$. The density function of Y is*

$$f(x) = \frac{\mu_2^{1/2}\phi(v)}{\sigma\xi'(v)}, \tag{17}$$

with $v = \xi^{-1}\left(\mu_2^{1/2}(x-\mu)/\sigma\right)$. The distribution function is smooth, and $f(x)$ is unimodal and satisfies $f(x) > 0$ for $x \in \mathbb{R}$.

Proof First note that

$$\xi'(u) = a_1 + 2a_2u + 3a_3u^2, \qquad \xi^{(2)}(u) = 2a_2 + 6a_3u, \qquad \xi^{(3)}(u) = 6a_3, \tag{18}$$

and higher order derivatives are zero. And

$$v'(x) = \frac{1}{\xi'(v)}\frac{\mathrm{d}v}{\mathrm{d}x}\frac{\mu_2^{1/2}(x-\mu)}{\sigma}(x) = \frac{\mu_2^{1/2}}{\sigma\xi'(v)}.$$

Thus, differentiating $F(\Phi(v))$, we get equation (17). Since $\phi(v) > 0$, $\sigma > 0$ and $\xi'(v) > 0$ for $v \in \mathbb{R}$, we have $f(x) > 0$ for $x \in \mathbb{R}$.

Differentiating equation (17) and substituting $\phi'(v) = -v\phi(v)$, we get

$$f'(v) = -\frac{\mu_2 \phi(v)}{\sigma^2 \left[\xi'(v)\right]^3} \left(v\xi'(v) + \xi^{(2)}(v)\right),$$

which is zero whenever $p(v) = v\xi'(v) + \xi^{(2)}(v) = 0$. So f is unimodal whenever $p(v)$ has only one root. Since $p(v)$ is a cubic polynomial it has either one or three roots and has only one root whenever $p'(v) > 0$ for $v \in \mathbb{R}$. And

$$\begin{aligned} p'(v) &= \xi'(v) + v\xi^{(2)}(v) + \xi^{(3)}(v) \\ &= 9a_3v^2 + 4a_2v + a_1 + 6a_3. \end{aligned}$$

This has discriminant $16a_2^2 - 36a_3(a_1 + 6a_3) = -4d(q, k)$ with

$$d(q, k) = 126q^2 - 153kq - 22q + 45k^2 + 9k.$$

We solve

$$\frac{\partial d}{\partial q} = 252q - 153k - 22 = 0, \qquad \frac{\partial d}{\partial k} = 90k - 153q + 9 = 0$$

to show $d(q, k)$ has a minimum, $188/81$, at $q = -67/81$, $k = -122/81$. Thus $d(q, k) > 0$ and so the discriminant is always negative. It follows that $p'(v)$ has no real roots. Since $p'(v) > 0$ for large enough v, we have $p'(v) > 0$ for $v \in \mathbb{R}$. It follows that f is unimodal.

It remains to show F is smooth. It is clearly continuous. So we need to show f and all its derivatives are also continuous. Define $g_0(v) = \mu_2^{1/2}\phi(v)/\sigma$ and

$$g_{n+1}(v) = \frac{\mu_2^{1/2}}{\sigma}\left[g_n'(v)\xi'(v) - (2n+1)g_n(v)\xi^{(2)}(v)\right]$$

for $n = 0, 1, 2, \ldots$. Then g_{n+1} is smooth if g_n is smooth. And g_0 is smooth. So g_n is smooth for all $n \geq 0$. And, since $\xi'(v) > 0$ and $v = \xi^{-1}\left(\mu_2^{1/2}(x-\mu)/\sigma\right)$ is smooth,

$$h_n(x) = \frac{g_n(v)}{\left[\xi'(v)\right]^{2n+1}}$$

is also smooth for all $n \geq 0$. We have defined $h_n(x)$ so that $h_0(x) = f^{(0)}(x) = f(x)$. Suppose it is not true that $f^{(n)}(x) = h_n(x)$ for all $n > 0$ and choose k minimal such that $h_{k+1}(x) \neq f^{(k+1)}(x)$. Then

$$\begin{aligned} f^{(k+1)}(x) &= \left(f^{(k)}\right)'(x) = (h_k)'(x) \\ &= \frac{g_k'(v)\left[\xi'(v)\right]^{2k+1} - (2k+1)g_k(v)\left[\xi'(v)\right]^{2k}\xi^{(2)}(v)}{\left[\xi'(v)\right]^{4k+2}} \frac{\mu_2^{1/2}}{\sigma\xi'(v)} \\ &= \frac{\mu_2^{1/2}}{\sigma} \frac{g_k'(v)\xi'(v) - (2k+1)g_k(v)\xi^{(2)}(v)}{\left[\xi'(v)\right]^{2k+3}} \\ &= \frac{g_{k+1}(v)}{\left[\xi'(v)\right]^{2(k+1)+1}} \\ &= h_{k+1}(x), \end{aligned}$$

contradicting the minimality of k. It follows that $f^{(n)}(x) = h_n(x)$ for all $n \geq 0$ and so F is smooth. ∎

Random variates and a goodness-of-fit test

Random variate generators (Devroye 1986) are important in simulation studies and much work has been done to find fast random variate generators. There are well-known methods (Marsaglia & Tsang 2000) for generating standard normal variates fast. And given a standard normal variate Z

$$\mu + \sigma\mu_2^{-1/2}\xi(Z) \quad \text{and} \quad \exp\left(\mu + \sigma\mu_2^{-1/2}\xi(Z)\right)$$

are fast random variate generators for the Cornish–Fisher distribution and log Cornish–Fisher distribution (see Section 6.2) with parameters $\mathbf{p}$. The log Cornish–Fisher distribution is plausibly useful, for example in discrete-event simulation, when we want to simulate service times, which cannot be negative. Otherwise, as is usual when we simulate times from a normal distribution, we must reject negative values and in effect sample from a left-truncated distribution.

The Anderson–Darling test (Anderson & Darling 1952) is a powerful general test of goodness-of-fit. The probability of obtaining an Anderson–Darling test statistic of at least a given size depends on the distribution and is only known by simulation. Cheng (2006) develops a bootstrap Anderson–Darling test that can be used when we can compute both random variates and the distribution function of random variable. Equation (16) gives the Cornish–Fisher distribution function and we have just described how to find random variates. So we can carry out a bootstrap Anderson–Darling goodness-of-fit test given a Cornish–Fisher distribution and data we think may fit it.

We implement the bootstrap Anderson–Darling test in the R-package provided in the supplementary material.

An application to investment fund distributions

To illustrate the Cornish–Fisher distribution we consider the monthly returns of 339 investment funds from January 2000 to December 2012. We choose these data because investment returns often plausibly have a unimodal density that is too skewed, platykurtic or leptokurtic to be normal. These data originate from hedge funds. We collected monthly returns of the various strategies listed in Table 1. The data come from Morningstar (2014), which describes the strategies in detail.

The points in the lower chart of Figure 1 show the skewness and kurtosis (estimated by k-statistics) for each investment except GM15, HFFms22 and SF47, which have skewness 5.672, -9.099 and 6.26 out of the range of the chart. The prefix on the labelled funds shows its type. We choose a range of types to ensure a range of values of skewness and kurtosis. Table 1 shows the prefixes and fund types.

We fit Cornish–Fisher distributions for as many of the funds as we can using the R-package in the supplementary material. We do this even if $(\kappa_3, \kappa_4) \notin \hat{R}$ and we compute a bootstrap Anderson–Darling test statistic whenever we can. We fail to fit Cornish–Fisher distributions for seven funds: GM2, GM15, HFFms9, HFFms22, LSeq24, MS8 and SF47. All are strongly skewed. GM4, LSeq22 and 13 SF funds also have $(\kappa_3, \kappa_4) \notin \hat{R}$. We find Cornish–Fisher

Table 1: Investment fund types

PREFIX	TYPE	PREFIX	TYPE
HFFeq	Hedge funds of funds for equity	MS	Multi-strategy
HFFms	Hedge funds of funds for multi-strategy	SF	Systematic futures
HFFmasy	Hedge funds of funds for macro systematic	GM	Global Macro
LSeq	Long-short equity strategy	ED	Event driven strategy

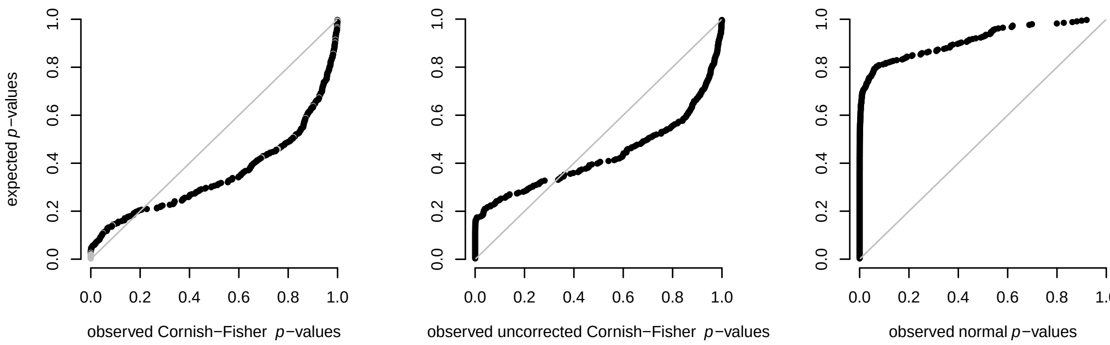

Figure 2: Anderson–Darling p-values for Cornish–Fisher and normal distributions

distributions for them when we relax the requirement that the distribution function is defined in its tails. We tried using a Jarque–Bera test in place of Anderson–Darling and found very similar results.

If we assume all fund returns have Cornish–Fisher distributions with known parameters and are independent then the p-values from the Anderson–Darling tests should be uniformly distributed on $[0, 1]$. Figure 2 (left) shows the observed and expected quantiles under this assumption. We record a p-value of 0 when we cannot fit a distribution. The funds with $(\kappa_3, \kappa_4) \in \hat{R}$ have black points and the rest grey. For comparison we show the Anderson–Darling p-values if we use uncorrected Cornish–Fisher distributions (centre), defined at the end of Section 2, and normal distributions (right). We omit the 59 cases where the uncorrected Cornish–Fisher expansion does not give a distribution function.

Both the Cornish–Fisher and uncorrected Cornish–Fisher fits have better than expected p-values for many funds. We think this is largely because the four parameters allow the distribution to fit the data better than its population. We note that the correlation between many pairs of fund returns is too high for them to be plausibly independent.

Figures 3–5 show more detail for some of the fitted Cornish–Fisher distributions. Each chart shows a quantile–quantile plot, the fitted distribution function together with an empirical distribution function, and a histogram together with the fitted density function (solid) and a density function estimated by kernel density estimation. The p shown is the p-value from the bootstrap Anderson–Darling test.

Depending on how accurate a fit is needed, 20–30 funds do not plausibly fit their (corrected) Cornish–Fisher distributions or do not have one. Most of these have one or two extreme values and $(\kappa_3, \kappa_4) \notin \hat{R}$, though ED12 (Figures 1 and 3 left) does not. MS19 (Figure 3 right) and

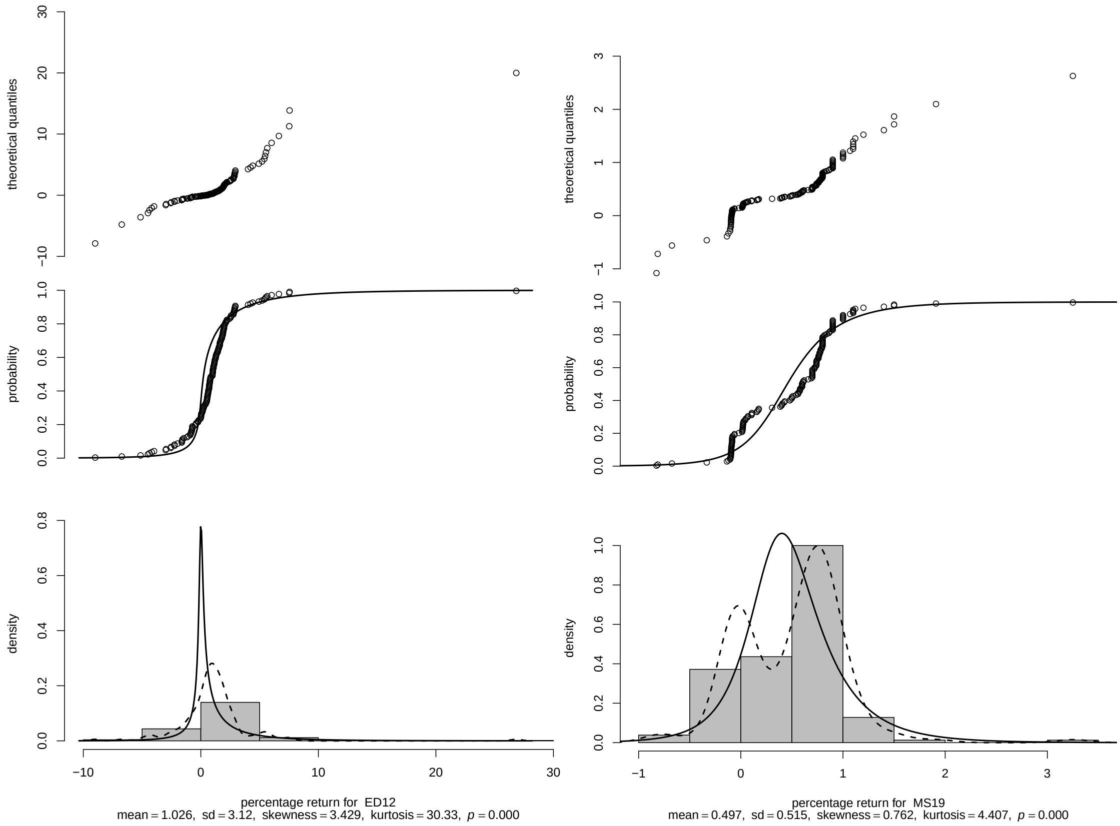

Figure 3: Poor-fit Cornish–Fisher distributions

HFFms42 (not shown) have $(\kappa_3, \kappa_4) \notin \hat{R}$ but density functions that are likely not unimodal.

Figure 4 shows two better-fit distributions. LSeq55 (left) is the 302nd best fit and SF64 (right), the 149th. Figure 5 shows two weaker-fit distribution. GM11 (left) is the 320th best fit. GM4 (right) has the least good fit of the 13 funds with negative kurtosis. We find nine of these funds have Anderson–Darling estimated p-values exceeding 0.9. To see why this might happen, consider the case $s = 0$. The Jacobian has a zero approximately at $k = -0.139$ corresponding to kurtosis about -1.31. And $\xi(u)$ is increasing between its roots at $\pm\sqrt{1 - 1/(3k)}$. Thus, in the worst case, when $k = -0.139$ the distribution is well defined for approximately 1.84 standard deviations on either side of the mean value. That is, only the extreme tails of the distribution function are undefined.

The dashed lines on Figure 1 approximate and interpolate skewness and kurtosis corresponding to some of the zeroes of the Jacobian. The lines are not simple. For example, we estimate three closely spaced zeros for skewness about ± 2.7. Nonetheless, we conjecture that the region including R on which G is invertible includes most of the points between $\hat{R}$ and this line. This would account for the very good fit of funds with small or negative kurtosis not in $\hat{R}$.

Section 3 showed we can estimate quantiles and tail-means using the Cornish–Fisher distribution. The estimates should be less influenced by outliers than, for example, the method of

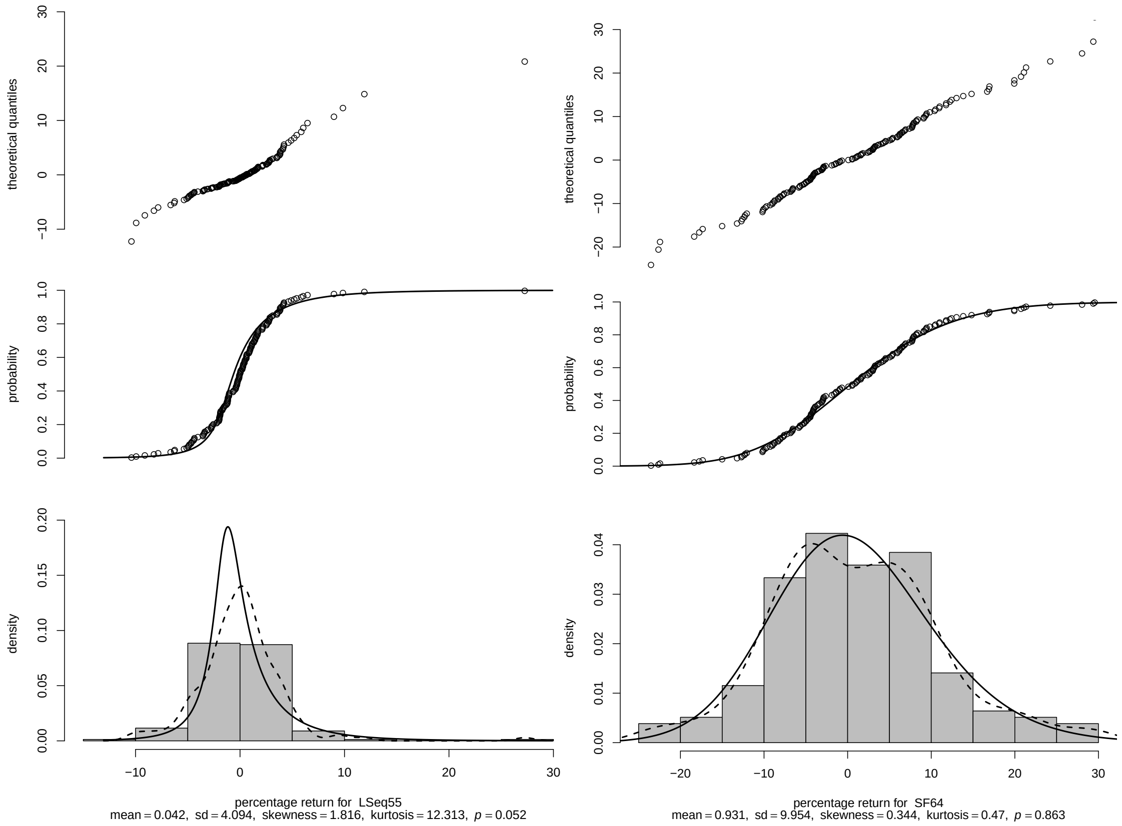

Figure 4: Good-fit Cornish–Fisher distributions

Acerbi (2007) for estimating CVaR (negative of tail mean) directly from the data.

We compute CVaR at $\alpha = 0.1$ using the Cornish–Fisher distribution, the uncorrected Cornish–Fisher distribution and the empirical method of Acerbi (2007). We exclude here the 21 from 339 cases where $(\kappa_3, \kappa_4) \notin \hat{R}$. We find the mean absolute percentage deviation of the Cornish–Fisher estimate from the empirical estimate is 5.7%. We find also a mean percentage deviation of -0.4%. We exclude 59 cases to compute the same percentages for the uncorrected Cornish–Fisher distribution and find a mean absolute percentage deviation of 6.0% and mean percentage deviation of -0.7%.

While there is no correct way to estimate CVaR, we find little evidence for inaccuracy or bias using either the Cornish–Fisher or the uncorrected Cornish–Fisher distribution. Although it takes longer to compute the corrected parameters—we found an average of 1.8 ms compared to 0.2 ms using R (a slow computer language) on a 3.5 GHz Intel i5 processor—, the extra computation is negligible.

Section 3 notes that we can sometimes estimate CVaR when $(\kappa_3, \kappa_4) \notin \hat{R}$. For example, for GM4 (see Figure 5 right) ξ is increasing between about -0.917 and 1.428 so that the Cornish–Fisher expansion defines a quantile function except in the extreme (< 0.005) tails. The Cornish–Fisher CVaR at 10% is 7.172 while the empirical value is 7.369.

