THE BERKOVITS COMPLEX AND SEMI-FREE EXTENSIONS OF KOSZUL ALGEBRAS

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ABSTRACT. In his extension [3] of W. Siegel's ideas on string quantization, N. Berkovits made several observations which deserve further study and development. Indeed, interesting accounts of this work have already appeared in the mathematical literature [8, 15] and in a different guise due to Avramov. In this paper we bridge between these three approaches, by providing a complex that is useful in the calculation of some homologies.

CONTENTS

1. INTRODUCTION

This paper began with an observation of the importance of Koszul duality theory for commutative algebras in the work of Berkovits [3] on string quantization and string/gauge theory duality. In Berkovits' paper the commutative algebra in question is the projective coordinate algebra S of the orthogonal Grassmannian $OG(5, 10)$, related to the spinor representation of the group $SO(10, \mathbb{C})$. This is a commutative quadratic algebra, and for such algebras it is an easy consequence of the definition that its Koszul dual is the universal enveloping algebra of a graded Lie superalgebra $L = \bigoplus_{i\geq 1} L_i$.

In a very interesting paper Movshev–Schwarz [15] observed that the algebra of syzygies of S is isomorphic to the cohomology $H^*(L_{\geq 2}, \mathbb{C})$ of the graded Lie superalgebra $L_{\geq 2} = \bigoplus_{i\geq 2} L_i$. They proposed a further complex in an attempt to describe an off-shell formulation of $N = 1$, $D = 10$ Yang–Mills theory, and posed the question of calculating its homology. Their construction is an iteration of the Koszul homology of a sequence of elements of an algebra, first applied to the algebra S and then applied to its syzygies. Namely, if A is finitely generated and presented as $\mathbb{C}[a_1, \dots, a_n]/I$, the Koszul homology with respect to the sequence $\{a_i\}$ is called the algebra of syzygies of A. The generators Γ_j

of the ideal I represent syzygies in the lowest degree. The Koszul homology of the algebra of syzygies with respect to the sequence $\{\Gamma_j\}$ is the algebra studied by Movshev–Schwarz.

The same construction has appeared in the study of *deviations* of a local ring (see [2] and section 2.3 below). These are a sequence of integers that are attached to a local ring, that measure how far it is from being regular or a complete intersection. In calculating them, Avramov constructs complexes that are analogous to Movshev–Schwarz. A consequence of Avramov's work is that if A is Koszul, the Berkovits homology can be described explicitly in terms of the graded Lie superalgebra L. Namely, for an arbitrary finitely generated commutative Koszul algebra, the Berkovits homology is isomorphic to the cohomology $H^*(L_{\geq 3}, \mathbb{C})$. This result was also shown by Movshev–Schwarz for the case $A = S$. Further, they suggested an algebraic technique to study the Berkovits homology of the algebra S by constructing an equivalent but "smaller" complex. This complex does not seem to appear as an accident; we believe that it is naturally attached to a commutative Koszul algebra. We construct such a "smaller" complex for a Koszul algebra attached to the Plücker embedding of the Grassmannian $G(2, 5)$.

At this point we would like to mention a connection with a conjecture made by Avramov: Conjecture C_{10} of [1] states that if A is not a complete intersection, then the appropriate Lie superalgebra L contains a free nonabelian graded Lie subalgebra. The algebra S above provides an example confirming the conjecture, according to [11, 15]. Indeed, the "smaller" complex allows one to calculate that $H^2(L_{\geq 3}, \mathbb{C})$ is trivial. Our construction of a "smaller" complex allows us to show that the Lie superalgebra $L_{\geq 3}$ is free for the projective coordinate algebra of $G(2, 5)$, providing another confirmation of C_{10}.

The algebra of syzygies of S is in fact non-quadratic, and yet the "smaller" complex still exists. So, we believe that our construction can be extended to a more general case, but requires (to our knowledge) a substantial extension of the notion of Koszulness to non-quadratic algebras.

The paper is organised as follows: in Section 2 we discuss some preliminaries including Koszul algebras and Lie superalgebras; in Section 3.2 we define the Berkovits complex and how it arises as a semi-free extension of the initial algebra. The main theorem of the paper is Theorem 3.11, which shows that for any commutative Koszul algebra, its Berkovits homology is isomorphic to $H^*(L_{\geq 3}, \mathbb{C})$. It is in Section 4 where we construct the "small" complex and prove that it calculates the Berkovits homology when we have a Koszul algebra attached to the Plücker embedding of the Grassmannian $G(2, 5)$ (Corollary 4.8).

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2. Preliminaries on commutative Koszul algebras

We fix our ground field $k = \mathbb{C}$. Unless otherwise stated, all algebras are graded $A = \bigoplus_{i \geq 0} A_i$ and locally finite dimensional, i.e. $\dim A_i < \infty$ for all $i \geq 0$. Further, we assume that $A_0 = \mathbb{C}$, so that $A = \mathbb{C} \oplus A_+$ is augmented, with augmentation ideal $A_+ = \bigoplus_{i>0} A_i$.

2.1. **Koszul Algebras.**

Definition 2.1. *Let A be a quadratic algebra defined by a presentation $A = T(V)/Q$, where V is a finite dimensional vector space of generators in degree one, and Q is a two-sided ideal generated by quadratic elements. The* Koszul dual algebra *$A^!$ is the graded algebra defined as*

$$A^! = T(V^*)/Q^\perp,$$

where V^* is concentrated in degree one and $Q^\perp \subset V^* \otimes V^*$ is the annihilator of Q.

Clearly $A^{!!} = A$.

The *Koszul complex* $(K(A), d_A)$ of a quadratic algebra A is defined by the sequence of right $A \otimes A^!$-modules

$$K_p(A) := A \otimes (A^!)^*_p \qquad\qquad (p \geq 0).$$

Here A is a right A-module via multiplication and $(A^!)^*$ is a right $A^!$-module via

$$(\varphi b)(c) := \varphi(bc), \qquad (\varphi \in (A^!)^*,\ b, c \in A^!).$$

The differential d_A is the action of the identity $\mathrm{id}_V \in \mathrm{Hom}(V, V) = V \otimes V^* \subset A \otimes A^!$. It has degree -1, and satisfies $d_A^2 = 0$ since A is quadratic.

Recall that since A is graded, the algebra $\mathrm{Ext}_A(\mathbb{C}, \mathbb{C})$ is bigraded, and decomposes as

$$\mathrm{Ext}^i_A(\mathbb{C}, \mathbb{C}) = \bigoplus_{j \geq 0} \mathrm{Ext}^{ij}_A(\mathbb{C}, \mathbb{C}),$$

where i is the cohomological grading and j is the grading descending from A.

Definition 2.2. *An augmented algebra A is a* Koszul algebra *if $\mathrm{Ext}^{ij}_A(\mathbb{C}, \mathbb{C}) = 0$ whenever $i \neq j$.*

The following theorem gives some equivalent definitions of a Koszul algebra. A proof may be found, for example, in [16, Chapter 2, Theorem 4.1].

Theorem 2.3. *Let A be a quadratic algebra. Then the following are equivalent:*

(1) A is Koszul,
(2) $A^!$ is Koszul,
(3) $\mathrm{Ext}_A(\mathbb{C}, \mathbb{C}) \cong A^!$ as algebras with $\mathrm{Ext}^i_A(\mathbb{C}, \mathbb{C}) = A^!_i$,
(4) $(K(A), d_A)$ is a resolution of $\mathbb{C}$.

Example 2.4. Consider the symmetric algebra, $S(V)$. It is a quadratic algebra and the ideal of relations Q inside $T(V)$ is precisely $(u \otimes v - v \otimes u \mid u, v \in V)$. The annihilator of Q inside $T(V^*)$ is generated by tensors of the form $u^* \otimes v^* + v^* \otimes u^*$ and so $S(V)^! = \bigwedge V^*$. Therefore the Koszul complex of $S(V)$ has graded components

$$K_p(S(V)) = S(V) \otimes \bigwedge^p W_1 \qquad\qquad (p \geq 0) \tag{1}$$

where W_1 is a copy of V inside $(S(V)^!)^*$. If $a_1, \ldots, a_n$ and $\xi_1, \ldots, \xi_n$ are bases of V and W_1 respectively, then the differential can be written as

$$\sum_i a_i \frac{\partial}{\partial \xi_i}.$$

This makes $K(S(V))$ into a DG algebra over $S(V)$, with the obvious product structure. It is a resolution of $\mathbb{C}$ (see [14, Proposition VII.2.1] for a proof), so $S(V)$ and $S(V)^! = \bigwedge V^*$ are Koszul algebras.

2.2. Lie Superalgebras.

Definition 2.5. *A* Lie superalgebra *over $\mathbb{C}$ is a $\mathbb{Z}_2$-graded vector space $L = L_{\bar{0}} \oplus L_{\bar{1}}$ with a map $[\cdot,\cdot] : L \otimes L \to L$ of $\mathbb{Z}_2$-graded spaces, satisfying:*

1. *(anti-symmetry) $[x, y] = -(-1)^{|x||y|}[y, x]$ for all homogeneous $x, y \in L$,*
2. *(Jacobi identity) $(-1)^{|x||z|}[x, [y, z]] + (-1)^{|y||x|}[y, [z, x]] + (-1)^{|z||y|}[z, [x, y]] = 0$ for all homogeneous $x, y, z \in L$.*

Here $|x| = i$ when $x \in L_{\bar{i}}$ for $i = 0, 1$, and an element x in $L_{\bar{0}}$ or $L_{\bar{1}}$ is termed even or odd respectively. We recover the familiar definition of a Lie algebra (over $\mathbb{C}$) in the case $L = L_{\bar{0}}$.