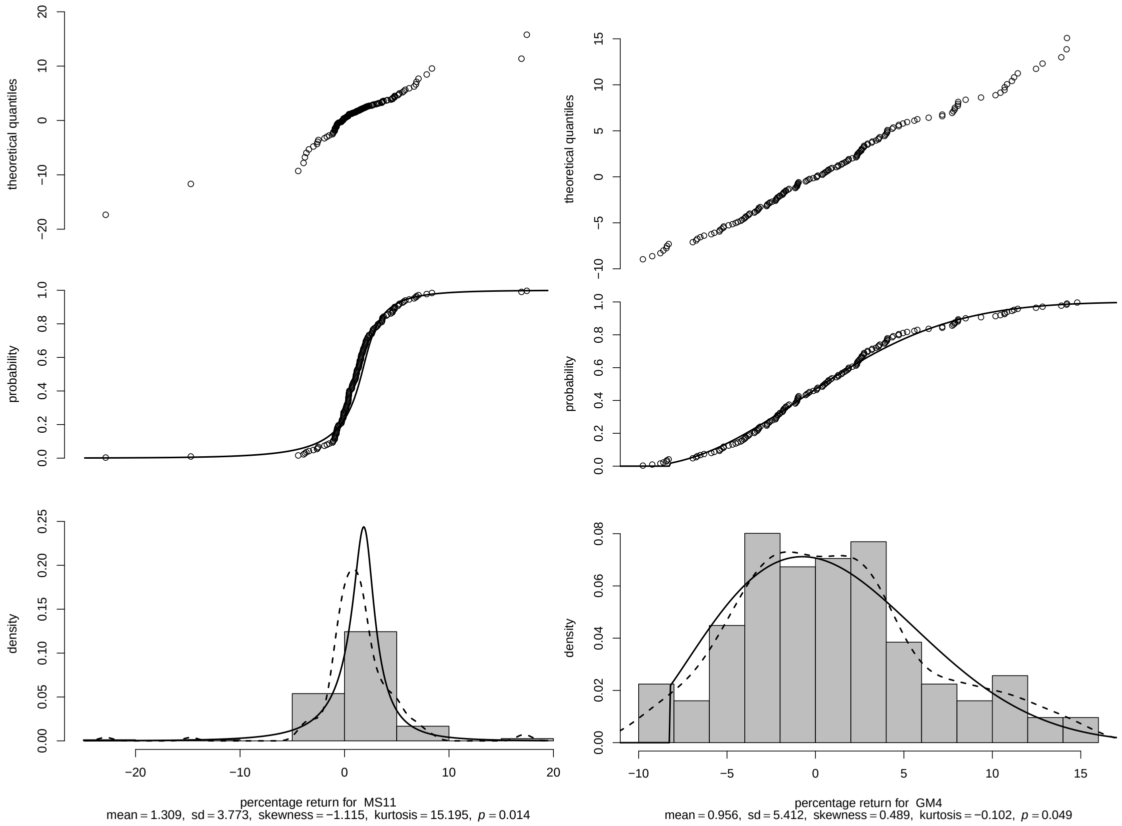

Figure 5: Weak-fit Cornish–Fisher distributions

Generalisations of the Cornish–Fisher distribution

Hill & Davis (1968) derive a general Cornish–Fisher expansion where Φ need not be the normal distribution function. So it is reasonable to ask whether we can derive four-parameter distribution families using some distribution other than the normal. We show here that a finite-order Cornish–Fisher expansion with a lognormal, gamma, or beta distribution function does not give a distribution function in the same way that the normal distribution does. We also show that for $X \sim \mathcal{F}(\mathbf{p})$, $\log X$ has a reasonably well behaved distribution function.

Cornish–Fisher expansions with non-normal Φ

In Appendix D we find the form of the Cornish–Fisher expansion when Φ is the gamma, beta or lognormal. From equations (D.33) and (D.34) of Appendix D it is evident that when Φ is gamma the fourth-order expansion has the form

$$x = b_0 u + \sum_{k=1}^{5} b_k u^{-k}$$

for some constants $b_0, \ldots, b_5$. For the expansion to define a distribution function we require $x = 0$ when $u = 0$. This only happens in the degenerate case when $b_1 = \cdots = b_k = 0$.

When Φ is lognormal the form of the fourth-order expansion is

$$x = u + \sum_{k=2}^{5} b_k(\log(u))u^{-k}$$

for some polynomials b_k of degree k (equations (D.33) and (D.36) of Appendix D). Again we require $x = 0$ when $u = 0$, which only happens in the degenerate case when $x = u$.

When Φ is beta the form of the fourth-order expansion is

$$x = u + b_0 u(u-1)^{-4} + \sum_{k=0}^{6}\sum_{l=0}^{6} b_{kl}u^{-k}(u-1)^{-l}$$

for some constants b_0 and b_{kl}, some of which are nonzero if the distribution we wish to fit does not have the same first four cumulants as a beta distribution (equations (D.33) and (D.35) of Appendix D). Again we require $x = 0$ when $u = 0$, which only happens in the degenerate case when all b_{kl} are equal to zero.

The fourth-order expansions fail to give distribution functions because they contain a term u^{-k} with nonzero coefficient for some positive integer k. It is likely easy to show that expansions of any order have the same problem for the gamma, beta or lognormal distributions.

Rather than use expansions in the cumulants of Φ, plausibly we might define

$$\zeta(U) = a_0 + a_1 U + a_2 U^2 + a_3 U^3 \tag{19}$$

for a random variable U with distribution Φ and seek values for a_0, a_1, a_2 and a_3 so that $X = \zeta(U)$ has specified moments. Appendix C shows that this gives an nonconvex optimisation problem that we know of no way to solve except when Φ is normal.

The log Cornish–Fisher distribution

If we want a distribution with similar properties to the lognormal distribution, then the logarithm of the Cornish–Fisher distribution is a reasonable candidate. We call it the *log Cornish–Fisher distribution* and define it by $Y = \log(X)$, where $X \sim \mathcal{F}(\mathbf{p})$. Note that the parameters $\mathbf{p}$ are the mean, standard deviation, skewness and kurtosis of X rather than Y.

The lognormal distribution is unimodal. So we might expect the log Cornish–Fisher distribution also to be unimodal. But it turns out that this cannot happen except in the degenerate case where skewness and kurtosis are zero. The density function of the log Cornish–Fisher distribution is $g(x) = x^{-1}f(\log(x))$, where f is the Cornish–Fisher density. Put $v = \xi^{-1}(\mu_2^{1/2}\log(x) - \mu)/\sigma)$. Then

$$\begin{aligned}
g'(x) &= x^{-2}\left[f'(\log(x)) - f(\log(x))\right] \\
&= x^{-2}\left[-\frac{\mu_2\phi(v)}{\sigma^2[\xi'(v)]^3}(vi'(v) + \xi^{(2)}(v)) - \frac{\mu_2^{1/2}\phi(v)}{\sigma[\xi'(v)]}i'(v)\right] \\
&= -x^{-2}\frac{\mu_2\phi(v)}{\sigma^2[\xi'(v)]^3}\left[v\xi'(v) + \xi^{(2)}(v) + \frac{\sigma}{\mu_2^{1/2}}\left[\xi'(v)\right]^2\right].
\end{aligned}$$

Hence $g'(x) = 0$ if and only if $d(v) = v\xi'(v) + \xi^{(2)}(v) + \sigma\left[\xi'(v)\right]^2/\mu_2^{1/2} = 0$. Using equations (18), we find $d(v)$ is a quartic polynomial in v in which the coefficient of v^4 is $\sigma a_3^2/\mu_2^{1/2} > 0$. Since the range of v for $x \in (0,\infty)$ is $(-\infty,\infty)$, $d(v)$ always has an even number of roots and so $g(v)$ cannot be unimodal.

Fitting sums of densities

We noted in Section 1 that fitting a sum of densities gives a plausible alternative to the Cornish–Fisher distribution. Here we discuss a method and its generalisations and show that it does not work well with the normal distribution.

Let $w(x)$ be a density function and suppose $p_n(x) = \sum_{k=0}^{n} p_{nk}x^k$ $(n = 0,1,2,\ldots)$ are orthogonal polynomials with respect to $w(x)$: that is,

$$\int_S p_m(x)p_n(x)w(x)\,\mathrm{d}x = \delta_{mn},$$

where S is the region over which the density is defined and δ_{mn} is the Kronecker delta. The following generalises a result of Bowers (1966) where $w(x)$ is a gamma density.

Proposition 3 *Suppose $f(x)$ is the density function of a distribution with finite moments*

$$\mu_k = \int_S x^k f(x)\,\mathrm{d}x, \qquad k = 0,1,2,\ldots.$$

Suppose also we can write

$$f(x) = w(x)\sum_{n=0}^{\infty} A_n p_n(x) \tag{20}$$

for constants $A_0, A_1, \ldots$. Then $A_n = \sum_{k=0}^{n} p_{nk}\mu_k$ and, for $N \geq 0$, the first $N+1$ moments of $\hat{f}(x)$ given by equation (20) are $\mu_0, \ldots, \mu_k$.

We prove this in Appendix E. Bowers (1966) uses a special case to derive an expression for a distribution with mean and variance α and some known higher moments as a sum of gamma densities. Bowers does not note that $\hat{f}$ is not a density for all values of $\mu_0, \ldots, \mu_N$. Although it is reasonably easy to check if a particular set of moments makes $\hat{f}$ a density, we have not found the general region in which it is. We do not consider the gamma distribution further.

The orthogonal polynomials with respect to the normal density are the Hermite polynomials, the first five of which are

$$p_0(x) = 1, \quad p_1(x) = x, \quad p_2(x) = x^2 - 1, \quad p_3(x) = x^3 - 3x, \quad p_4(x) = x^4 - 6x^2 + 3.$$

We can assume without loss of generality that $\mu_0 = 0$ and $\mu_2 = 1$, which gives us

$$A_0 = 1, \quad A_1 = A_2 = 0, \quad A_3 = \mu_3, \quad \text{and} \quad A_4 = \mu_4 - 3.$$

and

$$\begin{aligned}\hat{f}(x) &= \phi(x)\left[1+\mu_3(x^3-3x)+(\mu_4-3)(x^4-6x^2+3)\right]\\ &= \phi(x)\left[(\mu_4-3)x^4+\mu_3x^3-6(\mu_4-3)x^2-3\mu_3x+3\mu_4-8\right]\\ &= \phi(x)\left[\kappa_4x^4+\kappa_3x^3-6\kappa_4x^2-3\kappa_3x+3\kappa_4+1\right],\end{aligned}$$

substituting $\kappa_3 = \mu_3$ and $\kappa_4 = \mu_4 - 3$.

We must have $f(x) \geq 0$ for it to be a density. Equivalently $g(x) = \kappa_4x^4 + \kappa_3x^3 - 6\kappa_4x^2 - 3\kappa_3x + 3\kappa_4 + 1 \geq 0$. Put $u = 1$ or -1 according as $\kappa_3 \geq 0$ or $\kappa_3 < 0$. Then $g(u) = -2\kappa_4 - 2|\kappa_4| + 1$ and so κ_3 and κ_4 must satisfy $|\kappa_3| + \kappa_4 \leq 1/2$, which is too small a range for most practical use.

Conclusion

Maillard (2012) derives expressions for the skewness and kurtosis of the Cornish–Fisher expansion and suggests it is possible we can choose parameters s and k to obtain a distribution with desired skewness κ_3 and kurtosis κ_4. We correct and extend Maillard (2012), showing how to find s and k efficiently. Moreover we show a region $\hat{R}$ such that for $(\kappa_3, \kappa_4) \in \hat{R}$ the s and k we obtain are unique and give a Cornish–Fisher expansion with a valid distribution function. Thus we derive a family $\mathcal{F}(\mathbf{p})$ of Cornish–Fisher distributions with quantile functions in $C^2_{\mathbf{p}}$ whose mean, standard deviation, skewness and kurtosis are given by $\mathbf{p} = (\mu, \sigma, \kappa_3, \kappa_4)$.

We show the Cornish–Fisher distribution has, like the normal distribution, a unimodal density that is positive on all of $\mathbb{R}$. In contrast, the log Cornish–Fisher distribution is not unimodal unless it is also lognormal.

We show the Cornish–Fisher expansion may be useful even when it does not give a distribution function, in particular for modelling small negative kurtosis. We have not succeeded in extending our proof of uniqueness of s and k to cases where the distribution function is defined on a finite, but useful, range.

The region $\hat{R}$ of skewness and kurtosis on which we can use a Cornish–Fisher distribution is large enough to be practically useful. We illustrate this with an example of investment fund distributions and suggest future application to portfolio optimisation, simulation and as a replacement for BCPE in GAMLSS (Stasinopoulos & Rigby 2007).

We show that many of the plausible generalisations of the Cornish–Fisher expansions do not give useful distributions.

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Appendix A

We show G, defined by equation (9) is invertible on a region $S = S^+ \cup S^-$ containing R. We write $q = s^2$ and note that for $(s, k) \in R$ equation (4) gives

$$k \geq 2q, \tag{A.21}$$

because R is the region on which $\xi' > 0$, which requires $a_3 \geq 0$.

R is defined by

$$30q^2 + (7 - 33k)\, q + 9k^2 - 3k \leq 0$$

(Maillard 2012), a quadratic in $q \geq 0$. This is only nonpositive between its roots and its discriminant is $9k^2 - 102k + 49$. This must be nonnegative. And so we have

$$k \leq \frac{17 - 4\sqrt{15}}{3} < \frac{51}{100}$$

when $q = 3 - 11/\sqrt{15} < 4/25$. Thus we can be sure that any (q, k) satisfying $(\sqrt{q}, k) \in R$ lies below the line through $(0, 1/3)$ and $(4/25, 51/100)$: that is, the line $k = 53q/48 + 1/3$. Define S^+ to be the region given by $q < 3 - 2\sqrt{2}$, $s \geq 0$ and

$$\frac{1 + 11q - \sqrt{q^2 - 6q + 1}}{6} < k < \frac{53q}{48} + \frac{1}{3}. \tag{A.22}$$

Define $S^- = \{(s, k) : (-s, k) \in S^+\}$. Then, or $(s, r) \in R$ with $s \geq 0$, $(s, r) \in S^+$ and $(-s, r) \in S^-$.

Notice that we can restrict search for (s, k) satisfying $G(s, k) = (\hat{s}, \hat{k})$ to S^+ or S^-, because s and $\hat{s}$ have the same sign. Ideally we would like S^+ and S^- to be convex by replacing $k = 53q/48 + 1/3$ with $k = 53s/120 + 1/3$, but we find G is not invertible on such a reason. It is straightforward to restrict search to convex subsets of S^+ or S^- though we find in practice it is enough to restrict search to S^+ or S^-.

We have

$$\frac{\partial \hat{s}}{\partial s} = {\mu_2}^{-3/2} \frac{\partial \mu_3}{\partial s} - \frac{3}{2} \mu_3 {\mu_2}^{-5/2} \frac{\partial \mu_2}{\partial s}, \qquad \frac{\partial \hat{s}}{\partial k} = {\mu_2}^{-3/2} \frac{\partial \mu_3}{\partial k} - \frac{3}{2} \mu_3 {\mu_2}^{-5/2} \frac{\partial \mu_2}{\partial k},$$

$$\frac{\partial \hat{k}}{\partial s} = {\mu_2}^{-2} \frac{\partial \mu_4}{\partial s} - 2 \mu_4 {\mu_2}^{-3} \frac{\partial \mu_2}{\partial s}, \qquad \frac{\partial \hat{k}}{\partial k} = {\mu_2}^{-2} \frac{\partial \mu_4}{\partial k} - 2 \mu_4 {\mu_2}^{-3} \frac{\partial \mu_2}{\partial k}.$$

And

$$\begin{aligned} \frac{\partial \mu_1}{\partial s} &= 0, \qquad \frac{\partial \mu_2}{\partial s} = 100s^3 - 48sk, \\ \frac{\partial \mu_3}{\partial s} &= 2550s^4 - 1404ks^2 - 228s^2 + 108k^2 + 36k + 6, \\ \frac{\partial \hat{\mu}_4}{\partial s} &= 519960s^7 - 742320ks^5 - 14400s^5 + 353520k^2s^3 \\ &\quad + 32544ks^3 - 168s^3 - 56160k^3s - 12096k^2s - 1008ks. \end{aligned} \tag{A.23}$$

And

$$\begin{aligned}
&\frac{\partial \mu_1}{\partial k} = 0, \qquad \frac{\partial \mu_2}{\partial k} = 12k - 24s^2, \\
&\frac{\partial \mu_3}{\partial k} = -468s^3 + 216ks + 36s, \\
&\frac{\partial \mu_4}{\partial k} = -123720s^6 + 176760ks^4 + 8136s^4 - 84240k^2s^2 \\
&\qquad - 12096ks^2 - 504s^2 + 13392k^3 + 3888k^2 + 504k + 24.
\end{aligned} \tag{A.24}$$

Hence

$$\begin{aligned}
S(q,k) &= \frac{\mu_2^{9/2}|J|}{144} \\
&= -94775q^7 - 492005q^6 - (806490k^2 + 261735)q^5 \\
&\quad - 1543770k^2q^4 \\
&\quad - (286740k^4 + 582660k^2 + 209520k + 19565)q^3 \\
&\quad - 354780k^4q^2 \\
&\quad - (5832k^6 + 63180k^4 + 43200k^3 + 11250k^2 + 1236k + 53)q \\
&\quad + 477120kq^6 + 1333380kq^5 \\
&\quad + (659340k^3 + 619680k + 114475)q^4 \\
&\quad + 976320k^3q^3 \\
&\quad + (64152k^5 + 272160k^3 + 143100k^2 + 25800k + 1401)q^2 \\
&\quad + 69984k^5q \\
&\quad - 5832k^6 + 5832k^5 + 4860k^4 + 1620k^3 + 270k^2 + 24k + 1.
\end{aligned}$$

We have $k \geq 2q$ from inequality (A.21) and so

$$\frac{\partial \mu_2}{\partial k} = 12k - 24q \geq 24q - 24q = 0. \tag{A.25}$$

Thus μ_2 is maximum on R at $k = 0$, when

$$\mu_2 = 1 + 25q \geq 1. \tag{A.26}$$

Thus μ_2 is always positive on for $k \geq 0$. So $|J| > 0$ on when $S(q,k) > 0$ and $k \geq 0$. For $q_u \geq q \geq q_l \geq 0$ and $k \geq 0$ we have

$$\begin{aligned}
S(q,k) &\geq S(q_l, q_u, k) \\
&= -94775q_u^7 - 492005q_u^6 - (806490k^2 + 261735)q_u^5 \\
&\quad - 1543770k^2q_u^4 \\
&\quad - (286740k^4 + 582660k^2 + 209520k + 19565)q_u^3 \\
&\quad - 354780k^4q_u^2 \\
&\quad - (5832k^6 + 63180k^4 + 43200k^3 + 11250k^2 + 1236k + 53)q_u \\
&\quad + 477120kq_l^6 + 1333380kq_l^5 \\
&\quad + (659340k^3 + 619680k + 114475)q_l^4
\end{aligned}$$

$$
\begin{aligned}
&+ 976320k^3 q_l^3 \\
&+ (64152k^5 + 272160k^3 + 143100k^2 + 25800k + 1401)q_l^2 \\
&+ 69984k^5 q_l \\
&- 5832k^6 + 5832k^5 + 4860k^4 + 1620k^3 + 270k^2 + 24k + 1.
\end{aligned}
$$

From inequality (A.22), if we can find intervals $[q_l, q_u]$ covering $[0, 3-2\sqrt{2}]$ such that $S(q_l, q_u, k)$ is positive for k satisfying

$$
k_{\min} = \frac{1 + 11q_l - \sqrt{q_l^2 - 6q_l + 1}}{6} \le k \le \frac{53q_u}{48} + \frac{1}{3} = k_{\max}, \tag{A.27}
$$

then we have $|J| > 0$ on S^+.

We find such intervals using computer search in Maxima. In practice we find the intervals are small, and we checked 1164 intervals. For each we compute $S(q_l, q_u, k)$ as a polynomial in k. This polynomial always has degree 6 with a negative coefficient for k^6. We use Maxima to check that there are precisely two real root r_l and r_u. The polynomial is positive between these roots. We check the lower exceeds $k_{\min}$, and the upper is less than $k_{\max}$, by at least 10^{-7}, the tolerance on the values of the roots computed by Maxima. Maxima computes the roots using Sturm sequences, and we can be sure of the number of real roots and of the tolerance of the solution because $S(q_l, q_k, k)$ is a polynomial in k with rational coefficients. The following table shows the sequence of intervals. The roots r_l, r_u and values of $k_{\min}$ and $k_{\max}$ are rounded to 6 decimal places. Other values are exact. Note that $0.171573 > 3 - 2\sqrt{2}$.

q_l	q_u	$k_{\min}$	$k_{\max}$	r_l	r_u
0	0.010724	0	0.379071	−0.115382	1.53688
0.010724	0.015752	0.025102	0.388766	−0.094667	1.584687
0.015752	0.020622	0.036928	0.396758	−0.06975	1.584958
0.020622	0.02534	0.048421	0.40364	0.026366	1.585221
0.02534	0.027626	0.059591	0.406743	−0.041464	1.611068
0.027626	0.029876	0.065016	0.409674	−0.019502	1.611494
0.029876	0.032091	0.070366	0.412453	0.012241	1.611911
0.032091	0.034271	0.075641	0.415097	0.036656	1.612331
0.034271	0.036417	0.080841	0.417618	0.055297	1.612742
0.036417	0.038529	0.085969	0.420027	0.071076	1.613156
0.038529	0.040609	0.091024	0.422337	0.085242	1.61355
0.040609	0.041633	0.096012	0.423452	0.018219	1.627202
0.041633	0.042649	0.09847	0.424545	0.026645	1.627455
0.042649	0.043657	0.100912	0.425616	0.035097	1.627706
0.043657	0.044657	0.103336	0.426667	0.043325	1.627957
0.044657	0.045649	0.105743	0.427698	0.051179	1.628207
0.045649	0.046633	0.108133	0.42871	0.058613	1.628457
0.046633	0.047611	0.110506	0.429705	0.065871	1.628679
0.047611	0.048581	0.112866	0.430682	0.072528	1.628927
0.048581	0.049543	0.11521	0.431641	0.078852	1.629175
0.049543	0.050497	0.117536	0.432583	0.084883	1.629422
0.050497	0.051443	0.119844	0.433508	0.090655	1.629669
0.051443	0.052383	0.122136	0.434419	0.096404	1.629887
0.052383	0.053315	0.124414	0.435314	0.101752	1.630133
0.053315	0.054239	0.126676	0.436194	0.106917	1.630378
0.054239	0.055157	0.12892	0.437061	0.11211	1.630594
0.055157	0.056067	0.131152	0.437913	0.116965	1.630839
0.056067	0.056971	0.133366	0.438753	0.121871	1.631053
0.056971	0.057867	0.135567	0.439579	0.126465	1.631297
0.057867	0.058757	0.137751	0.440393	0.131127	1.631511
0.058757	0.059639	0.139923	0.441193	0.135496	1.631755
0.059639	0.060515	0.142077	0.441982	0.139945	1.631967
0.060515	0.061383	0.144218	0.442759	0.144114	1.632211
0.061383	0.061815	0.146342	0.443143	0.084647	1.639367
0.061815	0.062245	0.147399	0.443525	0.087108	1.639491
0.062245	0.062673	0.148452	0.443903	0.089541	1.639614
0.062673	0.063099	0.149501	0.444278	0.091945	1.639737
0.063099	0.063523	0.150546	0.44465	0.094319	1.63986
0.063523	0.063947	0.151586	0.445021	0.097153	1.639949
0.063947	0.064369	0.152626	0.445389	0.099477	1.640071
0.064369	0.064789	0.153662	0.445754	0.101769	1.640193
0.064789	0.065207	0.154694	0.446116	0.104028	1.640315
0.065207	0.065623	0.155721	0.446475	0.106254	1.640436

q_l	q_u	$k_{\min}$	$k_{\max}$	r_l	r_u
0.065623	0.066037	0.156744	0.446831	0.108448	1.640557
0.066037	0.066451	0.157762	0.447187	0.111093	1.640644
0.066451	0.066863	0.158781	0.447539	0.113235	1.640765
0.066863	0.067273	0.159796	0.447889	0.115346	1.640885
0.067273	0.067681	0.160806	0.448235	0.117425	1.641006
0.067681	0.068087	0.161811	0.44858	0.119474	1.641125
0.068087	0.068493	0.162812	0.448923	0.121966	1.64121
0.068493	0.068897	0.163814	0.449263	0.123967	1.64133
0.068897	0.069299	0.164811	0.449601	0.125939	1.641449
0.069299	0.069699	0.165803	0.449936	0.127883	1.641568
0.069699	0.070097	0.166792	0.450268	0.129799	1.641687
0.070097	0.070495	0.167775	0.4506	0.132151	1.641769
0.070495	0.070891	0.168759	0.450929	0.134025	1.641888
0.070891	0.071285	0.169739	0.451255	0.135873	1.642006
0.071285	0.071677	0.170714	0.451579	0.137696	1.642124
0.071677	0.072069	0.171685	0.451902	0.139949	1.642205
0.072069	0.072459	0.172656	0.452222	0.141733	1.642322
0.072459	0.072847	0.173623	0.45254	0.143493	1.64244
0.072847	0.073233	0.174585	0.452855	0.14523	1.642557
0.073233	0.073619	0.175543	0.45317	0.147393	1.642636
0.073619	0.074003	0.176501	0.453482	0.149095	1.642753
0.074003	0.074385	0.177455	0.453792	0.150775	1.64287
0.074385	0.074765	0.178404	0.454099	0.152433	1.642986
0.074765	0.075145	0.179348	0.454406	0.154515	1.643064
0.075145	0.075523	0.180294	0.45471	0.156142	1.64318
0.075523	0.075899	0.181234	0.455011	0.157748	1.643295
0.075899	0.076273	0.18217	0.455311	0.159333	1.643411
0.076273	0.076647	0.183102	0.45561	0.161341	1.643488
0.076647	0.077019	0.184034	0.455906	0.162897	1.643603
0.077019	0.077389	0.184961	0.4562	0.164435	1.643718
0.077389	0.077757	0.185884	0.456492	0.165953	1.643832
0.077757	0.078125	0.186803	0.456783	0.167892	1.643908
0.078125	0.078491	0.187722	0.457072	0.169384	1.644022
0.078491	0.078855	0.188636	0.457358	0.170857	1.644136
0.078855	0.079219	0.189546	0.457644	0.172751	1.644211
0.079219	0.079581	0.190456	0.457928	0.174198	1.644325
0.079581	0.079941	0.191362	0.458209	0.175628	1.644438
0.079941	0.080299	0.192263	0.458489	0.177041	1.644551
0.080299	0.080657	0.19316	0.458767	0.178875	1.644625
0.080657	0.081013	0.194057	0.459044	0.180264	1.644738
0.081013	0.081367	0.194949	0.459318	0.181636	1.644851
0.081367	0.081721	0.195837	0.459592	0.18343	1.644923
0.081721	0.082073	0.196726	0.459864	0.18478	1.645036

q_l	q_u	$k_{\min}$	$k_{\max}$	r_l	r_u
0.082073	0.082423	0.19761	0.460133	0.186113	1.645148
0.082423	0.082773	0.198489	0.460402	0.187869	1.645219
0.082773	0.083121	0.199368	0.460669	0.18918	1.645332
0.083121	0.083467	0.200244	0.460934	0.190476	1.645443
0.083467	0.083813	0.201114	0.461198	0.192195	1.645514
0.083813	0.084157	0.201985	0.46146	0.19347	1.645626
0.084157	0.084499	0.202851	0.46172	0.194729	1.645737
0.084499	0.084841	0.203713	0.46198	0.196414	1.645807
0.084841	0.085181	0.204575	0.462237	0.197653	1.645918
0.085181	0.085519	0.205433	0.462493	0.198877	1.646029
0.085519	0.085857	0.206286	0.462748	0.200528	1.646098
0.085857	0.086193	0.207139	0.463001	0.201733	1.646208
0.086193	0.086527	0.207988	0.463252	0.202922	1.646319
0.086527	0.086861	0.208832	0.463502	0.204541	1.646387
0.086861	0.087193	0.209677	0.463751	0.205712	1.646497
0.087193	0.087523	0.210517	0.463997	0.206868	1.646608
0.087523	0.087853	0.211352	0.464243	0.208456	1.646674
0.087853	0.088181	0.212188	0.464488	0.209594	1.646784
0.088181	0.088507	0.213019	0.46473	0.210717	1.646894
0.088507	0.088833	0.213846	0.464971	0.212275	1.64696
0.088833	0.089157	0.214673	0.465211	0.21338	1.64707
0.089157	0.089479	0.215495	0.465449	0.214471	1.647179
0.089479	0.089801	0.216313	0.465687	0.216	1.647244
0.089801	0.090121	0.217131	0.465922	0.217074	1.647354
0.090121	0.090281	0.217945	0.46604	0.161351	1.650969
0.090281	0.090441	0.218351	0.466158	0.162258	1.651008
0.090441	0.090601	0.218758	0.466275	0.163164	1.651047
0.090601	0.090761	0.219166	0.466392	0.164068	1.651085
0.090761	0.090919	0.219573	0.466508	0.16374	1.651169
0.090919	0.091077	0.219975	0.466624	0.164633	1.651207
0.091077	0.091235	0.220377	0.466739	0.165524	1.651245
0.091235	0.091393	0.22078	0.466855	0.166414	1.651283
0.091393	0.091551	0.221182	0.46697	0.167302	1.651322
0.091551	0.091709	0.221585	0.467085	0.168189	1.65136
0.091709	0.091865	0.221988	0.467199	0.167836	1.651443
0.091865	0.092021	0.222385	0.467313	0.168711	1.651481
0.092021	0.092177	0.222783	0.467426	0.169585	1.651519
0.092177	0.092333	0.223181	0.46754	0.170456	1.651556
0.092333	0.092489	0.223579	0.467653	0.171326	1.651594
0.092489	0.092645	0.223978	0.467766	0.172195	1.651632
0.092645	0.092801	0.224376	0.467879	0.173062	1.651669
0.092801	0.092955	0.224774	0.467991	0.172687	1.651753
0.092955	0.093109	0.225168	0.468102	0.173542	1.65179

q_l	q_u	$k_{\min}$	$k_{\max}$	r_l	r_u
0.093109	0.093263	0.225561	0.468214	0.174396	1.651827
0.093263	0.093417	0.225955	0.468325	0.175248	1.651864
0.093417	0.093571	0.226349	0.468436	0.176098	1.651901
0.093571	0.093725	0.226742	0.468548	0.176946	1.651939
0.093725	0.093879	0.227136	0.468659	0.177793	1.651976
0.093879	0.094031	0.22753	0.468768	0.177397	1.652059
0.094031	0.094183	0.227919	0.468878	0.178233	1.652096
0.094183	0.094335	0.228308	0.468987	0.179067	1.652132
0.094335	0.094487	0.228698	0.469096	0.179899	1.652169
0.094487	0.094639	0.229087	0.469205	0.180729	1.652206
0.094639	0.094791	0.229476	0.469314	0.181558	1.652242
0.094791	0.094941	0.229866	0.469422	0.18114	1.652326
0.094941	0.095091	0.23025	0.469529	0.181958	1.652362
0.095091	0.095241	0.230635	0.469637	0.182774	1.652398
0.095241	0.095391	0.23102	0.469744	0.183588	1.652434
0.095391	0.095541	0.231405	0.469851	0.1844	1.65247
0.095541	0.095691	0.23179	0.469958	0.185211	1.652506
0.095691	0.095841	0.232175	0.470065	0.18602	1.652543
0.095841	0.095989	0.23256	0.470171	0.185582	1.652626
0.095989	0.096137	0.23294	0.470276	0.186381	1.652661
0.096137	0.096285	0.23332	0.470382	0.187177	1.652697
0.096285	0.096433	0.2337	0.470487	0.187972	1.652733
0.096433	0.096581	0.234081	0.470592	0.188765	1.652768
0.096581	0.096729	0.234461	0.470697	0.189557	1.652804
0.096729	0.096877	0.234842	0.470802	0.190346	1.65284
0.096877	0.097023	0.235223	0.470906	0.18989	1.652923
0.097023	0.097169	0.235598	0.471009	0.190669	1.652958
0.097169	0.097315	0.235974	0.471113	0.191447	1.652993
0.097315	0.097461	0.23635	0.471216	0.192223	1.653028
0.097461	0.097607	0.236726	0.471319	0.192997	1.653063
0.097607	0.097753	0.237102	0.471423	0.19377	1.653098
0.097753	0.097899	0.237478	0.471526	0.194541	1.653134
0.097899	0.098043	0.237855	0.471627	0.194065	1.653217
0.098043	0.098187	0.238226	0.471729	0.194826	1.653252
0.098187	0.098331	0.238597	0.47183	0.195585	1.653286
0.098331	0.098475	0.238969	0.471932	0.196343	1.653321
0.098475	0.098619	0.23934	0.472033	0.197099	1.653355
0.098619	0.098763	0.239712	0.472134	0.197853	1.65339
0.098763	0.098907	0.240084	0.472235	0.198606	1.653425
0.098907	0.099049	0.240455	0.472335	0.198111	1.653508
0.099049	0.099191	0.240822	0.472434	0.198854	1.653542
0.099191	0.099333	0.241189	0.472534	0.199595	1.653576
0.099333	0.099475	0.241556	0.472633	0.200335	1.65361

q_l	q_u	$k_{\min}$	$k_{\max}$	r_l	r_u
0.099475	0.099617	0.241923	0.472733	0.201073	1.653644
0.099617	0.099759	0.24229	0.472832	0.20181	1.653678
0.099759	0.099901	0.242658	0.472931	0.202545	1.653712
0.099901	0.100041	0.243025	0.473029	0.202031	1.653796
0.100041	0.100181	0.243387	0.473127	0.202757	1.65383
0.100181	0.100321	0.24375	0.473225	0.203481	1.653863
0.100321	0.100461	0.244112	0.473322	0.204203	1.653897
0.100461	0.100601	0.244475	0.47342	0.204924	1.65393
0.100601	0.100741	0.244838	0.473517	0.205644	1.653964
0.100741	0.100881	0.245201	0.473614	0.206362	1.653997
0.100881	0.101021	0.245563	0.473712	0.207078	1.654031
0.101021	0.101159	0.245926	0.473808	0.206547	1.654115
0.101159	0.101297	0.246284	0.473903	0.207255	1.654148
0.101297	0.101435	0.246642	0.473999	0.20796	1.654181
0.101435	0.101573	0.247001	0.474095	0.208665	1.654214
0.101573	0.101711	0.247359	0.47419	0.209368	1.654247
0.101711	0.101849	0.247717	0.474286	0.210069	1.65428
0.101849	0.101987	0.248076	0.474381	0.210769	1.654313
0.101987	0.102123	0.248434	0.474475	0.210218	1.654396
0.102123	0.102259	0.248788	0.474569	0.210909	1.654429
0.102259	0.102395	0.249141	0.474663	0.211599	1.654462
0.102395	0.102531	0.249495	0.474757	0.212287	1.654494
0.102531	0.102667	0.249848	0.474851	0.212974	1.654527
0.102667	0.102803	0.250202	0.474945	0.21366	1.654559
0.102803	0.102939	0.250556	0.475038	0.214344	1.654592
0.102939	0.103075	0.25091	0.475132	0.215027	1.654624
0.103075	0.103209	0.251264	0.475224	0.214458	1.654708
0.103209	0.103343	0.251613	0.475316	0.215132	1.65474
0.103343	0.103477	0.251963	0.475408	0.215804	1.654772
0.103477	0.103611	0.252312	0.4755	0.216476	1.654804
0.103611	0.103745	0.252661	0.475592	0.217146	1.654836
0.103745	0.103879	0.253011	0.475684	0.217815	1.654868
0.103879	0.104013	0.25336	0.475775	0.218482	1.6549
0.104013	0.104145	0.25371	0.475866	0.217892	1.654984
0.104145	0.104277	0.254054	0.475956	0.218551	1.655016
0.104277	0.104409	0.254399	0.476046	0.219209	1.655047
0.104409	0.104541	0.254743	0.476137	0.219865	1.655079
0.104541	0.104673	0.255088	0.476227	0.22052	1.65511
0.104673	0.104805	0.255433	0.476317	0.221174	1.655142
0.104805	0.104937	0.255778	0.476407	0.221827	1.655173
0.104937	0.105069	0.256123	0.476497	0.222479	1.655205
0.105069	0.105199	0.256468	0.476585	0.22187	1.655289
0.105199	0.105329	0.256808	0.476674	0.222513	1.65532

q_l	q_u	$k_{\min}$	$k_{\max}$	r_l	r_u
0.105329	0.105459	0.257149	0.476762	0.223154	1.655351
0.105459	0.105589	0.257489	0.476851	0.223795	1.655382
0.105589	0.105719	0.257829	0.476939	0.224435	1.655413
0.105719	0.105849	0.25817	0.477027	0.225073	1.655444
0.105849	0.105979	0.25851	0.477115	0.22571	1.655475
0.105979	0.106109	0.258851	0.477203	0.226346	1.655506
0.106109	0.106237	0.259192	0.47729	0.225718	1.65559
0.106237	0.106365	0.259527	0.477377	0.226345	1.655621
0.106365	0.106493	0.259863	0.477464	0.226972	1.655651
0.106493	0.106621	0.260199	0.47755	0.227597	1.655682
0.106621	0.106749	0.260534	0.477637	0.228222	1.655712
0.106749	0.106877	0.26087	0.477723	0.228845	1.655743
0.106877	0.107005	0.261206	0.47781	0.229468	1.655773
0.107005	0.107133	0.261543	0.477896	0.230089	1.655804
0.107133	0.107259	0.261879	0.477981	0.22944	1.655888
0.107259	0.107385	0.26221	0.478066	0.230052	1.655918
0.107385	0.107511	0.262541	0.478151	0.230664	1.655948
0.107511	0.107637	0.262872	0.478236	0.231275	1.655978
0.107637	0.107763	0.263204	0.47832	0.231885	1.656008
0.107763	0.107889	0.263535	0.478405	0.232494	1.656038
0.107889	0.108015	0.263867	0.47849	0.233102	1.656068
0.108015	0.108141	0.264199	0.478575	0.233709	1.656098
0.108141	0.108265	0.26453	0.478658	0.233038	1.656182
0.108265	0.108389	0.264857	0.478741	0.233637	1.656212
0.108389	0.108513	0.265184	0.478824	0.234235	1.656242
0.108513	0.108637	0.26551	0.478907	0.234831	1.656271
0.108637	0.108761	0.265837	0.47899	0.235427	1.656301
0.108761	0.108885	0.266164	0.479073	0.236022	1.65633
0.108885	0.109009	0.266491	0.479156	0.236616	1.65636
0.109009	0.109133	0.266818	0.479239	0.237209	1.656389
0.109133	0.109255	0.267146	0.479321	0.236516	1.656474
0.109255	0.109377	0.267468	0.479402	0.237101	1.656503
0.109377	0.109499	0.26779	0.479484	0.237685	1.656532
0.109499	0.109621	0.268112	0.479565	0.238268	1.656561
0.109621	0.109743	0.268435	0.479646	0.23885	1.65659
0.109743	0.109865	0.268757	0.479728	0.239432	1.656619
0.109865	0.109987	0.26908	0.479809	0.240012	1.656648
0.109987	0.110109	0.269402	0.47989	0.240592	1.656677
0.110109	0.110231	0.269725	0.479971	0.24117	1.656706
0.110231	0.110351	0.270048	0.480051	0.240457	1.65679
0.110351	0.110471	0.270365	0.480131	0.241028	1.656819
0.110471	0.110591	0.270683	0.480211	0.241598	1.656848
0.110591	0.110711	0.271001	0.48029	0.242166	1.656876