A graded Lie superalgebra *is a Lie superalgebra L together with a grading compatible with the bracket and supergrading. That is, $L = \bigoplus_{m\geq 1} L_m$ such that $[L_i, L_j] \subset L_{i+j}$ and $L_{\bar{i}} = \bigoplus_{m\geq 1} L_{2m-i}$ for $i = 0, 1$.*

Remark 2.6. In the literature a graded Lie superalgebra is sometimes called simply a graded Lie algebra. However, this could also refer to an ordinary Lie algebra $L = L_{\bar{0}}$ with a grading compatible with the bracket. In order to avoid this ambiguity we prefer the terminology as in the definition.

Let V be a vector space with basis $a_1, \ldots, a_n$ and consider a quadratic commutative algebra $A = T(V)/Q$. Then, $Q = C \oplus I$ where C is the ideal $(u \otimes v - v \otimes u \mid u, v \in V)$ and $I = (\Gamma_1, \ldots, \Gamma_m) \subset S^2(V)$ so that we can write $A = S(V)/I$. Further, $Q \supseteq C$ implies that $Q^\perp \subseteq C^\perp = S^2(V^*)$ and so $Q^\perp$ is generated by certain linear combinations of anti-commutators $[a_i^*, a_j^*] = a_i^* a_j^* + a_j^* a_i^*$. We can therefore describe the Koszul dual as the universal envelope of a graded Lie superalgebra,

$$A^! = U(L), \qquad L = \bigoplus_{m\geq 1} L_m = \mathbb{L}(V^*)/J, \tag{2}$$

where $\mathbb{L}$ is the free Lie superalgebra functor, the space of (odd) generators V^* is concentrated in degree 1, and J is the Lie ideal with the same generators as $Q^\perp$ but viewed as linear combinations of supercommutators.

Definition 2.7. *For $k \geq 2$, we define the graded Lie superalgebras*

$$L_{\geq k} = \bigoplus_{m\geq k} L_m.$$

The interpretation of the algebras $L_{\geq k}$ for $k > 2$ was outlined in [3]. In this paper, we concentrate on the case $k = 3$.

2.3. **Deviations.** Consider the Hilbert series of the commutative algebra $A = S(V)/I$

$$H_A(t) := \sum_{r=0}^{\infty} \dim(A_r)\, t^r = \frac{h(t)}{(1-t)^n}, \tag{3}$$

where $h(t)$ is a polynomial such that $h(0) = 1$. Gauss' cyclotomic identity [7] allows us to expand this rational function into an infinite product

$$\frac{h(t)}{(1-t)^n} = \prod_{s=1}^{\infty} (1 - t^s)^{(-1)^s \varepsilon_s(A)}, \tag{4}$$

where the exponents $\varepsilon_s(A)$ are integers known as the *deviations* of A.

When A is a Koszul algebra, the exponents $\varepsilon_s(A)$ give the dimensions of the graded components L_s of the Lie superalgebra L (see for example [13]), and in particular we have $\varepsilon_1(A) = n$. Indeed, by the Poincaré–Birkhoff–Witt theorem for the universal enveloping algebra of a graded Lie superalgebra, the Hilbert series of $A^! = U(L)$ is equal to

$$H_{A^!}(t) = \prod_{s=1}^{\infty}(1-(-t)^s)^{(-1)^{s-1}\dim(L_s)}$$

and from the relation $H_A(t)\,H_{A^!}(-t) = 1$ it follows that $\varepsilon_s(A) = \dim(L_s)$ in (4). See [16, Section 2.2] for further details, and also Remark 3.10 below.

Dividing by

$$\prod_{s=1}^{k-1}(1-t^s)^{(-1)^s\varepsilon_s(A)}$$

we obtain the Hilbert series of $L_{\geq k}$ as the following infinite product

$$\prod_{s=k}^{\infty}(1-t^s)^{(-1)^s\varepsilon_s(A)}.$$

3. Koszul homology and Berkovits complex

3.1. **Koszul homology and the algebra of syzygies.** Let $A = S(V)/I$ be a commutative quadratic algebra with $\{a_1, \ldots, a_n\}$ a basis of V.

Definition 3.1. *The* Koszul homology *of A with respect to a sequence of elements $x_1, \ldots, x_k$ in A is the homology of the complex*

$$A[W_1] := \left(A \otimes \bigwedge W_1,\ \sum_{i=1}^{k} x_i \frac{\partial}{\partial \theta_i}\right)$$

where A has homological degree zero and W_1 is the vector space spanned by elements $\theta_1, \ldots, \theta_k$ in homological degree one.

The algebra of syzygies *of A is the Koszul homology of A with respect to the sequence $\{a_1, \ldots, a_n\}$.*

The algebra of syzygies of A is graded finite dimensional by Hilbert's syzygy theorem (c.f. [6, Theorem 1.1]).

Remark 3.2. We make the following trivial remarks:

(1) Suppose A is the symmetric algebra $S(V)$. Then $A^! = \bigwedge V^*$ and the complex $A[W_1]$ calculating the algebra of syzygies of A is precisely the Koszul complex, compare Example 2.4.