q_l	q_u	$k_{\min}$	$k_{\max}$	r_l	r_u
0.110711	0.110831	0.271319	0.48037	0.242735	1.656905
0.110831	0.110951	0.271637	0.480449	0.243302	1.656933
0.110951	0.111071	0.271955	0.480529	0.243868	1.656962
0.111071	0.111191	0.272273	0.480608	0.244434	1.65699
0.111191	0.111309	0.272591	0.480687	0.243697	1.657075
0.111309	0.111427	0.272905	0.480765	0.244255	1.657103
0.111427	0.111545	0.273218	0.480843	0.244812	1.657131
0.111545	0.111663	0.273531	0.480921	0.245368	1.657159
0.111663	0.111781	0.273844	0.480999	0.245923	1.657187
0.111781	0.111899	0.274158	0.481077	0.246478	1.657215
0.111899	0.112017	0.274471	0.481154	0.247031	1.657243
0.112017	0.112135	0.274785	0.481232	0.247584	1.657271
0.112135	0.112253	0.275099	0.48131	0.248137	1.657299
0.112253	0.112369	0.275412	0.481387	0.247377	1.657384
0.112369	0.112485	0.275721	0.481463	0.247921	1.657412
0.112485	0.112601	0.27603	0.481539	0.248465	1.657439
0.112601	0.112717	0.276339	0.481616	0.249008	1.657467
0.112717	0.112833	0.276647	0.481692	0.24955	1.657495
0.112833	0.112949	0.276956	0.481768	0.250092	1.657522
0.112949	0.113065	0.277265	0.481844	0.250632	1.65755
0.113065	0.113181	0.277575	0.481921	0.251172	1.657577
0.113181	0.113297	0.277884	0.481997	0.251712	1.657605
0.113297	0.113411	0.278193	0.482071	0.250929	1.65769
0.113411	0.113525	0.278497	0.482146	0.25146	1.657717
0.113525	0.113639	0.278801	0.482221	0.251991	1.657744
0.113639	0.113753	0.279106	0.482296	0.252521	1.657771
0.113753	0.113867	0.27941	0.48237	0.25305	1.657798
0.113867	0.113981	0.279714	0.482445	0.253579	1.657825
0.113981	0.114095	0.280019	0.482519	0.254107	1.657852
0.114095	0.114209	0.280324	0.482594	0.254634	1.657879
0.114209	0.114323	0.280628	0.482668	0.255161	1.657906
0.114323	0.114435	0.280933	0.482741	0.254353	1.657991
0.114435	0.114547	0.281233	0.482815	0.254872	1.658018
0.114547	0.114659	0.281532	0.482888	0.255391	1.658045
0.114659	0.114771	0.281832	0.482961	0.255908	1.658071
0.114771	0.114883	0.282132	0.483034	0.256425	1.658098
0.114883	0.114995	0.282432	0.483107	0.256942	1.658124
0.114995	0.115107	0.282732	0.483179	0.257457	1.658151
0.115107	0.115219	0.283032	0.483252	0.257973	1.658178
0.115219	0.115331	0.283332	0.483325	0.258487	1.658204
0.115331	0.115441	0.283632	0.483397	0.257654	1.65829
0.115441	0.115551	0.283927	0.483468	0.25816	1.658316
0.115551	0.115661	0.284222	0.48354	0.258667	1.658342

q_l	q_u	$k_{\min}$	$k_{\max}$	r_l	r_u
0.115661	0.115771	0.284518	0.483611	0.259172	1.658368
0.115771	0.115881	0.284813	0.483682	0.259677	1.658394
0.115881	0.115991	0.285108	0.483754	0.260181	1.65842
0.115991	0.116101	0.285404	0.483825	0.260685	1.658446
0.116101	0.116211	0.285699	0.483896	0.261188	1.658472
0.116211	0.116321	0.285995	0.483968	0.261691	1.658498
0.116321	0.116429	0.286291	0.484037	0.260831	1.658584
0.116429	0.116537	0.286581	0.484107	0.261326	1.658609
0.116537	0.116645	0.286872	0.484177	0.26182	1.658635
0.116645	0.116753	0.287162	0.484247	0.262314	1.658661
0.116753	0.116861	0.287453	0.484317	0.262807	1.658686
0.116861	0.116969	0.287744	0.484387	0.2633	1.658712
0.116969	0.117077	0.288035	0.484456	0.263792	1.658738
0.117077	0.117185	0.288326	0.484526	0.264284	1.658763
0.117185	0.117293	0.288617	0.484596	0.264775	1.658789
0.117293	0.117401	0.288908	0.484665	0.265265	1.658814
0.117401	0.117507	0.289199	0.484734	0.264379	1.6589
0.117507	0.117613	0.289485	0.484802	0.264862	1.658925
0.117613	0.117719	0.289771	0.48487	0.265344	1.65895
0.117719	0.117825	0.290057	0.484938	0.265826	1.658975
0.117825	0.117931	0.290343	0.485006	0.266307	1.659001
0.117931	0.118037	0.29063	0.485075	0.266788	1.659026
0.118037	0.118143	0.290916	0.485143	0.267268	1.659051
0.118143	0.118249	0.291202	0.485211	0.267748	1.659076
0.118249	0.118355	0.291489	0.485279	0.268227	1.659101
0.118355	0.118459	0.291776	0.485346	0.267312	1.659187
0.118459	0.118563	0.292057	0.485412	0.267784	1.659212
0.118563	0.118667	0.292338	0.485479	0.268255	1.659236
0.118667	0.118771	0.29262	0.485546	0.268725	1.659261
0.118771	0.118875	0.292901	0.485612	0.269195	1.659286
0.118875	0.118979	0.293183	0.485679	0.269665	1.65931
0.118979	0.119083	0.293465	0.485745	0.270134	1.659335
0.119083	0.119187	0.293747	0.485812	0.270603	1.659359
0.119187	0.119291	0.294029	0.485879	0.271071	1.659384
0.119291	0.119395	0.294311	0.485945	0.271539	1.659409
0.119395	0.119497	0.294593	0.48601	0.270595	1.659495
0.119497	0.119599	0.29487	0.486075	0.271055	1.659519
0.119599	0.119701	0.295146	0.48614	0.271515	1.659543
0.119701	0.119803	0.295423	0.486206	0.271974	1.659568
0.119803	0.119905	0.2957	0.486271	0.272433	1.659592
0.119905	0.120007	0.295977	0.486336	0.272891	1.659616
0.120007	0.120109	0.296255	0.486401	0.273349	1.65964
0.120109	0.120211	0.296532	0.486466	0.273807	1.659664

q_l	q_u	$k_{\min}$	$k_{\max}$	r_l	r_u
0.120211	0.120313	0.296809	0.486531	0.274264	1.659688
0.120313	0.120415	0.297087	0.486595	0.27472	1.659712
0.120415	0.120515	0.297364	0.486659	0.273747	1.659799
0.120515	0.120615	0.297636	0.486723	0.274196	1.659823
0.120615	0.120715	0.297909	0.486786	0.274644	1.659846
0.120715	0.120815	0.298181	0.48685	0.275092	1.65987
0.120815	0.120915	0.298454	0.486913	0.27554	1.659894
0.120915	0.121015	0.298726	0.486977	0.275987	1.659917
0.121015	0.121115	0.298999	0.48704	0.276434	1.659941
0.121115	0.121215	0.299271	0.487104	0.276881	1.659965
0.121215	0.121315	0.299544	0.487167	0.277327	1.659988
0.121315	0.121415	0.299817	0.487231	0.277772	1.660012
0.121415	0.121513	0.30009	0.487293	0.276767	1.660099
0.121513	0.121611	0.300358	0.487355	0.277205	1.660122
0.121611	0.121709	0.300625	0.487417	0.277643	1.660145
0.121709	0.121807	0.300893	0.487479	0.27808	1.660168
0.121807	0.121905	0.301161	0.487541	0.278517	1.660191
0.121905	0.122003	0.301429	0.487603	0.278954	1.660215
0.122003	0.122101	0.301697	0.487665	0.27939	1.660238
0.122101	0.122199	0.301965	0.487727	0.279826	1.660261
0.122199	0.122297	0.302233	0.487789	0.280261	1.660284
0.122297	0.122395	0.302501	0.48785	0.280696	1.660307
0.122395	0.122493	0.30277	0.487912	0.281131	1.660331
0.122493	0.122589	0.303038	0.487973	0.280095	1.660418
0.122589	0.122685	0.303301	0.488033	0.280522	1.66044
0.122685	0.122781	0.303564	0.488094	0.280948	1.660463
0.122781	0.122877	0.303828	0.488154	0.281375	1.660486
0.122877	0.122973	0.304091	0.488215	0.281801	1.660508
0.122973	0.123069	0.304354	0.488275	0.282226	1.660531
0.123069	0.123165	0.304618	0.488336	0.282651	1.660554
0.123165	0.123261	0.304881	0.488396	0.283076	1.660577
0.123261	0.123357	0.305145	0.488456	0.283501	1.660599
0.123357	0.123453	0.305409	0.488517	0.283925	1.660622
0.123453	0.123547	0.305672	0.488576	0.282855	1.660709
0.123547	0.123641	0.305931	0.488635	0.283271	1.660731
0.123641	0.123735	0.306189	0.488694	0.283688	1.660754
0.123735	0.123829	0.306448	0.488753	0.284103	1.660776
0.123829	0.123923	0.306706	0.488812	0.284519	1.660798
0.123923	0.124017	0.306965	0.488871	0.284934	1.66082
0.124017	0.124111	0.307224	0.48893	0.285349	1.660843
0.124111	0.124205	0.307483	0.488989	0.285764	1.660865
0.124205	0.124299	0.307742	0.489048	0.286178	1.660887
0.124299	0.124393	0.308001	0.489106	0.286592	1.660909

q_l	q_u	$k_{\min}$	$k_{\max}$	r_l	r_u
0.124393	0.124487	0.30826	0.489165	0.287006	1.660931
0.124487	0.124579	0.308519	0.489223	0.285901	1.661019
0.124579	0.124671	0.308773	0.48928	0.286307	1.661041
0.124671	0.124763	0.309027	0.489338	0.286713	1.661063
0.124763	0.124855	0.309281	0.489395	0.287118	1.661084
0.124855	0.124947	0.309535	0.489453	0.287524	1.661106
0.124947	0.125039	0.309789	0.48951	0.287928	1.661128
0.125039	0.125131	0.310043	0.489568	0.288333	1.66115
0.125131	0.125223	0.310297	0.489625	0.288737	1.661171
0.125223	0.125315	0.310552	0.489683	0.289141	1.661193
0.125315	0.125407	0.310806	0.48974	0.289544	1.661215
0.125407	0.125499	0.311061	0.489797	0.289948	1.661237
0.125499	0.125589	0.311315	0.489854	0.288807	1.661324
0.125589	0.125679	0.311564	0.48991	0.289203	1.661346
0.125679	0.125769	0.311814	0.489966	0.289599	1.661367
0.125769	0.125859	0.312063	0.490022	0.289994	1.661388
0.125859	0.125949	0.312312	0.490078	0.290389	1.66141
0.125949	0.126039	0.312562	0.490134	0.290783	1.661431
0.126039	0.126129	0.312811	0.49019	0.291178	1.661452
0.126129	0.126219	0.313061	0.490246	0.291572	1.661473
0.126219	0.126309	0.313311	0.490302	0.291965	1.661495
0.126309	0.126399	0.313561	0.490357	0.292359	1.661516
0.126399	0.126489	0.31381	0.490413	0.292752	1.661537
0.126489	0.126579	0.31406	0.490469	0.293145	1.661559
0.126579	0.126667	0.314311	0.490524	0.291968	1.661647
0.126667	0.126755	0.314555	0.490578	0.292353	1.661668
0.126755	0.126843	0.3148	0.490633	0.292739	1.661688
0.126843	0.126931	0.315044	0.490688	0.293123	1.661709
0.126931	0.127019	0.315289	0.490742	0.293508	1.66173
0.127019	0.127107	0.315534	0.490797	0.293892	1.661751
0.127107	0.127195	0.315779	0.490851	0.294276	1.661772
0.127195	0.127283	0.316024	0.490906	0.29466	1.661792
0.127283	0.127371	0.316269	0.49096	0.295043	1.661813
0.127371	0.127459	0.316515	0.491015	0.295427	1.661834
0.127459	0.127547	0.31676	0.491069	0.295809	1.661855
0.127547	0.127633	0.317005	0.491122	0.294593	1.661943
0.127633	0.127719	0.317245	0.491175	0.294969	1.661964
0.127719	0.127805	0.317485	0.491228	0.295344	1.661984
0.127805	0.127891	0.317725	0.491282	0.295719	1.662004
0.127891	0.127977	0.317965	0.491335	0.296094	1.662025
0.127977	0.128063	0.318206	0.491388	0.296468	1.662045
0.128063	0.128149	0.318446	0.491441	0.296842	1.662065
0.128149	0.128235	0.318686	0.491494	0.297216	1.662086

q_l	q_u	$k_{\min}$	$k_{\max}$	r_l	r_u
0.128235	0.128321	0.318927	0.491547	0.29759	1.662106
0.128321	0.128407	0.319167	0.4916	0.297963	1.662126
0.128407	0.128493	0.319408	0.491653	0.298337	1.662147
0.128493	0.128579	0.319648	0.491706	0.298709	1.662167
0.128579	0.128663	0.319889	0.491758	0.297453	1.662256
0.128663	0.128747	0.320124	0.491809	0.297818	1.662276
0.128747	0.128831	0.32036	0.491861	0.298184	1.662295
0.128831	0.128915	0.320595	0.491913	0.298549	1.662315
0.128915	0.128999	0.320831	0.491964	0.298914	1.662335
0.128999	0.129083	0.321066	0.492016	0.299278	1.662355
0.129083	0.129167	0.321302	0.492067	0.299643	1.662375
0.129167	0.129251	0.321538	0.492119	0.300007	1.662395
0.129251	0.129335	0.321773	0.492171	0.300371	1.662415
0.129335	0.129419	0.322009	0.492222	0.300734	1.662434
0.129419	0.129503	0.322245	0.492274	0.301097	1.662454
0.129503	0.129587	0.322481	0.492325	0.301461	1.662474
0.129587	0.129671	0.322717	0.492377	0.301823	1.662494
0.129671	0.129753	0.322954	0.492427	0.300526	1.662583
0.129753	0.129835	0.323184	0.492477	0.300882	1.662602
0.129835	0.129917	0.323415	0.492528	0.301237	1.662622
0.129917	0.129999	0.323646	0.492578	0.301592	1.662641
0.129999	0.130081	0.323877	0.492628	0.301947	1.662661
0.130081	0.130163	0.324108	0.492678	0.302302	1.66268
0.130163	0.130245	0.324339	0.492729	0.302656	1.662699
0.130245	0.130327	0.32457	0.492779	0.30301	1.662719
0.130327	0.130409	0.324801	0.492829	0.303364	1.662738
0.130409	0.130491	0.325032	0.492879	0.303718	1.662758
0.130491	0.130573	0.325263	0.492929	0.304071	1.662777
0.130573	0.130655	0.325495	0.492979	0.304424	1.662796
0.130655	0.130735	0.325726	0.493028	0.303082	1.662886
0.130735	0.130815	0.325952	0.493077	0.303428	1.662905
0.130815	0.130895	0.326178	0.493126	0.303774	1.662924
0.130895	0.130975	0.326404	0.493175	0.30412	1.662943
0.130975	0.131055	0.326631	0.493223	0.304465	1.662962
0.131055	0.131135	0.326857	0.493272	0.30481	1.66298
0.131135	0.131215	0.327083	0.493321	0.305155	1.662999
0.131215	0.131295	0.32731	0.49337	0.3055	1.663018
0.131295	0.131375	0.327536	0.493418	0.305844	1.663037
0.131375	0.131455	0.327763	0.493467	0.306188	1.663056
0.131455	0.131535	0.327989	0.493516	0.306532	1.663075
0.131535	0.131615	0.328216	0.493565	0.306876	1.663094
0.131615	0.131695	0.328443	0.493613	0.30722	1.663113
0.131695	0.131773	0.32867	0.493661	0.305831	1.663203

q_l	q_u	$k_{\min}$	$k_{\max}$	r_l	r_u
0.131773	0.131851	0.328891	0.493708	0.306168	1.663221
0.131851	0.131929	0.329112	0.493756	0.306504	1.66324
0.131929	0.132007	0.329334	0.493803	0.30684	1.663258
0.132007	0.132085	0.329555	0.49385	0.307176	1.663276
0.132085	0.132163	0.329777	0.493898	0.307511	1.663295
0.132163	0.132241	0.329998	0.493945	0.307847	1.663313
0.132241	0.132319	0.33022	0.493993	0.308182	1.663332
0.132319	0.132397	0.330442	0.49404	0.308517	1.66335
0.132397	0.132475	0.330664	0.494087	0.308852	1.663369
0.132475	0.132553	0.330886	0.494135	0.309186	1.663387
0.132553	0.132631	0.331108	0.494182	0.309521	1.663406
0.132631	0.132709	0.33133	0.494229	0.309855	1.663424
0.132709	0.132785	0.331552	0.494275	0.308418	1.663514
0.132785	0.132861	0.331769	0.494321	0.308745	1.663532
0.132861	0.132937	0.331985	0.494367	0.309071	1.66355
0.132937	0.133013	0.332202	0.494413	0.309398	1.663568
0.133013	0.133089	0.332419	0.494459	0.309724	1.663586
0.133089	0.133165	0.332636	0.494505	0.310051	1.663604
0.133165	0.133241	0.332853	0.494551	0.310377	1.663622
0.133241	0.133317	0.33307	0.494597	0.310703	1.66364
0.133317	0.133393	0.333287	0.494643	0.311028	1.663658
0.133393	0.133469	0.333504	0.494689	0.311354	1.663676
0.133469	0.133545	0.333721	0.494735	0.311679	1.663694
0.133545	0.133621	0.333938	0.494781	0.312004	1.663712
0.133621	0.133697	0.334156	0.494827	0.312329	1.66373
0.133697	0.133771	0.334373	0.494872	0.31084	1.66382
0.133771	0.133845	0.334585	0.494916	0.311157	1.663838
0.133845	0.133919	0.334797	0.494961	0.311475	1.663855
0.133919	0.133993	0.335009	0.495006	0.311792	1.663873
0.133993	0.134067	0.335221	0.49505	0.31211	1.66389
0.134067	0.134141	0.335433	0.495095	0.312427	1.663908
0.134141	0.134215	0.335645	0.49514	0.312743	1.663925
0.134215	0.134289	0.335858	0.495184	0.31306	1.663943
0.134289	0.134363	0.33607	0.495229	0.313376	1.66396
0.134363	0.134437	0.336282	0.495273	0.313693	1.663978
0.134437	0.134511	0.336495	0.495318	0.314009	1.663995
0.134511	0.134585	0.336707	0.495362	0.314325	1.664013
0.134585	0.134659	0.33692	0.495407	0.31464	1.66403
0.134659	0.134733	0.337133	0.495451	0.314956	1.664048
0.134733	0.134805	0.337345	0.495495	0.313413	1.664139
0.134805	0.134877	0.337553	0.495538	0.313722	1.664156
0.134877	0.134949	0.33776	0.495581	0.31403	1.664173
0.134949	0.135021	0.337967	0.495625	0.314338	1.66419

q_l	q_u	$k_{\min}$	$k_{\max}$	r_l	r_u
0.135021	0.135093	0.338174	0.495668	0.314646	1.664207
0.135093	0.135165	0.338382	0.495711	0.314954	1.664224
0.135165	0.135237	0.338589	0.495754	0.315262	1.664241
0.135237	0.135309	0.338797	0.495798	0.315569	1.664258
0.135309	0.135381	0.339004	0.495841	0.315876	1.664275
0.135381	0.135453	0.339212	0.495884	0.316183	1.664292
0.135453	0.135525	0.33942	0.495927	0.31649	1.664309
0.135525	0.135597	0.339628	0.49597	0.316797	1.664326
0.135597	0.135669	0.339836	0.496014	0.317104	1.664343
0.135669	0.135741	0.340044	0.496057	0.31741	1.66436
0.135741	0.135811	0.340252	0.496099	0.31581	1.664452
0.135811	0.135881	0.340454	0.496141	0.31611	1.664468
0.135881	0.135951	0.340656	0.496183	0.316409	1.664485
0.135951	0.136021	0.340859	0.496225	0.316708	1.664501
0.136021	0.136091	0.341062	0.496266	0.317007	1.664518
0.136091	0.136161	0.341264	0.496308	0.317306	1.664535
0.136161	0.136231	0.341467	0.49635	0.317604	1.664551
0.136231	0.136301	0.34167	0.496392	0.317902	1.664568
0.136301	0.136371	0.341873	0.496434	0.318201	1.664584
0.136371	0.136441	0.342075	0.496476	0.318499	1.664601
0.136441	0.136511	0.342278	0.496518	0.318797	1.664618
0.136511	0.136581	0.342482	0.496559	0.319094	1.664634
0.136581	0.136651	0.342685	0.496601	0.319392	1.664651
0.136651	0.136721	0.342888	0.496643	0.319689	1.664667
0.136721	0.136791	0.343091	0.496685	0.319987	1.664684
0.136791	0.136859	0.343295	0.496726	0.318327	1.664775
0.136859	0.136927	0.343492	0.496766	0.318617	1.664792
0.136927	0.136995	0.34369	0.496807	0.318908	1.664808
0.136995	0.137063	0.343888	0.496847	0.319198	1.664824
0.137063	0.137131	0.344086	0.496888	0.319487	1.66484
0.137131	0.137199	0.344283	0.496928	0.319777	1.664856
0.137199	0.137267	0.344481	0.496969	0.320067	1.664872
0.137267	0.137335	0.34468	0.497009	0.320356	1.664888
0.137335	0.137403	0.344878	0.49705	0.320645	1.664904
0.137403	0.137471	0.345076	0.49709	0.320934	1.66492
0.137471	0.137539	0.345274	0.497131	0.321223	1.664937
0.137539	0.137607	0.345472	0.497171	0.321512	1.664953
0.137607	0.137675	0.345671	0.497212	0.3218	1.664969
0.137675	0.137743	0.345869	0.497252	0.322089	1.664985
0.137743	0.137811	0.346068	0.497293	0.322377	1.665001
0.137811	0.137877	0.346267	0.497332	0.320654	1.665093
0.137877	0.137943	0.346459	0.497371	0.320935	1.665109
0.137943	0.138009	0.346652	0.497411	0.321216	1.665124