(2) The complex $A[W_1]$ is a DG algebra. The algebra structure is given by

$$(a \otimes \omega) \cdot (b \otimes \eta) = ab \otimes \omega \wedge \eta,$$

where $a, b \in A$ and $\omega, \eta \in \bigwedge W_1$. By a theorem of Kadeishvili [12], the Koszul homology, and in particular the algebra of syzygies, inherits an A_∞-algebra structure, which is unique up to (non-unique) isomorphism. We call this the *A_∞-minimal model* for $A[W_1]$ (the term minimal model will be reserved for something else).

(3) When A is graded, the Koszul homology is bigraded. In the case of Koszul homology with respect to a sequence of generators, the homological grading will be called the *order* of the syzygies and the sum of the homological grading and the grading on A will be called the *degree* of the syzygies.

We observe that $\mathrm{Tor}^{S(V)}(A,\mathbb{C})$ is also calculated by the complex $A[W_1]$, and so coincides with the algebra of syzygies. Indeed, $K(S(V))$ is a resolution of $\mathbb{C}$ by $S(V)$-modules, so $\mathrm{Tor}^{S(V)}(A,\mathbb{C})$ is the homology of the complex

$$A\otimes_{S(V)}K(S(V)) = A\otimes_{S(V)}S(V)\otimes_{\mathbb{C}}\bigwedge W_1 = A\otimes_{\mathbb{C}}\bigwedge W_1 \tag{5}$$

Alternatively, suppose we have a minimal free resolution of A,

$$\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow A \longrightarrow 0,$$

by graded free $S(V)$-modules

$$F_p = \bigoplus_q R_{p,q}\otimes S(V).$$

Here minimality means that the differential vanishes on tensoring this complex with the trivial $S(V)$-module $\mathbb{C}$, and hence

$$\mathrm{Tor}_p^{S(V)}(A,\mathbb{C}) \;=\; R_p := \bigoplus_q R_{p,q}.$$

Thus $R_{p,q}$ is the finite dimensional vector space of p-th order syzygies of degree q for A.

Example 3.3. Consider the Plücker embedding of the Grassmannian $G(2,5)$,

$$\begin{array}{rcl} G(2,5) & \longrightarrow & \mathbb{P}\left(\bigwedge^2\mathbb{C}^5\right) \\ \langle v_1,v_2\rangle_{\mathbb{C}} & \longmapsto & [v_1\wedge v_2]. \end{array}$$

Let $\{e_1,\ldots,e_5\}$ be a basis of $\mathbb{C}^5$, then $G(2,5)$ can be written as the intersection of five quadrics $\Gamma_i=0$ where

$$\begin{aligned} \Gamma_1 &= -e_{24}e_{35}+e_{23}e_{45}+e_{25}e_{34}\\ \Gamma_2 &= -e_{14}e_{35}+e_{13}e_{45}+e_{15}e_{34}\\ \Gamma_3 &= -e_{14}e_{25}+e_{12}e_{45}+e_{15}e_{24}\\ \Gamma_4 &= -e_{13}e_{25}+e_{12}e_{35}+e_{15}e_{23}\\ \Gamma_5 &= -e_{13}e_{24}+e_{12}e_{34}+e_{14}e_{23}. \end{aligned}$$

Here e_{12} is the coordinate function dual to $e_1\wedge e_2$, etc. The projective coordinate algebra of $G(2,5)$ is $A=S(V)/I$ where V is the vector space with basis $\{e_{ij}\}$ and $I=(\Gamma_1,\ldots,\Gamma_5)$.

From [8, Example 2.3.3] we know the dimensions of the syzygies and we can explicitly construct a minimal free resolution by $S(V)$ modules,

$$\begin{array}{ccccccccccc} 0 \longrightarrow & F_3 & \longrightarrow & F_2 & \longrightarrow & F_1 & \longrightarrow & F_0 & \longrightarrow & A & \longrightarrow 0.\\ & \langle c^*\rangle & & \langle\tilde\Lambda_1,\ldots,\tilde\Lambda_5\rangle & & \langle\tilde\Gamma_1,\ldots,\tilde\Gamma_5\rangle & & \langle c\rangle & & S(V)/I & \end{array}$$

In particular, $F_0=R_{0,0}\otimes S(V)\cong S(V)$ and the zeroth order syzygy $R_{0,0}$ is given by a copy of $\mathbb{C}$, generated by c. For exactness at F_0 we take $F_1=R_{1,2}\otimes S(V)$ where $R_{1,2}$ is spanned by $\tilde\Gamma_1,\ldots,\tilde\Gamma_5$ and $\partial\tilde\Gamma_i=\Gamma_i\cdot c$. The kernel of this boundary map is spanned by