q_l	q_u	$k_{\min}$	$k_{\max}$	r_l	r_u
0.138009	0.138075	0.346845	0.49745	0.321497	1.66514
0.138075	0.138141	0.347038	0.497489	0.321778	1.665156
0.138141	0.138207	0.347232	0.497528	0.322059	1.665171
0.138207	0.138273	0.347425	0.497567	0.32234	1.665187
0.138273	0.138339	0.347618	0.497607	0.32262	1.665203
0.138339	0.138405	0.347811	0.497646	0.322901	1.665218
0.138405	0.138471	0.348005	0.497685	0.323181	1.665234
0.138471	0.138537	0.348198	0.497724	0.323461	1.66525
0.138537	0.138603	0.348392	0.497763	0.323741	1.665265
0.138603	0.138669	0.348586	0.497802	0.32402	1.665281
0.138669	0.138735	0.348779	0.497842	0.3243	1.665296
0.138735	0.138801	0.348973	0.497881	0.324579	1.665312
0.138801	0.138867	0.349167	0.49792	0.324859	1.665328
0.138867	0.138931	0.349361	0.497958	0.323068	1.66542
0.138931	0.138995	0.349549	0.497996	0.323341	1.665435
0.138995	0.139059	0.349737	0.498034	0.323613	1.665451
0.139059	0.139123	0.349925	0.498071	0.323885	1.665466
0.139123	0.139187	0.350114	0.498109	0.324157	1.665481
0.139187	0.139251	0.350302	0.498147	0.324429	1.665496
0.139251	0.139315	0.350491	0.498185	0.324701	1.665511
0.139315	0.139379	0.350679	0.498223	0.324972	1.665527
0.139379	0.139443	0.350868	0.498261	0.325244	1.665542
0.139443	0.139507	0.351056	0.498299	0.325515	1.665557
0.139507	0.139571	0.351245	0.498336	0.325786	1.665572
0.139571	0.139635	0.351434	0.498374	0.326058	1.665587
0.139635	0.139699	0.351623	0.498412	0.326328	1.665602
0.139699	0.139763	0.351812	0.49845	0.326599	1.665618
0.139763	0.139827	0.352001	0.498488	0.32687	1.665633
0.139827	0.139891	0.35219	0.498526	0.32714	1.665648
0.139891	0.139953	0.352379	0.498562	0.325277	1.665741
0.139953	0.140015	0.352562	0.498599	0.325541	1.665756
0.140015	0.140077	0.352746	0.498635	0.325805	1.66577
0.140077	0.140139	0.352929	0.498672	0.326068	1.665785
0.140139	0.140201	0.353113	0.498708	0.326331	1.6658
0.140201	0.140263	0.353296	0.498745	0.326594	1.665814
0.140263	0.140325	0.35348	0.498782	0.326857	1.665829
0.140325	0.140387	0.353664	0.498818	0.32712	1.665844
0.140387	0.140449	0.353847	0.498855	0.327383	1.665859
0.140449	0.140511	0.354031	0.498891	0.327646	1.665873
0.140511	0.140573	0.354215	0.498928	0.327908	1.665888
0.140573	0.140635	0.354399	0.498964	0.328171	1.665903
0.140635	0.140697	0.354583	0.499001	0.328433	1.665917
0.140697	0.140759	0.354767	0.499037	0.328695	1.665932

q_l	q_u	$k_{\min}$	$k_{\max}$	r_l	r_u
0.140759	0.140821	0.354952	0.499074	0.328957	1.665947
0.140821	0.140883	0.355136	0.49911	0.329219	1.665962
0.140883	0.140943	0.35532	0.499145	0.327277	1.666055
0.140943	0.141003	0.355499	0.499181	0.327532	1.666069
0.141003	0.141063	0.355677	0.499216	0.327787	1.666083
0.141063	0.141123	0.355856	0.499251	0.328042	1.666098
0.141123	0.141183	0.356034	0.499287	0.328297	1.666112
0.141183	0.141243	0.356213	0.499322	0.328552	1.666126
0.141243	0.141303	0.356392	0.499357	0.328806	1.66614
0.141303	0.141363	0.356571	0.499392	0.32906	1.666155
0.141363	0.141423	0.35675	0.499428	0.329315	1.666169
0.141423	0.141483	0.356929	0.499463	0.329569	1.666183
0.141483	0.141543	0.357108	0.499498	0.329823	1.666197
0.141543	0.141603	0.357287	0.499533	0.330077	1.666212
0.141603	0.141663	0.357466	0.499568	0.330331	1.666226
0.141663	0.141723	0.357645	0.499604	0.330584	1.66624
0.141723	0.141783	0.357825	0.499639	0.330838	1.666254
0.141783	0.141843	0.358004	0.499674	0.331091	1.666269
0.141843	0.141903	0.358184	0.499709	0.331344	1.666283
0.141903	0.141961	0.358363	0.499743	0.329319	1.666377
0.141961	0.142019	0.358537	0.499777	0.329565	1.666391
0.142019	0.142077	0.358711	0.499811	0.329812	1.666404
0.142077	0.142135	0.358884	0.499845	0.330058	1.666418
0.142135	0.142193	0.359058	0.499879	0.330304	1.666432
0.142193	0.142251	0.359232	0.499913	0.33055	1.666446
0.142251	0.142309	0.359406	0.499947	0.330796	1.666459
0.142309	0.142367	0.35958	0.499981	0.331042	1.666473
0.142367	0.142425	0.359754	0.500015	0.331288	1.666487
0.142425	0.142483	0.359928	0.500049	0.331534	1.666501
0.142483	0.142541	0.360102	0.500083	0.331779	1.666515
0.142541	0.142599	0.360277	0.500117	0.332024	1.666528
0.142599	0.142657	0.360451	0.500151	0.33227	1.666542
0.142657	0.142715	0.360626	0.500185	0.332515	1.666556
0.142715	0.142773	0.3608	0.500218	0.33276	1.66657
0.142773	0.142831	0.360975	0.500252	0.333005	1.666584
0.142831	0.142889	0.361149	0.500286	0.33325	1.666597
0.142889	0.142947	0.361324	0.50032	0.333495	1.666611
0.142947	0.143003	0.361499	0.500353	0.331379	1.666705
0.143003	0.143059	0.361667	0.500386	0.331617	1.666719
0.143059	0.143115	0.361836	0.500418	0.331855	1.666732
0.143115	0.143171	0.362005	0.500451	0.332093	1.666745
0.143171	0.143227	0.362174	0.500484	0.332331	1.666759
0.143227	0.143283	0.362343	0.500516	0.332568	1.666772

q_l	q_u	$k_{\min}$	$k_{\max}$	r_l	r_u
0.143283	0.143339	0.362512	0.500549	0.332806	1.666785
0.143339	0.143395	0.362682	0.500582	0.333043	1.666799
0.143395	0.143451	0.362851	0.500614	0.333281	1.666812
0.143451	0.143507	0.36302	0.500647	0.333518	1.666825
0.143507	0.143563	0.36319	0.50068	0.333755	1.666839
0.143563	0.143619	0.363359	0.500712	0.333992	1.666852
0.143619	0.143675	0.363528	0.500745	0.334229	1.666865
0.143675	0.143731	0.363698	0.500777	0.334465	1.666879
0.143731	0.143787	0.363868	0.50081	0.334702	1.666892
0.143787	0.143843	0.364037	0.500843	0.334939	1.666905
0.143843	0.143899	0.364207	0.500875	0.335175	1.666918
0.143899	0.143955	0.364377	0.500908	0.335411	1.666932
0.143955	0.144009	0.364547	0.500939	0.333197	1.667027
0.144009	0.144063	0.364711	0.500971	0.333427	1.667039
0.144063	0.144117	0.364875	0.501002	0.333657	1.667052
0.144117	0.144171	0.365039	0.501034	0.333886	1.667065
0.144171	0.144225	0.365203	0.501065	0.334116	1.667078
0.144225	0.144279	0.365367	0.501096	0.334345	1.667091
0.144279	0.144333	0.365531	0.501128	0.334574	1.667104
0.144333	0.144387	0.365696	0.501159	0.334803	1.667117
0.144387	0.144441	0.36586	0.50119	0.335032	1.667129
0.144441	0.144495	0.366024	0.501222	0.335261	1.667142
0.144495	0.144549	0.366189	0.501253	0.33549	1.667155
0.144549	0.144603	0.366353	0.501285	0.335719	1.667168
0.144603	0.144657	0.366518	0.501316	0.335947	1.667181
0.144657	0.144711	0.366683	0.501347	0.336176	1.667194
0.144711	0.144765	0.366847	0.501379	0.336404	1.667207
0.144765	0.144819	0.367012	0.50141	0.336633	1.667219
0.144819	0.144873	0.367177	0.501441	0.336861	1.667232
0.144873	0.144927	0.367342	0.501473	0.337089	1.667245
0.144927	0.144981	0.367507	0.501504	0.337317	1.667258
0.144981	0.145033	0.367672	0.501534	0.334996	1.667353
0.145033	0.145085	0.367831	0.501564	0.335218	1.667366
0.145085	0.145137	0.36799	0.501594	0.335439	1.667378
0.145137	0.145189	0.368149	0.501625	0.33566	1.66739
0.145189	0.145241	0.368308	0.501655	0.335881	1.667403
0.145241	0.145293	0.368468	0.501685	0.336103	1.667415
0.145293	0.145345	0.368627	0.501715	0.336323	1.667428
0.145345	0.145397	0.368786	0.501745	0.336544	1.66744
0.145397	0.145449	0.368946	0.501775	0.336765	1.667452
0.145449	0.145501	0.369105	0.501805	0.336986	1.667465
0.145501	0.145553	0.369265	0.501835	0.337206	1.667477
0.145553	0.145605	0.369425	0.501865	0.337427	1.66749

q_l	q_u	$k_{\min}$	$k_{\max}$	r_l	r_u
0.145605	0.145657	0.369584	0.501896	0.337647	1.667502
0.145657	0.145709	0.369744	0.501926	0.337867	1.667514
0.145709	0.145761	0.369904	0.501956	0.338088	1.667527
0.145761	0.145813	0.370064	0.501986	0.338308	1.667539
0.145813	0.145865	0.370224	0.502016	0.338528	1.667551
0.145865	0.145917	0.370384	0.502046	0.338748	1.667564
0.145917	0.145969	0.370544	0.502076	0.338967	1.667576
0.145969	0.146021	0.370704	0.502106	0.339187	1.667589
0.146021	0.146071	0.370864	0.502135	0.33675	1.667684
0.146071	0.146121	0.371018	0.502164	0.336964	1.667696
0.146121	0.146171	0.371172	0.502193	0.337177	1.667708
0.146171	0.146221	0.371327	0.502222	0.33739	1.66772
0.146221	0.146271	0.371481	0.50225	0.337603	1.667732
0.146271	0.146321	0.371635	0.502279	0.337816	1.667744
0.146321	0.146371	0.37179	0.502308	0.338029	1.667756
0.146371	0.146421	0.371944	0.502337	0.338241	1.667768
0.146421	0.146471	0.372099	0.502366	0.338454	1.66778
0.146471	0.146521	0.372253	0.502395	0.338666	1.667792
0.146521	0.146571	0.372408	0.502424	0.338879	1.667804
0.146571	0.146621	0.372563	0.502452	0.339091	1.667816
0.146621	0.146671	0.372718	0.502481	0.339303	1.667828
0.146671	0.146721	0.372872	0.50251	0.339516	1.667839
0.146721	0.146771	0.373027	0.502539	0.339728	1.667851
0.146771	0.146821	0.373182	0.502568	0.33994	1.667863
0.146821	0.146871	0.373337	0.502597	0.340151	1.667875
0.146871	0.146921	0.373492	0.502625	0.340363	1.667887
0.146921	0.146971	0.373648	0.502654	0.340575	1.667899
0.146971	0.147021	0.373803	0.502683	0.340786	1.667911
0.147021	0.147069	0.373958	0.502711	0.33822	1.668007
0.147069	0.147117	0.374107	0.502738	0.338425	1.668019
0.147117	0.147165	0.374256	0.502766	0.33863	1.66803
0.147165	0.147213	0.374406	0.502794	0.338835	1.668042
0.147213	0.147261	0.374555	0.502821	0.33904	1.668053
0.147261	0.147309	0.374704	0.502849	0.339245	1.668065
0.147309	0.147357	0.374854	0.502876	0.33945	1.668076
0.147357	0.147405	0.375003	0.502904	0.339655	1.668088
0.147405	0.147453	0.375153	0.502932	0.339859	1.668099
0.147453	0.147501	0.375302	0.502959	0.340064	1.66811
0.147501	0.147549	0.375452	0.502987	0.340268	1.668122
0.147549	0.147597	0.375602	0.503014	0.340473	1.668133
0.147597	0.147645	0.375752	0.503042	0.340677	1.668145
0.147645	0.147693	0.375902	0.50307	0.340881	1.668156
0.147693	0.147741	0.376051	0.503097	0.341085	1.668168

q_l	q_u	$k_{\min}$	$k_{\max}$	r_l	r_u
0.147741	0.147789	0.376201	0.503125	0.341289	1.668179
0.147789	0.147837	0.376352	0.503152	0.341493	1.668191
0.147837	0.147885	0.376502	0.50318	0.341697	1.668202
0.147885	0.147933	0.376652	0.503207	0.341901	1.668214
0.147933	0.147981	0.376802	0.503235	0.342104	1.668225
0.147981	0.148029	0.376952	0.503263	0.342308	1.668236
0.148029	0.148075	0.377103	0.503289	0.339598	1.668333
0.148075	0.148121	0.377247	0.503315	0.339795	1.668344
0.148121	0.148167	0.377391	0.503342	0.339992	1.668355
0.148167	0.148213	0.377535	0.503368	0.340189	1.668366
0.148213	0.148259	0.37768	0.503395	0.340386	1.668377
0.148259	0.148305	0.377824	0.503421	0.340583	1.668388
0.148305	0.148351	0.377968	0.503447	0.34078	1.668399
0.148351	0.148397	0.378113	0.503474	0.340977	1.66841
0.148397	0.148443	0.378258	0.5035	0.341174	1.668421
0.148443	0.148489	0.378402	0.503526	0.341371	1.668432
0.148489	0.148535	0.378547	0.503553	0.341567	1.668443
0.148535	0.148581	0.378692	0.503579	0.341764	1.668454
0.148581	0.148627	0.378836	0.503605	0.34196	1.668465
0.148627	0.148673	0.378981	0.503632	0.342156	1.668476
0.148673	0.148719	0.379126	0.503658	0.342352	1.668487
0.148719	0.148765	0.379271	0.503684	0.342549	1.668498
0.148765	0.148811	0.379416	0.503711	0.342745	1.668509
0.148811	0.148857	0.379561	0.503737	0.342941	1.66852
0.148857	0.148903	0.379706	0.503763	0.343137	1.668531
0.148903	0.148949	0.379852	0.50379	0.343332	1.668542
0.148949	0.148995	0.379997	0.503816	0.343528	1.668553
0.148995	0.149041	0.380142	0.503842	0.343724	1.668564
0.149041	0.149087	0.380288	0.503869	0.343919	1.668575
0.149087	0.149131	0.380433	0.503894	0.341051	1.668672
0.149131	0.149175	0.380572	0.503919	0.34124	1.668683
0.149175	0.149219	0.380712	0.503944	0.34143	1.668694
0.149219	0.149263	0.380851	0.503969	0.341619	1.668704
0.149263	0.149307	0.38099	0.503994	0.341808	1.668715
0.149307	0.149351	0.38113	0.50402	0.341997	1.668725
0.149351	0.149395	0.381269	0.504045	0.342186	1.668736
0.149395	0.149439	0.381409	0.50407	0.342375	1.668746
0.149439	0.149483	0.381548	0.504095	0.342564	1.668757
0.149483	0.149527	0.381688	0.50412	0.342753	1.668767
0.149527	0.149571	0.381828	0.504145	0.342942	1.668778
0.149571	0.149615	0.381967	0.50417	0.34313	1.668788
0.149615	0.149659	0.382107	0.504196	0.343319	1.668799
0.149659	0.149703	0.382247	0.504221	0.343507	1.668809

q_l	q_u	$k_{\min}$	$k_{\max}$	r_l	r_u
0.149703	0.149747	0.382387	0.504246	0.343696	1.66882
0.149747	0.149791	0.382527	0.504271	0.343884	1.66883
0.149791	0.149835	0.382667	0.504296	0.344072	1.668841
0.149835	0.149879	0.382807	0.504321	0.34426	1.668852
0.149879	0.149923	0.382948	0.504346	0.344448	1.668862
0.149923	0.149967	0.383088	0.504371	0.344636	1.668873
0.149967	0.150011	0.383228	0.504396	0.344824	1.668883
0.150011	0.150055	0.383368	0.504421	0.345012	1.668894
0.150055	0.150099	0.383509	0.504447	0.3452	1.668904
0.150099	0.150141	0.383649	0.50447	0.342151	1.669002
0.150141	0.150183	0.383784	0.504494	0.342333	1.669012
0.150183	0.150225	0.383918	0.504518	0.342514	1.669022
0.150225	0.150267	0.384052	0.504542	0.342696	1.669032
0.150267	0.150309	0.384186	0.504566	0.342877	1.669042
0.150309	0.150351	0.384321	0.50459	0.343059	1.669052
0.150351	0.150393	0.384455	0.504614	0.34324	1.669062
0.150393	0.150435	0.38459	0.504638	0.343422	1.669072
0.150435	0.150477	0.384724	0.504662	0.343603	1.669083
0.150477	0.150519	0.384859	0.504686	0.343784	1.669093
0.150519	0.150561	0.384993	0.50471	0.343965	1.669103
0.150561	0.150603	0.385128	0.504734	0.344146	1.669113
0.150603	0.150645	0.385263	0.504757	0.344327	1.669123
0.150645	0.150687	0.385398	0.504781	0.344508	1.669133
0.150687	0.150729	0.385533	0.504805	0.344688	1.669143
0.150729	0.150771	0.385668	0.504829	0.344869	1.669153
0.150771	0.150813	0.385803	0.504853	0.34505	1.669163
0.150813	0.150855	0.385938	0.504877	0.34523	1.669173
0.150855	0.150897	0.386073	0.504901	0.345411	1.669183
0.150897	0.150939	0.386208	0.504925	0.345591	1.669193
0.150939	0.150981	0.386343	0.504949	0.345771	1.669203
0.150981	0.151023	0.386479	0.504972	0.345951	1.669213
0.151023	0.151065	0.386614	0.504996	0.346132	1.669223
0.151065	0.151107	0.386749	0.50502	0.346312	1.669233
0.151107	0.151187	0.386885	0.505066	0.386407	1.667561
0.151187	0.151267	0.387143	0.505111	0.386719	1.667578
0.151267	0.151347	0.387401	0.505156	0.387031	1.667596
0.151347	0.151427	0.38766	0.505202	0.387343	1.667613
0.151427	0.151507	0.387919	0.505247	0.387655	1.667631
0.151507	0.151587	0.388178	0.505293	0.387966	1.667648
0.151587	0.151667	0.388437	0.505338	0.388278	1.667666
0.151667	0.151745	0.388697	0.505382	0.387091	1.667773
0.151745	0.151823	0.38895	0.505426	0.387395	1.66779
0.151823	0.151901	0.389204	0.505471	0.387698	1.667807

q_l	q_u	$k_{\min}$	$k_{\max}$	r_l	r_u
0.151901	0.151979	0.389457	0.505515	0.388002	1.667824
0.151979	0.152057	0.389711	0.505559	0.388306	1.667842
0.152057	0.152135	0.389965	0.505603	0.38861	1.667859
0.152135	0.152211	0.39022	0.505646	0.387378	1.667966
0.152211	0.152287	0.390468	0.505689	0.387674	1.667982
0.152287	0.152363	0.390716	0.505732	0.38797	1.667999
0.152363	0.152439	0.390964	0.505775	0.388266	1.668016
0.152439	0.152515	0.391213	0.505818	0.388562	1.668033
0.152515	0.152591	0.391462	0.505861	0.388858	1.668049
0.152591	0.152667	0.391711	0.505904	0.389154	1.668066
0.152667	0.152741	0.39196	0.505946	0.387877	1.668173
0.152741	0.152815	0.392203	0.505988	0.388165	1.66819
0.152815	0.152889	0.392447	0.50603	0.388453	1.668206
0.152889	0.152963	0.39269	0.506071	0.388741	1.668222
0.152963	0.153037	0.392934	0.506113	0.389029	1.668239
0.153037	0.153111	0.393177	0.506155	0.389317	1.668255
0.153111	0.153185	0.393421	0.506197	0.389605	1.668272
0.153185	0.153257	0.393666	0.506237	0.38828	1.668379
0.153257	0.153329	0.393903	0.506278	0.38856	1.668395
0.153329	0.153401	0.394142	0.506318	0.38884	1.668411
0.153401	0.153473	0.39438	0.506359	0.389121	1.668427
0.153473	0.153545	0.394618	0.5064	0.389401	1.668443
0.153545	0.153617	0.394857	0.50644	0.389681	1.668459
0.153617	0.153689	0.395096	0.506481	0.389961	1.668475
0.153689	0.153759	0.395335	0.50652	0.388585	1.668582
0.153759	0.153829	0.395568	0.50656	0.388858	1.668598
0.153829	0.153899	0.395801	0.506599	0.38913	1.668613
0.153899	0.153969	0.396034	0.506638	0.389402	1.668629
0.153969	0.154039	0.396267	0.506678	0.389675	1.668644
0.154039	0.154109	0.3965	0.506717	0.389947	1.66866
0.154109	0.154179	0.396734	0.506757	0.390219	1.668675
0.154179	0.154247	0.396968	0.506795	0.38879	1.668783
0.154247	0.154315	0.397195	0.506833	0.389055	1.668798
0.154315	0.154383	0.397423	0.506871	0.38932	1.668813
0.154383	0.154451	0.397651	0.506909	0.389584	1.668828
0.154451	0.154519	0.397879	0.506948	0.389849	1.668844
0.154519	0.154587	0.398107	0.506986	0.390113	1.668859
0.154587	0.154655	0.398336	0.507024	0.390378	1.668874
0.154655	0.154723	0.398564	0.507062	0.390642	1.668889
0.154723	0.154789	0.398793	0.507099	0.389157	1.668997
0.154789	0.154855	0.399015	0.507136	0.389414	1.669011
0.154855	0.154921	0.399238	0.507173	0.389671	1.669026
0.154921	0.154987	0.399461	0.50721	0.389928	1.669041

q_l	q_u	$k_{\min}$	$k_{\max}$	r_l	r_u
0.154987	0.155053	0.399684	0.507247	0.390185	1.669056
0.155053	0.155119	0.399907	0.507284	0.390442	1.66907
0.155119	0.155185	0.40013	0.507321	0.390698	1.669085
0.155185	0.155251	0.400354	0.507358	0.390955	1.6691
0.155251	0.155315	0.400577	0.507394	0.38941	1.669208
0.155315	0.155379	0.400794	0.50743	0.38966	1.669222
0.155379	0.155443	0.401012	0.507466	0.389909	1.669236
0.155443	0.155507	0.401229	0.507502	0.390158	1.669251
0.155507	0.155571	0.401447	0.507538	0.390407	1.669265
0.155571	0.155635	0.401665	0.507573	0.390656	1.669279
0.155635	0.155699	0.401883	0.507609	0.390905	1.669294
0.155699	0.155763	0.402101	0.507645	0.391154	1.669308
0.155763	0.155825	0.40232	0.50768	0.389546	1.669416
0.155825	0.155887	0.402532	0.507714	0.389788	1.66943
0.155887	0.155949	0.402744	0.507749	0.39003	1.669444
0.155949	0.156011	0.402956	0.507784	0.390271	1.669458
0.156011	0.156073	0.403169	0.507819	0.390512	1.669472
0.156073	0.156135	0.403381	0.507853	0.390754	1.669486
0.156135	0.156197	0.403594	0.507888	0.390995	1.6695
0.156197	0.156259	0.403807	0.507922	0.391236	1.669514
0.156259	0.156319	0.404021	0.507956	0.389561	1.669622
0.156319	0.156379	0.404227	0.507989	0.389795	1.669635
0.156379	0.156439	0.404434	0.508023	0.390029	1.669649
0.156439	0.156499	0.404641	0.508056	0.390263	1.669662
0.156499	0.156559	0.404848	0.50809	0.390497	1.669676
0.156559	0.156619	0.405055	0.508123	0.390731	1.669689
0.156619	0.156679	0.405263	0.508157	0.390964	1.669703
0.156679	0.156739	0.405471	0.50819	0.391198	1.669716
0.156739	0.156797	0.405679	0.508223	0.389451	1.669825
0.156797	0.156855	0.40588	0.508255	0.389677	1.669838
0.156855	0.156913	0.406081	0.508287	0.389904	1.669851
0.156913	0.156971	0.406283	0.50832	0.39013	1.669864
0.156971	0.157029	0.406484	0.508352	0.390356	1.669877
0.157029	0.157087	0.406686	0.508384	0.390582	1.66989
0.157087	0.157145	0.406889	0.508417	0.390808	1.669903
0.157145	0.157203	0.407091	0.508449	0.391034	1.669916
0.157203	0.157261	0.407293	0.508481	0.39126	1.66993
0.157261	0.157317	0.407496	0.508512	0.389436	1.670038
0.157317	0.157373	0.407692	0.508544	0.389655	1.67005
0.157373	0.157429	0.407888	0.508575	0.389874	1.670063
0.157429	0.157485	0.408084	0.508606	0.390093	1.670076
0.157485	0.157541	0.408281	0.508637	0.390311	1.670089
0.157541	0.157597	0.408477	0.508668	0.39053	1.670101

q_l	q_u	$k_{\min}$	$k_{\max}$	r_l	r_u
0.157597	0.157653	0.408674	0.508699	0.390748	1.670114
0.157653	0.157709	0.408871	0.508731	0.390967	1.670127
0.157709	0.157765	0.409068	0.508762	0.391185	1.670139
0.157765	0.157819	0.409266	0.508792	0.389278	1.670248
0.157819	0.157873	0.409456	0.508822	0.38949	1.67026
0.157873	0.157927	0.409647	0.508852	0.389701	1.670272
0.157927	0.157981	0.409838	0.508882	0.389912	1.670285
0.157981	0.158035	0.410029	0.508912	0.390123	1.670297
0.158035	0.158089	0.41022	0.508942	0.390334	1.670309
0.158089	0.158143	0.410411	0.508972	0.390545	1.670322
0.158143	0.158197	0.410603	0.509002	0.390756	1.670334
0.158197	0.158251	0.410795	0.509032	0.390967	1.670346
0.158251	0.158305	0.410987	0.509062	0.391178	1.670358
0.158305	0.158357	0.411179	0.509091	0.389181	1.670467
0.158357	0.158409	0.411364	0.509119	0.389385	1.670479
0.158409	0.158461	0.411549	0.509148	0.389589	1.670491
0.158461	0.158513	0.411735	0.509177	0.389792	1.670503
0.158513	0.158565	0.411921	0.509206	0.389996	1.670514
0.158565	0.158617	0.412107	0.509235	0.390199	1.670526
0.158617	0.158669	0.412293	0.509264	0.390403	1.670538
0.158669	0.158721	0.412479	0.509292	0.390606	1.67055
0.158721	0.158773	0.412666	0.509321	0.39081	1.670562
0.158773	0.158823	0.412852	0.509349	0.388715	1.67067
0.158823	0.158873	0.413032	0.509377	0.388912	1.670682
0.158873	0.158923	0.413212	0.509404	0.389108	1.670693
0.158923	0.158973	0.413392	0.509432	0.389304	1.670705
0.158973	0.159023	0.413572	0.50946	0.3895	1.670716
0.159023	0.159073	0.413752	0.509487	0.389696	1.670728
0.159073	0.159123	0.413933	0.509515	0.389892	1.670739
0.159123	0.159173	0.414113	0.509543	0.390088	1.670751
0.159173	0.159223	0.414294	0.509571	0.390284	1.670762
0.159223	0.159273	0.414475	0.509598	0.39048	1.670774
0.159273	0.159323	0.414656	0.509626	0.390676	1.670785
0.159323	0.159371	0.414838	0.509652	0.388476	1.670894
0.159371	0.159419	0.415012	0.509679	0.388665	1.670905
0.159419	0.159467	0.415187	0.509705	0.388853	1.670916
0.159467	0.159515	0.415361	0.509732	0.389042	1.670927
0.159515	0.159563	0.415536	0.509759	0.389231	1.670938
0.159563	0.159611	0.415711	0.509785	0.38942	1.670949
0.159611	0.159659	0.415886	0.509812	0.389608	1.67096
0.159659	0.159707	0.416062	0.509838	0.389797	1.670971
0.159707	0.159755	0.416237	0.509865	0.389985	1.670982
0.159755	0.159803	0.416413	0.509891	0.390174	1.670993

q_l	q_u	$k_{\min}$	$k_{\max}$	r_l	r_u
0.159803	0.159849	0.416589	0.509917	0.387855	1.671102
0.159849	0.159895	0.416758	0.509942	0.388037	1.671113
0.159895	0.159941	0.416926	0.509967	0.388218	1.671123
0.159941	0.159987	0.417095	0.509993	0.3884	1.671134
0.159987	0.160033	0.417265	0.510018	0.388581	1.671144
0.160033	0.160079	0.417434	0.510044	0.388763	1.671155
0.160079	0.160125	0.417604	0.510069	0.388944	1.671166
0.160125	0.160171	0.417773	0.510094	0.389125	1.671176
0.160171	0.160217	0.417943	0.51012	0.389307	1.671187
0.160217	0.160263	0.418113	0.510145	0.389488	1.671197
0.160263	0.160309	0.418283	0.510171	0.389669	1.671208
0.160309	0.160353	0.418454	0.510195	0.38722	1.671317
0.160353	0.160397	0.418617	0.510219	0.387394	1.671327
0.160397	0.160441	0.41878	0.510243	0.387568	1.671337
0.160441	0.160485	0.418944	0.510268	0.387742	1.671348
0.160485	0.160529	0.419108	0.510292	0.387917	1.671358
0.160529	0.160573	0.419271	0.510316	0.388091	1.671368
0.160573	0.160617	0.419435	0.51034	0.388265	1.671378
0.160617	0.160661	0.419599	0.510365	0.388439	1.671388
0.160661	0.160705	0.419764	0.510389	0.388613	1.671398
0.160705	0.160749	0.419928	0.510413	0.388787	1.671409
0.160749	0.160793	0.420093	0.510437	0.388961	1.671419
0.160793	0.160837	0.420258	0.510461	0.389135	1.671429
0.160837	0.160879	0.420423	0.510485	0.38654	1.671538
0.160879	0.160921	0.42058	0.510508	0.386707	1.671548
0.160921	0.160963	0.420738	0.510531	0.386874	1.671558
0.160963	0.161005	0.420896	0.510554	0.387041	1.671567
0.161005	0.161047	0.421054	0.510577	0.387208	1.671577
0.161047	0.161089	0.421213	0.5106	0.387375	1.671587
0.161089	0.161131	0.421371	0.510623	0.387542	1.671597
0.161131	0.161173	0.42153	0.510646	0.387709	1.671606
0.161173	0.161215	0.421688	0.51067	0.387876	1.671616
0.161215	0.161257	0.421847	0.510693	0.388043	1.671626
0.161257	0.161299	0.422006	0.510716	0.388209	1.671636
0.161299	0.161341	0.422166	0.510739	0.388376	1.671645
0.161341	0.161381	0.422325	0.510761	0.385617	1.671755
0.161381	0.161421	0.422477	0.510783	0.385777	1.671764
0.161421	0.161461	0.422629	0.510805	0.385937	1.671773
0.161461	0.161501	0.422782	0.510827	0.386097	1.671783
0.161501	0.161541	0.422934	0.510849	0.386257	1.671792
0.161541	0.161581	0.423087	0.510871	0.386417	1.671801
0.161581	0.161621	0.423239	0.510893	0.386576	1.67181
0.161621	0.161661	0.423392	0.510915	0.386736	1.67182

q_l	q_u	$k_{\min}$	$k_{\max}$	r_l	r_u
0.161661	0.161701	0.423545	0.510937	0.386896	1.671829
0.161701	0.161741	0.423699	0.510959	0.387056	1.671838
0.161741	0.161781	0.423852	0.510981	0.387215	1.671848
0.161781	0.161821	0.424006	0.511002	0.387375	1.671857
0.161821	0.161861	0.424159	0.511024	0.387534	1.671866
0.161861	0.161899	0.424313	0.511045	0.384588	1.671976
0.161899	0.161937	0.42446	0.511066	0.384741	1.671985
0.161937	0.161975	0.424606	0.511087	0.384894	1.671994
0.161975	0.162013	0.424753	0.511108	0.385047	1.672002
0.162013	0.162051	0.4249	0.511129	0.3852	1.672011
0.162051	0.162089	0.425047	0.51115	0.385353	1.67202
0.162089	0.162127	0.425194	0.51117	0.385505	1.672029
0.162127	0.162201	0.425341	0.511211	0.424253	1.670227
0.162201	0.162275	0.425628	0.511252	0.424535	1.670243
0.162275	0.162349	0.425916	0.511292	0.424816	1.670259
0.162349	0.162423	0.426205	0.511333	0.425097	1.670274
0.162423	0.162495	0.426494	0.511372	0.423794	1.670391
0.162495	0.162567	0.426775	0.511412	0.424068	1.670406
0.162567	0.162639	0.427058	0.511451	0.424342	1.670422
0.162639	0.162709	0.427341	0.511489	0.422992	1.670539
0.162709	0.162779	0.427617	0.511528	0.423258	1.670554
0.162779	0.162849	0.427893	0.511566	0.423524	1.670569
0.162849	0.162919	0.42817	0.511604	0.42379	1.670584
0.162919	0.162987	0.428447	0.511641	0.42239	1.6707
0.162987	0.163055	0.428718	0.511679	0.422648	1.670715
0.163055	0.163123	0.428988	0.511716	0.422907	1.67073
0.163123	0.163191	0.42926	0.511753	0.423165	1.670744
0.163191	0.163257	0.429532	0.511789	0.421713	1.670861
0.163257	0.163323	0.429796	0.511825	0.421964	1.670875
0.163323	0.163389	0.430061	0.511861	0.422215	1.670889
0.163389	0.163453	0.430327	0.511896	0.420707	1.671006
0.163453	0.163517	0.430585	0.511931	0.420951	1.67102
0.163517	0.163581	0.430844	0.511966	0.421194	1.671034
0.163581	0.163645	0.431103	0.512001	0.421437	1.671048
0.163645	0.163707	0.431363	0.512035	0.419871	1.671164
0.163707	0.163769	0.431615	0.512069	0.420107	1.671178
0.163769	0.163831	0.431868	0.512103	0.420343	1.671191
0.163831	0.163893	0.432121	0.512136	0.420579	1.671205
0.163893	0.163953	0.432375	0.512169	0.41895	1.671321
0.163953	0.164013	0.432621	0.512202	0.419178	1.671334
0.164013	0.164073	0.432868	0.512235	0.419407	1.671347
0.164073	0.164133	0.433115	0.512267	0.419635	1.67136
0.164133	0.164193	0.433363	0.5123	0.419863	1.671373

q_l	q_u	$k_{\min}$	$k_{\max}$	r_l	r_u
0.164193	0.164251	0.433612	0.512332	0.418168	1.67149
0.164251	0.164309	0.433852	0.512363	0.418389	1.671502
0.164309	0.164367	0.434093	0.512395	0.41861	1.671515
0.164367	0.164425	0.434335	0.512426	0.418831	1.671528
0.164425	0.164481	0.434577	0.512457	0.417064	1.671644
0.164481	0.164537	0.434812	0.512487	0.417278	1.671656
0.164537	0.164593	0.435047	0.512518	0.417491	1.671669
0.164593	0.164649	0.435282	0.512548	0.417704	1.671681
0.164649	0.164705	0.435518	0.512579	0.417918	1.671694
0.164705	0.164759	0.435755	0.512608	0.416075	1.67181
0.164759	0.164813	0.435984	0.512637	0.416281	1.671822
0.164813	0.164867	0.436213	0.512667	0.416487	1.671834
0.164867	0.164921	0.436442	0.512696	0.416693	1.671846
0.164921	0.164973	0.436673	0.512725	0.414767	1.671961
0.164973	0.165025	0.436895	0.512753	0.414965	1.671973
0.165025	0.165077	0.437117	0.512781	0.415164	1.671985
0.165077	0.165129	0.437341	0.512809	0.415363	1.671996
0.165129	0.165181	0.437564	0.512838	0.415561	1.672008
0.165181	0.165231	0.437789	0.512865	0.413545	1.672124
0.165231	0.165281	0.438005	0.512892	0.413736	1.672135
0.165281	0.165331	0.438221	0.512919	0.413928	1.672146
0.165331	0.165381	0.438438	0.512946	0.414119	1.672157
0.165381	0.165431	0.438656	0.512973	0.41431	1.672168
0.165431	0.165479	0.438874	0.512999	0.412196	1.672284
0.165479	0.165527	0.439084	0.513025	0.41238	1.672295
0.165527	0.165575	0.439294	0.513052	0.412564	1.672306
0.165575	0.165623	0.439505	0.513078	0.412748	1.672317
0.165623	0.165671	0.439716	0.513104	0.412932	1.672327
0.165671	0.165719	0.439928	0.51313	0.413116	1.672338
0.165719	0.165765	0.44014	0.513155	0.410894	1.672454
0.165765	0.165811	0.440344	0.51318	0.411071	1.672464
0.165811	0.165857	0.440548	0.513204	0.411248	1.672474
0.165857	0.165903	0.440753	0.513229	0.411425	1.672485
0.165903	0.165949	0.440959	0.513254	0.411601	1.672495
0.165949	0.165993	0.441165	0.513278	0.40926	1.672611
0.165993	0.166037	0.441362	0.513302	0.409429	1.672621
0.166037	0.166081	0.44156	0.513326	0.409599	1.672631
0.166081	0.166125	0.441758	0.51335	0.409769	1.672641
0.166125	0.166169	0.441957	0.513374	0.409938	1.672651
0.166169	0.166213	0.442156	0.513397	0.410108	1.672661
0.166213	0.166255	0.442356	0.51342	0.407633	1.672776
0.166255	0.166297	0.442547	0.513443	0.407796	1.672786
0.166297	0.166339	0.442738	0.513466	0.407958	1.672795