$$\begin{aligned} \Lambda_1 &= \phantom{e_{12}\tilde\Gamma_1} -e_{12}\tilde\Gamma_2+e_{13}\tilde\Gamma_3 -e_{14}\tilde\Gamma_4+e_{15}\tilde\Gamma_5\\ \Lambda_2 &= e_{12}\tilde\Gamma_1 -e_{23}\tilde\Gamma_3+e_{24}\tilde\Gamma_4-e_{25}\tilde\Gamma_5\\ \Lambda_3 &= -e_{13}\tilde\Gamma_1+e_{23}\tilde\Gamma_2 -e_{34}\tilde\Gamma_4+e_{35}\tilde\Gamma_5\\ \Lambda_4 &= e_{14}\tilde\Gamma_1-e_{24}\tilde\Gamma_2+e_{34}\tilde\Gamma_3 -e_{45}\tilde\Gamma_5\\ \Lambda_5 &= -e_{15}\tilde\Gamma_1+e_{25}\tilde\Gamma_2-e_{35}\tilde\Gamma_3+e_{45}\tilde\Gamma_4 \end{aligned}$$

so for exactness at F_1 we take $F_2 = R_{2,3} \otimes S(V)$ where $R_{2,3}$ is spanned by $\tilde{\Lambda}_1, \ldots, \tilde{\Lambda}_5$ and

$$\begin{aligned}
\partial\tilde{\Lambda}_1 &= (0, -e_{12}, e_{13}, -e_{14}, e_{15})\\
\partial\tilde{\Lambda}_2 &= (e_{12}, 0, -e_{23}, e_{24}, -e_{25})\\
\partial\tilde{\Lambda}_3 &= (-e_{13}, e_{23}, 0, -e_{34}, e_{35})\\
\partial\tilde{\Lambda}_4 &= (e_{14}, -e_{24}, e_{34}, 0, -e_{45})\\
\partial\tilde{\Lambda}_5 &= (-e_{15}, e_{25}, -e_{35}, e_{45}, 0)
\end{aligned}$$

The third and highest order syzygy $R_{3,5}$ has a single generator c^* with

$$\partial c^* = \sum_{i=1}^{5} \Gamma_i \, \tilde{\Lambda}_i.$$

We return to this example in Example 4.4, where we write down the algebra structure on these syzygies.

3.2. **The Berkovits complex.** Let A be a commutative quadratic algebra given by $S(V)/I$, where V is a vector space with basis $a_1, \ldots, a_n$, and let W_1 be the vector space concentrated in degree 1 with basis $\theta_1, \ldots, \theta_n$.

Lemma 3.4. *Suppose the quadratic ideal I is spanned by*

$$\Gamma_k = \sum_{i,j=1}^{n} \Gamma_{ij}^k a_i a_j, \qquad (1 \le k \le m).$$

Then the first order syzygies are given by the following homology classes in $A[W_1]$,

$$\tilde{\Gamma}_k = \sum_{i,j=1}^{n} \Gamma_{ij}^k a_i \theta_j, \qquad (1 \le k \le m).$$

This leads to the following definition.

Definition 3.5. *The* Berkovits complex *of a commutative quadratic algebra A is*

$$A[W_1 \oplus W_2] := \left(A \otimes \bigwedge W_1 \otimes S(W_2), \ \sum_{i=1}^{n} a_i \frac{\partial}{\partial \theta_i} + \sum_{k=1}^{m} \tilde{\Gamma}_k \frac{\partial}{\partial y_k} \right),$$

where W_2 is the vector space spanned by elements $y_1, \ldots, y_m$ in homological degree two.

The Berkovits homology *of A is the homology of this complex.*

Definitions 3.1 and 3.5, which calculate Koszul and Berkovits homology respectively, are examples of *semi-free extensions* of commutative DG algebras, or *relative Sullivan algebras* (see for example [2, 10, 17, 18]):

Definition 3.6. *A* semi-free extension *of A is an inclusion of commutative DG algebras*

$$A \to A \otimes S_\bullet(W)$$

for some positively graded vector space W. We will write $A[W]$ for $A \otimes S_\bullet(W)$.

By $S_\bullet$ we mean the symmetric algebra in the graded sense,

$$S_\bullet(W) = S(W_{\text{even}}) \otimes \bigwedge W_{\text{odd}}.$$

For instance Definition 3.5 would read $A[W_1 \oplus W_2] = A \otimes S_\bullet(W_1 \oplus W_2)$.

3.3. **Minimal models for commutative Koszul algebras.** Assume that A is a commutative Koszul algebra with $A^! = U(L)$ as above. We construct a resolution of A in the category of DG algebras from the Chevalley-Eilenberg complex of L.

Definition 3.7. *The Chevalley-Eilenberg complex of L is the cochain complex with*

$$\mathrm{CE}^i(L) = \left(\bigwedge^i L\right)^*,$$

and the differential $d_C : \mathrm{CE}^k(L) \to \mathrm{CE}^{k+1}(L)$ is defined on $\varphi \in \left(\bigwedge^k L\right)^*$ by

$$(d_C\varphi)(x_0, \dots, x_k) = \sum_{i<j} (-1)^{j+\varepsilon(i,j)} \varphi(x_0, \dots, x_{i-1}, [x_i, x_j], x_{i+1}, \dots, \widehat{x_j}, \dots, x_k),$$

where each x_r is homogeneous, $\widehat{x_j}$ means omit x_j and $\varepsilon(i,j) = |x_j|(|x_{i+1}| + \cdots + |x_{j-1}|)$.