q_l	q_u	$k_{\min}$	$k_{\max}$	r_l	r_u
0.166339	0.166381	0.44293	0.513488	0.408121	1.672805
0.166381	0.166423	0.443123	0.513511	0.408284	1.672814
0.166423	0.166465	0.443316	0.513534	0.408446	1.672824
0.166465	0.166505	0.443509	0.513556	0.405821	1.672939
0.166505	0.166545	0.443694	0.513577	0.405977	1.672948
0.166545	0.166585	0.443879	0.513599	0.406133	1.672957
0.166585	0.166625	0.444064	0.51362	0.406288	1.672967
0.166625	0.166665	0.44425	0.513642	0.406444	1.672976
0.166665	0.166705	0.444437	0.513664	0.406599	1.672985
0.166705	0.166745	0.444624	0.513685	0.406755	1.672994
0.166745	0.166821	0.444811	0.513726	0.443278	1.671092
0.166821	0.166897	0.445168	0.513768	0.443564	1.671107
0.166897	0.166971	0.445527	0.513808	0.442305	1.67123
0.166971	0.167043	0.445878	0.513846	0.441002	1.671352
0.167043	0.167115	0.446222	0.513885	0.441273	1.671367
0.167115	0.167185	0.446567	0.513923	0.439925	1.671489
0.167185	0.167255	0.446904	0.513961	0.440189	1.671503
0.167255	0.167323	0.447243	0.513998	0.438793	1.671625
0.167323	0.167391	0.447574	0.514034	0.439049	1.671639
0.167391	0.167457	0.447906	0.51407	0.437602	1.671761
0.167457	0.167523	0.448231	0.514106	0.437851	1.671775
0.167523	0.167587	0.448557	0.51414	0.43635	1.671896
0.167587	0.167651	0.448874	0.514175	0.436591	1.67191
0.167651	0.167713	0.449194	0.514208	0.435035	1.672031
0.167713	0.167775	0.449504	0.514242	0.435268	1.672044
0.167775	0.167835	0.449817	0.514274	0.433651	1.672165
0.167835	0.167895	0.450121	0.514306	0.433877	1.672178
0.167895	0.167953	0.450426	0.514337	0.432196	1.672299
0.167953	0.168011	0.450723	0.514369	0.432415	1.672311
0.168011	0.168067	0.451022	0.514399	0.430665	1.672431
0.168067	0.168123	0.451311	0.514429	0.430876	1.672444
0.168123	0.168177	0.451602	0.514458	0.429052	1.672564
0.168177	0.168231	0.451884	0.514487	0.429256	1.672576
0.168231	0.168285	0.452168	0.514516	0.42946	1.672587
0.168285	0.168337	0.452453	0.514544	0.427556	1.672707
0.168337	0.168389	0.452729	0.514572	0.427753	1.672719
0.168389	0.168439	0.453006	0.514599	0.425763	1.672838
0.168439	0.168489	0.453274	0.514626	0.425952	1.672849
0.168489	0.168539	0.453544	0.514653	0.426141	1.67286
0.168539	0.168587	0.453815	0.514679	0.424056	1.67298
0.168587	0.168635	0.454076	0.514705	0.424239	1.672991
0.168635	0.168681	0.454339	0.514729	0.42205	1.67311
0.168681	0.168727	0.454593	0.514754	0.422225	1.67312

q_l	q_u	$k_{\min}$	$k_{\max}$	r_l	r_u
0.168727	0.168773	0.454847	0.514779	0.4224	1.67313
0.168773	0.168817	0.455103	0.514802	0.420096	1.673249
0.168817	0.168861	0.45535	0.514826	0.420264	1.673259
0.168861	0.168905	0.455597	0.51485	0.420432	1.673269
0.168905	0.168947	0.455846	0.514872	0.418001	1.673387
0.168947	0.168989	0.456085	0.514895	0.418162	1.673397
0.168989	0.169071	0.456325	0.514939	0.45617	1.671225
0.169071	0.169151	0.456798	0.514982	0.455025	1.671351
0.169151	0.169227	0.457265	0.515023	0.452329	1.671586
0.169227	0.169301	0.457713	0.515062	0.451067	1.671711
0.169301	0.169373	0.458154	0.515101	0.449763	1.671836
0.169373	0.169443	0.458588	0.515139	0.448413	1.671961
0.169443	0.169511	0.459015	0.515175	0.447017	1.672085
0.169511	0.169577	0.459434	0.51521	0.44557	1.672209
0.169577	0.169641	0.459845	0.515245	0.444072	1.672333
0.169641	0.169703	0.460249	0.515278	0.442517	1.672456
0.169703	0.169763	0.460645	0.51531	0.440903	1.672579
0.169763	0.169821	0.461032	0.515341	0.439227	1.672702
0.169821	0.169877	0.461411	0.515371	0.437482	1.672825
0.169877	0.169931	0.461781	0.5154	0.435665	1.672947
0.169931	0.169983	0.462142	0.515428	0.43377	1.673069
0.169983	0.170033	0.462494	0.515455	0.431789	1.67319
0.170033	0.170083	0.462836	0.515482	0.431978	1.673201
0.170083	0.170131	0.463183	0.515507	0.429905	1.673322
0.170131	0.170177	0.463519	0.515532	0.42773	1.673443
0.170177	0.170221	0.463846	0.515555	0.425442	1.673564
0.170221	0.170307	0.464162	0.515602	0.463626	1.671249
0.170307	0.170387	0.464791	0.515644	0.459667	1.671599
0.170387	0.170463	0.465391	0.515685	0.456967	1.671837
0.170463	0.170533	0.465975	0.515722	0.452499	1.672186
0.170533	0.170599	0.466526	0.515758	0.449401	1.672423
0.170599	0.170661	0.46706	0.515791	0.446108	1.672659
0.170661	0.170719	0.467574	0.515822	0.442596	1.672895
0.170719	0.170773	0.468067	0.515851	0.43883	1.67313
0.170773	0.170823	0.468539	0.515877	0.434766	1.673364
0.170823	0.170871	0.468986	0.515903	0.432698	1.673486
0.170871	0.170959	0.469428	0.51595	0.467445	1.671264
0.170959	0.171037	0.47027	0.515992	0.460627	1.671841
0.171037	0.171105	0.471058	0.516028	0.452988	1.672417
0.171105	0.171165	0.471784	0.51606	0.446159	1.672878
0.171165	0.171217	0.472463	0.516088	0.438411	1.673338
0.171217	0.171307	0.473086	0.516136	0.470099	1.671219
0.171307	0.171375	0.474266	0.516172	0.453998	1.672473

q_l	q_u	$k_{\min}$	$k_{\max}$	r_l	r_u
0.171375	0.171425	0.475278	0.516199	0.437031	1.673496
0.171425	0.171499	0.476125	0.516238	0.459305	1.67216
0.171499	0.171573	0.477674	0.516278	0.459582	1.672175

It follows that $|J| > 0$ on S^+ and so, by symmetry, on S^-.

We note that $\partial\hat{s}/\partial s$ can take both positive and negative values on S^+.

We can use the fact that μ_2 is positive further to find conditions under which the derivatives of $\hat{s}$ and $\hat{k}$ with respect to s and k are positive or negative.

We have

$$\frac{\partial\hat{s}}{\partial k} = \mu_2^{-5/2}\left(\mu_2\frac{\partial\mu_3}{\partial k} - \frac{3}{2}\mu_3\frac{\partial\mu_2}{\partial k}\right) = 36\mu_2^{-5/2}sS_{sk}(q,k),$$

where

$$S_{sk}(q,k) = 185q^3 + (120k^2 + 50k)q \\ - (261k + 51)q^2 - 7q - 18k^3 - 12k^2 + 3k + 1.$$

At $q = 0$ we need only consider $k \leq 1/3$, because of equation (6). And $S_{sk}(0, 1/3) = 0$. So we cannot have $S_{sk}(q,k) > 0$ on S^+. But we can have $S_{sk}(q,k) \geq 0$. Define

$$S'_{sk}(q,k) = \frac{\partial}{\partial k}S_{sk}(q,k) = 240kq + 50q - 261q^2 - 54k^2 - 24k + 3.$$

At $(0, 1/3)$ we have $S'_{sk}(q,k) = -11 < 0$. So we can reasonably hope that in some region of S^+ close to $(0, 1/3)$ we have $S_{sk}(q,k) \geq 0$.

Put

$$S'_{sk}(q_l, q_u, k) = 240kq_u + 50q_u - 261q_l^2 - 54k^2 - 24k + 3.$$

Then

$$S'_{sk}(0, 1/50, k) = -54k^2 - \frac{96k}{5} + 4,$$

which is a quadratic in k whose larger root is $(\sqrt{214} - 8)/45 < 3/20$. So $S'_{sk} < 0$ for $q \leq 1/50$ and $k > 3/20$.

On the line $k = 53q/48 + 1/3$ we have

$$S_{sk}(q,k) = \frac{q(116019q^2 - 190672q + 66688)}{6144},$$

which has three roots,

$$0, \quad -\frac{88\sqrt{174571} - 95336}{116019} \approx 0.5048 \quad \text{and} \quad \frac{88\sqrt{174571} + 95336}{116019} \approx 1.1386.$$

Hence $S_{sk}(q,k) \geq 0$ for points on $k = 53q/48 + 1/3$ between $q = 0$ and $q = 3 - 2\sqrt{2} < 0.5048$. It follows that for $q \leq 1/50$ and $3/20 \leq k \leq 53q/48 + 1/3$ we have $S_{sk}(q,k) \geq 0$ with equality only at $(0, 1/3)$.

Put

$$\begin{aligned} S_{sk}(q_l, q_u, k) = &185q_l^3 + 120k^2 q_l + 50kq_l + 3k + 1 \\ &- (261k + 51)q_u^2 - 7q_u - 18k^3 - 12k^2. \end{aligned}$$

Then, as before, we seek intervals q_l, q_u such that $S_{sk}(q_l, q_u, k)$ is positive between the lower bound $k_{\min}$ and upper bound $k_{\max}$ of expression (A.27). $S_{sk}(q_l, q_u, l)$ is cubic in k with a negative coefficient of k^3. So, when there are three real roots, it is positive between the second root r_l and the third r_u. The following table shows these roots and the values of $k_{\min}$ and $k_{\max}$ to six decimal places together with the exact values of q_u, q_l for intervals covering $[0, 3 - 2\sqrt{2}]$. As before we compute values with Maxima.

q_l	q_u	$k_{\min}$	$k_{\max}$	r_l	r_u
0	0.02	0	0.15	−0.795199	0.314841
0.02	0.024738	0.046951	0.360648	−0.161921	0.369249
0.024738	0.033916	0.058163	0.370782	−0.145165	0.371486
0.033916	0.04252	0.079994	0.380282	−0.125118	0.389093
0.04252	0.050586	0.100601	0.389189	−0.106186	0.405802
0.050586	0.058148	0.12006	0.397538	−0.08836	0.421671
0.058148	0.072327	0.138437	0.413194	−0.05526	0.414638
0.072327	0.084733	0.173296	0.426893	−0.023018	0.442433
0.084733	0.095589	0.204303	0.43888	0.004662	0.468202
0.095589	0.105087	0.231913	0.449367	0.028191	0.492037
0.105087	0.113399	0.256515	0.458545	0.048083	0.513956
0.113399	0.127943	0.278465	0.474604	0.105492	0.48453
0.127943	0.138851	0.318111	0.486648	0.12612	0.531085
0.138851	0.147033	0.349314	0.495682	0.138362	0.569463
0.147033	0.159303	0.373995	0.50923	0.198417	0.540556
0.159303	0.171573	0.414765	0.481217	0.241581	0.551643

Note that we choose $q_l = 0$, $q_u = 1/50$ and $k_{\max} = 3/20$ in the first line because we have already shown $S_{sk}(q,k) \geq 0$ for $q \in [0, 1/50]$ and $k \geq 3/20$. Since the second and third roots all fall outside $[k_{\min}, k_{\max}]$ we conclude that, for $(s,k) \in S^+$, $\partial\hat{s}/\partial k \geq 0$, with equality only only at $(0, 1/3)$.

We have

$$\begin{aligned} \frac{\partial \hat{k}}{\partial k} &= \mu_2^{-3}\left(\mu_2 \frac{\partial \mu_4}{\partial k} - 2\mu_4 \frac{\partial \mu_2}{\partial k}\right) \\ &= 24\mu_2^{-3} S_{kk}(q,k), \end{aligned}$$

where

$$S_{kk}(q,k) = 1115q^5 + 3675q^4 + 5040k^2q^3 + (7428k + 364)q^2$$

$$
\begin{aligned}
&+ (324k^4 + 1728k^3)q \\
&- 4590kq^4 - (2064k + 5764)q^3 - (2160k^3 + 2052k^2)q^2 \\
&- (3132k^2 + 480k + 15)q \\
&- 324k^4 + 432k^3 + 144k^2 + 18k + 1.
\end{aligned}
$$

Write

$$
\begin{aligned}
S_{kk}(q_l, q_u, k) = {}& 1115q_l^5 + 3675q_l^4 + 5040k^2q_l^3 + (7428k + 364)q_l^2 \\
&+ (324k^4 + 1728k^3)q_l \\
&- 4590kq_u^4 - (2064k + 5764)q_u^3 - (2160k^3 + 2052k^2)q_u^2 \\
&- (3132k^2 + 480k + 15)q_u \\
&- 324k^4 + 432k^3 + 144k^2 + 18k + 1.
\end{aligned}
$$

Putting $q_l = 0$ and $q_u = 0.171753 > 3 - 2\sqrt{2}$, we get a polynomial in k:

$$
\begin{aligned}
&\frac{171753(324k^4 + 1728k^3)}{1000000} - 324k^4 + 432k^3 + \frac{2119193136459210951k^2}{12500000000000000} \\
&+ \frac{29499093009(7428k + 364)}{1000000000000} + 18k + \frac{872923744154436943855264780439}{200000000000000000000000000000}.
\end{aligned}
$$

This has degree four and the coefficient of k^4 is negative. Using Maxima we find two real roots (to within 10^{-7}): -0.0658944 and 3.0234489. The polynomial must be positive between the roots. It follows that

$$
\frac{\partial \hat{k}}{\partial k} > 0
$$

on S^+.

We have

$$
\begin{aligned}
\frac{\partial \hat{k}}{\partial s} &= \mu_2^{-3} \left(\mu_2 \frac{\partial \mu_4}{\partial s} - 2\mu_4 \frac{\partial \mu_2}{\partial s} \right) \\
&= -48s\mu_2^{-3} S_{ks}(q, k),
\end{aligned}
$$

where

$$
\begin{aligned}
S_{ks}(q, k) = {}& 1115kq^4 + 14550kq^3 + (5040k^3 + 13890k + 300)q^2 + 7380k^3q \\
&- 2500q^4 - (4590k^2 + 10920)q^3 - 17100k^2q^2 \\
&- (2160k^4 + 5790k^2 - 578k + 16)q \\
&+ 324k^5 - 1080k^4 + 792k^3 + 204k^2 + 15k.
\end{aligned}
$$

When $q = 0$, $S_{ks}(q, k) = 3k(108k^4 - 360k^3 + 264k^2 + 68k + 5)$, which has one real root at $k = 0$. So we cannot have $S_{ks}(q, k) > 0$. But we may hope to show $S_{ks}(q, k) \geq 0$.

Put

$$
S'_{ks}(q, k) = \frac{\partial}{\partial k} S_{ks}(k, s)
$$

$$
\begin{aligned}
&= 1620k^4 + (15120q^2 + 22140q)k^2 + 2376k^2 + 408k \\
&\quad - (8640q + 4320)k^3 - (9180q^3 + 34200q^2 + 11580q)k \\
&\quad + 1115q^4 + 14550q^3 + 13890q^2 + 578q + 15
\end{aligned}
$$

and

$$
\begin{aligned}
S'_{ks}(q, k_l, k_u) &= 1620k_l^4 + (15120q^2 + 22140q)k_l^2 + 2376k_l^2 + 408k_l \\
&\quad - (8640q + 4320)k_u^3 - (9180q^3 + 34200q^2 + 11580q)k_u \\
&\quad + 1115q^4 + 14550q^3 + 13890q^2 + 578q + 15.
\end{aligned}
$$

Then, for $k \in [k_l, k_u]$ $(0 \leq k_l < k_u)$ and $q \geq 0$, $S'_{ks}(q, s) > S'_{ks}(q, k_l, k_u)$.

$$
\begin{aligned}
S'_{ks}(q, 0, 51/100) &= 1115q^4 + 14550q^3 \\
&\quad + \frac{51(-9180q^3 - 34200q^2 - 11580q)}{100} \\
&\quad + 13890q^2 + 578q + \frac{132651(-8640q - 4320)}{1000000} + 15.
\end{aligned}
$$

We use Maxima to find the real roots to within 10^{-7}. They are approximately -9.1303203, -0.0920217 and 0.9798701. Since $S'_{ks}(q, 0, 51/100)$ is a polynomial of degree four in q in which q^4 has positive coefficient and $k < 51/100$ for $(s, k) \in S^+$, it follows that $S'_{ks}(q, k) > 0$ on S^+.

We have $\sqrt{q^2 - 6q + 1} \leq 1$. So on S^+ we must have

$$
k \geq \frac{1 + 11q - \sqrt{q^2 - 6q + 1}}{6} \geq \frac{11q}{6}.
$$

We have

$$
S_{ks}(q, 11q/6) = -\frac{143q^5}{8} - 25q^4 - \frac{71q^3}{2} + \frac{6136q^2}{3} + \frac{23q}{2}.
$$

This has a root at 0. We estimate the remaining real roots with Maxima: these are approximately -0.0056220 and 4.3095737. Thus $S_{ks}(q, 11q/6) \geq 0$ for $q \in [0, 3 - 2\sqrt{2}]$. Since $S'_{ks}(q, k) \geq 0$ on S^+ it follows that $S_{ks}(s, k) \geq 0$ on S^+ also. Thus

$$
\frac{\partial \hat{k}}{\partial s} \leq 0
$$

on S^+ with equality only when $s = 0$.

For $(s, k) \in S^+$ with $s > 0$ we have

$$
\frac{\partial \hat{k}}{\partial k} > 0, \qquad \frac{\partial \hat{k}}{\partial s} < 0, \quad \text{and} \quad \frac{\partial \hat{s}}{\partial k} > 0. \tag{A.28}
$$

Since

$$
|J| = \frac{\partial \hat{s}}{\partial s}\frac{\partial \hat{k}}{\partial k} - \frac{\partial \hat{s}}{\partial k}\frac{\partial \hat{k}}{\partial s} > 0
$$

we have

$$\frac{\partial \hat{s}}{\partial s}\frac{\partial \hat{k}}{\partial k} > \frac{\partial \hat{s}}{\partial k}\frac{\partial \hat{k}}{\partial s}.$$

Combining this with inequalities (A.28), we get

$$\frac{\partial \hat{s}}{\partial s}\Big/\frac{\partial \hat{k}}{\partial s} < \frac{\partial \hat{s}}{\partial k}\Big/\frac{\partial \hat{k}}{\partial k}.$$

But

$$\frac{\partial \hat{s}}{\partial k}\Big/\frac{\partial \hat{k}}{\partial k} \quad \text{and} \quad \frac{\partial \hat{s}}{\partial s}\Big/\frac{\partial \hat{k}}{\partial s}$$

are the gradients in the $\hat{k}$–$\hat{s}$ plane of the lines given by $(\hat{k}(s,k), \hat{s}(s,k))$ for constant k and s respectively. Since the first gradient exceeds the second, it follows that the lines have at most one point in common. That is, for $(s,k) \in S^+$ with $s > 0$ there is a unique $(\hat{s}, \hat{k}) \in G(S^+)$ such that $G((s,k)) = (\hat{s}, \hat{k})$. The case $s = 0$ is straightforward. From equations (7) and (8), we find that $\hat{s} = 0 \Leftrightarrow s = 0$ and that $\hat{k}$ is an increasing function of k when $s = 0$. So there is also a unique $(0, \hat{k}) \in G(S^+)$ such that $G((0,k)) = (0, \hat{k})$.

It follows by symmetry that $G:\ S \to G(S)$ has a unique inverse.

Appendix B

We derive expressions we can use to calculate the first and second partial derivatives of $F^{-1}(u;\mathbf{p})$ with respect to κ_3 and κ_4.