The Chevalley-Eilenberg complex is a cochain complex that calculates the cohomology of L with trivial coefficients, $H^*(L, \mathbb{C})$. Chevalley and Eilenberg in [5] introduced a *shuffle product* $\odot$ with respect to which the differential d_C is a derivation and gives this complex an algebra structure that descends to the algebra structure on the cohomology of L.

For graded Lie superalgebras with finite dimensional graded components, one can prove that the shuffle product $\odot$ is skew-supersymmetric and that there is an algebra isomorphism

$$\left(\bigwedge L^*, \wedge\right) \;\cong\; \left(\left(\bigwedge L\right)^*, \odot\right).$$

Hence, the Chevalley-Eilenberg complex can be alternatively defined as the exterior algebra on L^*, with differential d_C the extension as a derivation of the map dual to $[\ ,\ ] : L \wedge L \to L$.

As well as being a cochain complex whose cohomology is that of L, the Chevalley-Eilenberg complex may also be considered a chain complex which defines a resolution of A. The chain complex is given by defining L_p^* to have homological grading $p-1$, so that the homological and cohomological gradings together give the total degree in $\bigwedge L^*$. We illustrate the Chevalley-Eilenberg complex as a chain complex with homological grading as follows:

$$\begin{array}{cccccccccc}
\cdots\ \mathrm{CE}_4(L) & \xrightarrow{d_C} & \mathrm{CE}_3(L) & \xrightarrow{d_C} & \mathrm{CE}_2(L) & \xrightarrow{d_C} & \mathrm{CE}_1(L) & \xrightarrow{d_C} & \mathrm{CE}_0(L) \\
 & & & & & & 0 & \longrightarrow & L_1^* \\
 & & & & 0 & \longrightarrow & L_2^* & \longrightarrow & \bigwedge^2 L_1^* \\
 & & 0 & \longrightarrow & L_3^* & \longrightarrow & L_2^* \wedge L_1^* & \longrightarrow & \bigwedge^3 L_1^* \\
0 & \longrightarrow & L_4^* & \longrightarrow & L_3^* \wedge L_1^* \oplus L_2^* \wedge L_2^* & \longrightarrow & L_2^* \wedge L_1^* \wedge L_1^* & \longrightarrow & \bigwedge^4 L_1^*
\end{array}$$

The original cohomological grading is seen on the diagonals: $\mathrm{CE}^1(L) = L^*$ is the first non-zero diagonal and $\mathrm{CE}^2(L)$ is the diagonal containing exterior products of two factors.

Definition 3.8. *A* minimal model *of a commutative algebra $A = S(V)/I$ is a semi-free extension $S(V) \to S(V)[W]$ such that*

(1) the canonical quotient $S(V)[W] \to S(V) \to A$ is an isomorphism in homology,
(2) the differential is decomposable, $\partial W \subseteq S(V) \otimes S_\bullet^{\geq 2} W$.

This concept arose in rational homotopy theory, see for example [10, Definition 1.10] or [2, Section 7.2].

Proposition 3.9. *For a commutative Koszul algebra A with $A^! = U(L)$, the Chevalley-Eilenberg complex of L with homological grading is a minimal model of A.*

Proof. We observe that $\mathrm{CE}_0(L) = \bigwedge L_1^* = S(V)$. If we take $W_k = L_{k+1}^*$ for $k \geq 1$ then $\mathrm{CE}(L)$ is the semi-free extension $S(V)[W]$.

As L is graded, so is $\mathrm{Ext}^*_{U(L)}(\mathbb{C}, \mathbb{C})$, and since A is Koszul

$$H^i(L, \mathbb{C})_j \cong \mathrm{Ext}^{ij}_{U(L)}(\mathbb{C}, \mathbb{C}) = \begin{cases} A & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases},$$

where j is the total degree. Using our homological grading of $\mathrm{CE}(L)$ we have therefore,

$$H_i(\mathrm{CE}(L)) = \begin{cases} A & \text{if } i = 0 \\ 0 & \text{if } i \neq 0 \end{cases}$$

and the Chevalley-Eilenberg complex of L is a resolution of A.

Finally, the differential is decomposable, since $d_C(L_n^*) \subseteq \sum_{i=1}^{n-2} L_i^* \wedge L_{n-i}^*$. □

Remark 3.10. It follows that the Hilbert series of A may also be calculated as the Euler characteristic of the Chevalley-Eilenberg complex with homological grading, and we recover (4).

We will now give a proof of the following theorem, based on a more general result of Avramov.