Let $p, q \in \{\mu, \sigma\}$. Then

$$\begin{aligned}\frac{\partial}{\partial p}F^{-1}(u;\mathbf{p}) = \frac{\partial \mu}{\partial p} &+ {\mu_2}^{-1/2}\xi\left(\Phi^{-1}(u)\right)\frac{\partial \sigma}{\partial p} \\ &- \frac{1}{2}\sigma{\mu_2}^{-3/2}\xi\left(\Phi^{-1}(u)\right)\frac{\partial \mu_2}{\partial p} + \sigma{\mu_2}^{-1/2}\frac{\partial}{\partial p}\xi\left(\Phi^{-1}(u)\right).\end{aligned}$$

Since $\partial\mu/\partial p = 1$ whenever $p = \mu$ and $\partial\sigma/\partial p = 1$ whenever $p = \sigma$, we have $\partial^2\mu/(\partial p\partial q) = \partial^2\sigma/(\partial p\partial q) = 0$ and

$$\begin{aligned}\frac{\partial^2}{\partial p\partial q}F^{-1}(u;\mathbf{p}) = &-\frac{1}{2}{\mu_2}^{-3/2}\xi\left(\Phi^{-1}(u)\right)\left(\frac{\partial \sigma}{\partial p}\frac{\partial \mu_2}{\partial q} + \frac{\partial \sigma}{\partial q}\frac{\partial \mu_2}{\partial p}\right) \\ &- \frac{1}{2}\sigma{\mu_2}^{-3/2}\left(\frac{\partial \mu_2}{\partial p}\frac{\partial}{\partial q}\xi\left(\Phi^{-1}(u)\right) + \frac{\partial \mu_2}{\partial q}\frac{\partial}{\partial p}\xi\left(\Phi^{-1}(u)\right)\right) \\ &+ \frac{3}{4}\sigma{\mu_2}^{-5/2}\xi\left(\Phi^{-1}(u)\right)\frac{\partial^2 \mu_2}{\partial p\partial q} \\ &+ \sigma{\mu_2}^{-1/2}\frac{\partial^2}{\partial p\partial q}\xi\left(\Phi^{-1}(u)\right).\end{aligned}$$

The partial derivatives of μ_2 and ξ with respect to μ and σ are zero. Otherwise we have, for $\nu \in \{\mu_2, \xi\}$ and $p, q \in \{\kappa_3, \kappa_4\}$

$$\frac{\partial \nu}{\partial p} = \frac{\partial \nu}{\partial s}\frac{\partial s}{\partial p} + \frac{\partial \nu}{\partial k}\frac{\partial k}{\partial p} \quad \text{and}$$
$$\frac{\partial^2 \nu}{\partial p \partial q} = \frac{\partial^2 \nu}{\partial s^2}\frac{\partial s}{\partial p}\frac{\partial s}{\partial q} + \frac{\partial^2 \nu}{\partial s \partial k}\left(\frac{\partial s}{\partial p}\frac{\partial k}{\partial q} + \frac{\partial k}{\partial p}\frac{\partial s}{\partial q}\right) + \frac{\partial^2 \nu}{\partial k^2}\frac{\partial k}{\partial p}\frac{\partial k}{\partial q} + \frac{\partial \nu}{\partial s}\frac{\partial^2 s}{\partial p \partial q} + \frac{\partial \nu}{\partial k}\frac{\partial^2 k}{\partial p \partial q}.$$

The partial derivatives of μ_2 with respect to s and k are

$$\frac{\partial \mu_2}{\partial s} = 100s^3 - 48sk, \qquad \frac{\partial \mu_2}{\partial k} = \frac{k}{3} - 24s^2,$$
$$\frac{\partial^2 \mu_2}{\partial s^2} = 300s^2 - 48k, \qquad \frac{\partial^2 \mu_2}{\partial s \partial k} = -48s, \qquad \frac{\partial^2 \mu_2}{\partial k^2} = \frac{1}{3}.$$

And those of ξ are

$$\frac{\partial \xi(u)}{\partial s} = -1 + 10su + u^2 - 4su^3, \qquad \frac{\partial \xi(u)}{\partial k} = u^3 - u,$$
$$\frac{\partial^2 \xi(u)}{\partial s^2} = 10u - 12u^2, \qquad \frac{\partial^2 \xi(u)}{\partial s \partial k} = \frac{\partial^2 \xi(u)}{\partial k^2} = 0.$$

Since $\hat{s} = \kappa_3/6$ and $\hat{k} = \kappa_4/24$, for $t \in \{s, k\}$ we have

$$\frac{\partial t}{\partial \kappa_3} = \frac{1}{6}\frac{\partial t}{\partial \hat{s}}, \qquad \frac{\partial t}{\partial \kappa_4} = \frac{1}{24}\frac{\partial t}{\partial \hat{k}},$$
$$\frac{\partial^2 t}{\partial \kappa_3^2} = \frac{1}{36}\frac{\partial^2 t}{\partial \hat{s}^2}, \qquad \frac{\partial^2 t}{\partial \kappa_3 \partial \kappa_4} = \frac{1}{144}\frac{\partial^2 t}{\partial \hat{s} \partial \hat{k}}, \qquad \frac{\partial^2 t}{\partial \kappa_4^2} = \frac{1}{576}\frac{\partial^2 t}{\partial \hat{k}^2}.$$

So we need the partial derivatives of s and k with respect to $\hat{s}$ and $\hat{k}$. Given

$$\hat{s} = \hat{s}(s, k) \quad \text{and} \quad \hat{k} = \hat{k}(s, k),$$

we have

$$\begin{aligned} \frac{\partial \hat{s}}{\partial \hat{s}} &= \frac{\partial \hat{s}}{\partial s}\frac{\partial s}{\partial \hat{s}} + \frac{\partial \hat{s}}{\partial k}\frac{\partial k}{\partial \hat{s}} = 1, \qquad \frac{\partial \hat{k}}{\partial \hat{k}} = \frac{\partial \hat{k}}{\partial k}\frac{\partial s}{\partial \hat{k}} + \frac{\partial \hat{k}}{\partial k}\frac{\partial k}{\partial \hat{k}} = 1, \\ \frac{\partial \hat{s}}{\partial \hat{k}} &= \frac{\partial \hat{s}}{\partial s}\frac{\partial s}{\partial \hat{k}} + \frac{\partial \hat{s}}{\partial k}\frac{\partial k}{\partial \hat{k}} = 0, \qquad \frac{\partial \hat{k}}{\partial \hat{s}} = \frac{\partial \hat{k}}{\partial s}\frac{\partial s}{\partial \hat{s}} + \frac{\partial \hat{k}}{\partial k}\frac{\partial k}{\partial \hat{s}} = 0. \end{aligned} \tag{B.29}$$

Putting

$$\hat{J} = \begin{bmatrix} \frac{\partial \hat{s}}{\partial s} & \frac{\partial \hat{s}}{\partial k} \\ \frac{\partial \hat{k}}{\partial s} & \frac{\partial \hat{k}}{\partial k} \end{bmatrix} \quad \text{and} \quad J = \begin{bmatrix} \frac{\partial s}{\partial \hat{s}} & \frac{\partial s}{\partial \hat{k}} \\ \frac{\partial k}{\partial \hat{s}} & \frac{\partial k}{\partial \hat{k}} \end{bmatrix},$$

we can solve

$$J\hat{J} = I$$

to find the partial derivatives of the Jacobian matrix J. Put

$$A = \begin{bmatrix} \left(\frac{\partial s}{\partial \hat{s}}\right)^2 & 2\frac{\partial s}{\partial \hat{s}}\frac{\partial k}{\partial \hat{s}} & \left(\frac{\partial k}{\partial \hat{s}}\right)^2 \\ \frac{\partial s}{\partial \hat{s}}\frac{\partial s}{\partial \hat{k}} & \frac{\partial s}{\partial \hat{s}}\frac{\partial k}{\partial \hat{k}} + \frac{\partial s}{\partial \hat{k}}\frac{\partial k}{\partial \hat{s}} & \frac{\partial k}{\partial \hat{s}}\frac{\partial k}{\partial \hat{k}} \\ \left(\frac{\partial s}{\partial \hat{s}}\right)^2 & 2\frac{\partial s}{\partial \hat{k}}\frac{\partial k}{\partial \hat{k}} & \left(\frac{\partial k}{\partial \hat{k}}\right)^2 \end{bmatrix},$$

$$B = \begin{bmatrix} \frac{\partial^2 s}{\partial \hat{s}^2} & \frac{\partial^2 k}{\partial \hat{s}^2} \\ \frac{\partial^2 s}{\partial \hat{s}\partial \hat{k}} & \frac{\partial^2 k}{\partial \hat{s}\partial \hat{k}} \\ \frac{\partial^2 s}{\partial \hat{k}^2} & \frac{\partial^2 k}{\partial \hat{k}^2} \end{bmatrix} \quad \text{and} \quad \hat{B} = \begin{bmatrix} \frac{\partial^2 \hat{s}}{\partial s^2} & \frac{\partial^2 \hat{k}}{\partial s^2} \\ \frac{\partial^2 \hat{s}}{\partial s\partial k} & \frac{\partial^2 \hat{k}}{\partial s\partial k} \\ \frac{\partial^2 \hat{s}}{\partial k^2} & \frac{\partial^2 \hat{k}}{\partial k^2} \end{bmatrix}.$$

Differentiating equations (B.29) again and rearranging the resulting equations, we get

$$A\hat{B} = -B\hat{J}.$$

Since A, $\hat{B}$ and J are known, we can use this to find B and hence the second partial derivives of s and k with respect to $\hat{s}$ and $\hat{k}$.

Appendix C

Suppose we have a random variable U with known distribution and ζ defined in equation (19). Then $X = \zeta(U)$ defines a distribution if ζ' is strictly increasing: that is, if

$$a_3 > 0 \quad \text{and} \quad a_2^2 > a_1 a_3. \tag{C.30}$$

Suppose we wish to find X with given skewness and kurtosis. This is equivalent to finding X with specified third and fourth moments μ_3 and μ_4. That is, we require a_0, a_1, a_2 and a_3 satisfying

$$\mathbb{E}[X^3] = \mu_3 \quad \text{and} \quad \mathbb{E}[X^4] = \mu_3.$$

The left sides of these equations are third and fourth degree polynomials in a_0, a_1, a_2 and a_3. In general these and constrain (C.30) are nonconvex and do not uniquely specify a_0, a_1, a_2 and a_3. We know of no easy way to solve them or even to demonstrate whether or not a solution exists.

If U is standard normal we can simplify the problem by specifying

$$\mathbb{E}[X] = a_0 + a_1\mathbb{E}[U] + a_2\mathbb{E}[U^2] + a_3\mathbb{E}[U^3] = a_0 + a_3 = 0.$$

Then we must solve

$$\begin{aligned} 270a_2a_3^2 + 72a_1a_2a_3 + 8a_2^3 + 6a_1^2a_2 &= \mu_3 \\ 10395a_3^4 + 3780a_1a_3^3 + 4500a_2^2a_3^2 + 630a_1^2a_3^2 & \\ +936a_1a_2^2a_3 + 60a_1^3a_3 + 60a_1^2a_2^2 + 3a_1^4 &= \mu_4. \end{aligned}$$

We do not know how to solve even this except by substituting

$$a_1 = 1 + 5s^2 - 3k, \quad a_2 = s, \quad \text{and} \quad a_3 = k - 2s^2$$

so that ζ becomes ξ of equation (4).

Appendix D

Hill & Davis (1968) derive a general Cornish–Fisher inverse expansion where Φ of (1) need not be standard normal.

We can write any partition as

$$\pi = [s_1^{\rho_1}, \ldots, s_k^{\rho_k}] \tag{D.31}$$

where $s_i \in \{1, 2, 3, \ldots\}$, $\rho_i \in \{0, 1, 2, \ldots\}$ $(i = 1, \ldots, k)$ and $s_i \neq s_j$ unless $i \neq j$. We say π partitions m into l parts where

$$l(\pi) = l = \sum_{i=1}^{k} \rho_i \quad \text{and} \quad m(\pi) = m = \sum_{i=1}^{k} s_i \rho_i.$$

We say π is non-empty if $\rho_i > 0$ for some $i \in \{1, \ldots, k\}$. And the representation of π is unique if $s_1 < \cdots < s_k$ and $\rho_i > 0$ $(i = 1, \ldots, k)$. Given a partition π represented as in equation (D.31) and an integer $r > 0$ we write

$$\pi = \bigcup_{j=1}^{r} \pi_j = \pi_1 \cup \cdots \cup \pi_j$$

to indicate that $\pi_1, \ldots, \pi_r$ are (not necessarily distinct) partitions that we can represent as $\pi_j = [s_1^{\rho_{1j}}, \ldots, s_k^{\rho_{kj}}]$ $(j = 1, \ldots, r)$ with $\rho_i = \sum_{j=1}^{r} \rho_{ij}$ $(i = 1, \ldots, k)$. We may assume that precisely the first $R \leq r$ of $\pi_1, \ldots, \pi_r$ are distinct, write σ_j for the number of times that π_j occurs in $\pi_1 \cup \cdots \cup \pi_r$ $(j = 1, \ldots, R)$ and note $\sigma_1 + \cdots + \sigma_R = r$. We also define $m_j = \sum_{i=1}^{k} s_i \rho_{ij}$, the integer that π_j partitions.

Write

$$(n_1, \ldots, n_k)! = \frac{(n_1 + \cdots + n_k)!}{n_1! \cdots n_k!}$$

as notation for a multinomial coefficient. Given $\pi = \pi_1 \cup \cdots \cup \pi_j$, Hill & Davis (1968) define the partition function

$$p(\pi_1, \ldots, \pi_r) = \prod_{i=1}^{k} (\rho_{i1}, \ldots, \rho_{ir})!$$

and also

$$q(\pi_1, \ldots, \pi_r) = \frac{p(\pi_1, \ldots, \pi_r)}{\sigma_1! \cdots \sigma_R!}.$$

Define $\psi(u) = -\mathrm{D}_u \phi(u)/\phi(u)$, $\psi_\pi^{(1)} = \psi_{m(\pi)}$,

$$\psi_\pi^{(r)} = \sum_{\pi_1 \cup \cdots \cup \pi_r = \pi} q(\pi_1, \ldots, \pi_r) \psi_{m(\pi_1)} \cdots \psi_{m(\pi_r)}, \qquad (r = 2, 3, \ldots) \tag{D.32}$$

where the sum is taken over all sets $\{\pi_1, \ldots, \pi_r\}$ of r nonempty partitions with union π, $\psi_1(x) \equiv 1$,

$$\psi_r(u) = (\psi(u) - \mathrm{D}_u)\psi_{r-1}(u), \qquad (r = 2, 3, \ldots),$$

$$\mathrm{D}_u = \frac{\mathrm{d}}{\mathrm{d}u}.$$

and $\phi(u) = \mathrm{D}_u\Phi(u)$. Define also

$$\lambda_\pi = \frac{\lambda_{s_1}^{\rho_1} \cdots \lambda_{s_k}^{\rho_k}}{(s_1!)^{\rho_1} \cdots (s_k!)^{\rho_k} \rho_1! \cdots \rho_k!},$$

and

$$\lambda_r = \kappa_r - \gamma_r$$

where κ_r and γ_r are the cumulants of F_X and Φ of equation (1). Then Hill & Davis (1968) derive the Cornish–Fisher inverse expansion as

$$x = u + \sum_\pi \lambda_\pi P_\pi(u)$$

taken over all partitions of positive integers with

$$P_\pi(u) = \sum_{r=1}^{l(\pi)} (-1)^{r-1} \mathrm{D}_{(r)} \psi_\pi^{(r)}(u),$$

$\mathrm{D}_{(1)} = 1$ and

$$\mathrm{D}_{(r)} = (u - \mathrm{D}_u)(2u - \mathrm{D}_u) \cdots ((r-1)u - \mathrm{D}_u) \qquad (r > 1),$$

The fourth-order expansion is based on the assumption that, for some n, $\lambda_1 = O\left(n^{-1/2}\right)$, $\lambda_2 = O\left(n^{-1}\right)$ and $\lambda_r = O\left(n^{1-r/2}\right)$ for $r > 2$ and terms λ_π smaller than $O\left(n^{-1}\right)$ can be ignored. When Φ is standard normal and X has mean 0 and variance 1 we immediately get $\lambda_1 = \lambda_2$ and the truncated expansion reduces to equation (4). If Φ is a gamma, lognormal of beta distribution function with mean μ and variance σ we can still assume $\lambda_1 = \lambda_2 = 0$. If X has mean μ_X and variance σ_X^2 and we can use the Cornish–Fisher inverse expansion to find a distribution Y with the same skewness and kurtosis as X and mean μ and variance σ then $Y' = \sigma_X(Y - \mu)/\sigma + \mu_X$ will have the same mean, variance, skewness and kurtosis as X because skewness and kurtosis are invariant under shift and scale operations. Thus we can use the truncated fourth-order expansion,

$$x = u + \frac{1}{6}\lambda_3\psi_3(u) + \frac{1}{24}\lambda_4\psi_4(u) + \frac{1}{72}\lambda_3^2\left(\psi_6(u) - (u - \mathrm{D}_u)(\psi_3(u))^2\right). \qquad \text{(D.33)}$$

For the gamma distribution with parameters α and β we have

$$\phi(u) = \frac{u^{\alpha-1}}{\Gamma(\alpha)}\exp(-u) \quad \text{and} \quad \psi(u) = \beta + (1-\alpha)u^{-1}.$$

It is not hard to show that

$$\psi_n(u) = \sum_{k=0}^{n-1} \binom{n-1}{k} \beta^{n-1-k} \prod_{j=1}^{k} (j-\alpha) u^{-k}, \tag{D.34}$$

where we define $\prod_{j=1}^{0}(1-\alpha) = 1$.

For the beta distribution with parameters α and β we have

$$\phi(u) = \frac{u^{\alpha-1}(1-u)^{\beta-1}}{B(\alpha,\beta)} \quad \text{and} \quad \psi(u) = \frac{\alpha-1}{u} + \frac{\beta-1}{u-1}.$$

It is not hard to show that

$$\psi_n(u) = \sum_{k=0}^{n-1} \binom{n-1}{k} \prod_{i=1}^{n-k-1} (\alpha+i-2) \prod_{j=1}^{k} (\beta+j-2) u^{k+1-n} (1-u)^{-k}. \tag{D.35}$$

For the lognormal distribution with parameters μ and σ we have

$$\phi(u) = \frac{1}{\sigma u \sqrt{2\pi}} \exp\left(-\frac{1}{2} \left[\frac{\log(u)-\mu}{\sigma} \right]^2 \right) \quad \text{and} \quad \psi(u) = \frac{\log(u)+\sigma^2-\mu}{\sigma^2 u}.$$

Now put $p_2(v) = v/\sigma^2 + 1 - \mu/\sigma^2$ so that $\psi_2(u) = p_2(\log(u))/u$. Then $p_2(v)$ is a polynomial in v with leading coefficient $1/\sigma^2$. Suppose, for some $n > 2$, we can write $\psi_{n-1}(u) = p_n(\log(u))/u^n$ where $p_n(v)$ is a polynomial in $\log(u)$ with leading coefficient is $1/\sigma^{2n} > 0$. Then

$$\begin{aligned} \psi_n(u) &= \psi(u)\psi_{n-1}(u) - \mathrm{D}_u \psi_{n-1}(u) \\ &= \frac{p_{n-1}(\log(u)) p_2(\log(u)) + p_{n-1}(u) p'_{n-1}(\log(u))}{u^{n+1}} \\ &= \frac{p_n(\log(u))}{u^{n+1}}, \end{aligned} \tag{D.36}$$

where $p_n(v) = p_{n-1}(v)p_2(v) + p_{n-1}(v)p'_{n-1}(v)$ is a polynomial with leading coefficient $1/\sigma^{2n} > 0$.

Appendix E

Proof of Proposition 3. For any $n \geq 0$ we have

$$A_n = \sum_{m=0}^{\infty} A_m \int_S p_m(x) p_n(x) w(x) \,\mathrm{d}x = \int_S f(x) p_n(x) \,\mathrm{d}x = \sum_{k=0}^{n} p_{nk} \mu_k.$$

Write $\hat{\mu}_k$ for the kth moment of $\hat{f}$ with respect to $w(x)$. Then, for $n = 0, \dots, N$,

$$\sum_{k=0}^{n} p_{nk} \hat{\mu}_k = \int_S \hat{f}(x) p_m(x) \,\mathrm{d}x = \sum_{m=0}^{N} A_m \int_S p_m(x) p_n(x) w(x) \,\mathrm{d}x = A_n$$

We have $\mu_0 = \hat{\mu}_0 = 0$ by definition. For $n = 1, \ldots, N$,

$$\begin{aligned}
\hat{\mu}_n - \mu_n &= p_{nn}\hat{\mu}_n - p_{nn}\mu_n \\
&= \left(\sum_{k=0}^{n} p_{nk}\hat{\mu}_k - \sum_{k=0}^{n-1} p_{nk}\hat{\mu}_k\right) - \left(\sum_{k=0}^{n} p_{nk}\mu_k - \sum_{k=0}^{n-1} p_{nk}\mu_k\right) \\
&= \left(\sum_{k=0}^{n} p_{nk}\hat{\mu}_k - \sum_{k=0}^{n} p_{nk}\mu_k\right) - \left(\sum_{k=0}^{n-1} p_{nk}\hat{\mu}_k - \sum_{k=0}^{n-1} p_{nk}\mu_k\right) \\
&= 0.
\end{aligned}$$

So $\hat{\mu}_n = \mu_n$ for $n = 0, \ldots, N$. ∎

Note that if $\hat{f} \geq 0$ then it is a density function because its integral is $\mu_0 = 1$. But, in general, there is no guarantee that $\hat{f} \geq 0$.