Theorem 3.11. *The algebra of syzygies of A is isomorphic to $H^*(L_{\geq 2}, \mathbb{C})$ and the Berkovits homology is isomorphic to $H^*(L_{\geq 3}, \mathbb{C})$.*

Recall first the following definition from [2, Section 6.3].

Definition 3.12. *The* acyclic closure *of a commutative DG algebra A is a semi-free extension $A \to A[W]$ such that $A[W]$ is a resolution of $\mathbb{C}$ and $\{\partial w \mid w \in W_{n+1}\}$ minimally generates the reduced homology $\widetilde{H}_n(A[W_{\leq n}])$ for each $n \geq 0$.*

Thus the complexes calculating the Koszul and Berkovits homology are simply the first two steps of the acyclic closure of A. The following result of Avramov may be used to relate a general stage of the acyclic closure to the minimal model of A.

Recall that two DG algebras X and X' are quasi-isomorphic if there is a sequence of quasi-isomorphisms of DG algebras $X \sim X^1 \sim \cdots \sim X^n \sim X'$ pointing in either direction.

Theorem 3.13 ([2, Theorem 7.2.6])**.** *Suppose A is a commutative DG algebra with acyclic closure $A[W]$ and minimal model $S(V)[W']$. Then, for each $n \geq 1$, the DG algebras $A[W_{\leq n}]$ and $S(V)[W']/(W'_{<n})$ are quasi-isomorphic.*

If A is Koszul with $A^! = U(L)$ then the Chevalley-Eilenberg complex of L with homological grading is a minimal model $S(V)[W']$ of A, where $W'_i = L^*_{i+1}$. Since $S(V)[W']/(W'_{<n}) = \mathrm{CE}(L_{\geq n+1})$ the theorem gives a quasi-isomorphism

$$A[W_{\leq n}] \;\sim\; \mathrm{CE}(L_{\geq n+1}).$$

Theorem 3.11 is just the cases $n = 1$ and 2.

4. A smaller complex

Let A be a Koszul commutative algebra, with Koszul dual $A^!$ the universal enveloping algebra of a graded Lie superalgebra L. We have seen above that the Chevalley-Eilenberg complex of L (with homological grading) is a resolution of A, and the Chevalley-Eilenberg complexes of $L_{\geq 2}$ and $L_{\geq 3}$ calculate the algebra of syzygies R and the Berkovits homology respectively.

The Berkovits complex is not a module over $\mathbb{C}[y_1, \ldots, y_m]$, which makes its homology difficult to calculate in practice. There is a construction for the case of the orthogonal Grassmannian $OG(5, 10)$ alluded to in the introduction which appears in Movshev–Schwarz [15], where they found an alternative complex quasi-isomorphic to the Berkovits complex, which is a module over $\mathbb{C}[y_1, \ldots, y_m]$. This gives them a more manageable complex and they are able to calculate its homology directly.

Inspired by this construction, we sketch an alternative complex for calculating the Berkovits homology for a specific example where the algebra of syzygies can be explicitly described.

4.1. **The Grassmannians $G(2, N)$.** To that effect consider the Grassmannian as a homogeneous space $\mathrm{SL}_N(\mathbb{C})/P$, where P is some minimal complex parabolic subgroup that is not a Borel. In the usual way, the Plücker embedding of $G(2, N)$ is given by the orbit of the highest weight vector in a highest weight representation of $\mathrm{SL}_N(\mathbb{C})$. The corresponding projective coordinate algebra $A_{G(2,N)}$ is quadratic and Koszul. According to Gorodentsev et al. in [8] its algebra of syzygies of $A_{G(2,N)}$ is also quadratic and Koszul, but it was pointed out to us by the referee that this result is incorrect. The algebra of syzygies $R_{G(2,N)}$ in fact admits an non-trivial A_∞-algebra structure for $N > 5$, see Example 4.5 below.

Gorodentsev et al. give a full set of generators and relations for $A_{G(2,N)}$ in terms of semistandard Young tableaux. We recall some definitions regarding such tableaux and state this result here.

Definition 4.1. *A partition $\lambda = (\lambda_1, \ldots, \lambda_N)$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0$, is represented by a Young diagram with rows of lengths $\lambda_1, \ldots, \lambda_N$. The transposed diagram is denoted $\lambda' = (\lambda'_1, \ldots, \lambda'_{N'})$. A partition λ is written in Frobenius notation as*

$$(\alpha_1, \ldots, \alpha_p | \beta_1, \ldots, \beta_p) \qquad \lambda_i = \alpha_i + i, \ \lambda'_i = \beta_i + i.$$

For example, the partitions $(6, 4, 3, 3, 1, 0)$ and $(6, 4, 3, 3, 1, 0)' = (5, 4, 4, 2, 1, 1)$ are depicted

and are written $(5, 2, 0|4, 2, 1)$ and $(4, 2, 1|5, 2, 0)$ in Frobenius notation.

A semistandard Young tableau is a Young diagram in which the boxes are labelled with numbers $1, \ldots, N$ weakly increasing across rows and strictly increasing down columns. For instance,

(6)

1	1	2	4	5	5
3	3	4	6		
4	5	5			
5	6	6			
6					

The set of all semistandard Young tableaux of shape $(\alpha_1, \ldots, \alpha_p | \beta_1, \ldots, \beta_p)$ is denoted

$$\pi_{(\alpha_1, \ldots, \alpha_p | \beta_1, \ldots, \beta_p)}.$$

For each partition λ there is an irreducible $\mathrm{GL}_N(\mathbb{C})$-module (and in fact $\mathrm{SL}_N(\mathbb{C})$-module), also denoted by $\pi_{(\alpha_1, \ldots, \alpha_p | \beta_1, \ldots, \beta_p)}$, obtained as the quotient space

$$\left(\bigwedge^{\lambda'_1} V \otimes \cdots \otimes \bigwedge^{\lambda'_{k'}} V \right) \Big/ \text{column exchange relations},$$

where $V \cong \mathbb{C}^N$, the standard module for $\mathrm{GL}_N(\mathbb{C})$ or $\mathrm{SL}_N(\mathbb{C})$. We do not make these relations explicit, except to note that the semistandard Young tableaux in $\pi_{(\alpha_1,\ldots,\alpha_p|\beta_1,\ldots,\beta_p)}$ define basis vectors in this representation. For example the basis vector for (6) is

$$(e_1 \wedge e_3 \wedge e_4 \wedge e_5 \wedge e_6) \otimes (e_1 \wedge e_3 \wedge e_5 \wedge e_6) \otimes (e_2 \wedge e_4 \wedge e_5 \wedge e_6) \otimes (e_4 \wedge e_6) \otimes e_5 \otimes e_5,$$

where $e_1, \ldots, e_6$ are a basis of the standard module for $\mathrm{GL}_6(\mathbb{C})$ or $\mathrm{SL}_6(\mathbb{C})$. Further, $\pi_{(\lambda_1,\ldots,\lambda_N)} \cong \pi_{(\lambda_1-\lambda_N,\ldots,\lambda_{N-1}-\lambda_N,0)} \otimes \det^{\lambda_N}$.

If $\mathfrak{h}$ is the standard Cartan subalgebra of diagonal matrices in $\mathfrak{gl}_N = \mathrm{Lie}(\mathrm{GL}_N(\mathbb{C}))$, we let $\varepsilon_1, \ldots, \varepsilon_N$ be the standard basis of $\mathfrak{h}^*$. The highest weight of the irreducible representation corresponding to a partition λ is $(\lambda_1-\lambda_N)\varepsilon_1 + \cdots + (\lambda_{N-1}-\lambda_N)\varepsilon_{N-1}$. We also note that the fundamental weights of $\mathrm{GL}_N(\mathbb{C})$ are $\omega_i = \varepsilon_1 + \cdots + \varepsilon_i$ for $i = 1, \ldots, N-1$. The highest weight of the representation defined by Young diagrams of shape depicted in (6) is $6\varepsilon_1 + 4\varepsilon_2 + 3\varepsilon_3 + 3\varepsilon_4 + \varepsilon_5$ or $[2, 1, 0, 2, 1]$ written in a basis of fundamental weights.

One introduces the algebra

$$\mathsf{A}(N) = \bigoplus_{p,q} \mathsf{A}_{p,q}(N)$$

where

$$\mathsf{A}_{p,q}(N) := \bigoplus_{\substack{N-2 \geq i_1 > \cdots > i_p \geq 2 \\ i_1 + \cdots + i_p = q}} \pi_{((i_1-2),\ldots,(i_p-2)|(i_1+1),\ldots,(i_p+1))}.$$

This algebra is generated by

$$\mathsf{A}_{1,r} = \pi_{(r-2|r+1)},$$

for $2 \leq r \leq N-2$. The multiplication is given by the projection onto the relevant irreducible component in the tensor product representation. We have the following:

Theorem 4.2. [8, Theorem 4.4.1] *The algebra of syzygies of $G(2,5)$ form a bigraded supercommutative Frobenius Koszul algebra isomorphic to $\mathsf{A}(5)$ with*

$$(R_{G(2,5)})_{-p+q,q} = \mathsf{A}_{p,q}(5).$$

Remark 4.3. The theorem as stated in [8] claims that the syzygies of $G(2, N)$ form a bigraded supercommutative Frobenius Koszul algebra. We suggest why this is not true for $N = 6$. In particular the existence of a non-trivial ternary product implies that it is not Koszul.

Example 4.4. We return to Example 3.3 and consider the syzygies of $G(2,5)$, in which case

$$\begin{array}{rcccccccc} \mathsf{A}(5) & = & \mathsf{A}_{0,0}(5) & \oplus & \mathsf{A}_{1,2}(5) & \oplus & \mathsf{A}_{1,3}(5) & \oplus & \mathsf{A}_{2,5}(5) \\ & = & \pi_\emptyset & \oplus & \pi_{(0|3)} & \oplus & \pi_{(1|4)} & \oplus & \pi_{(10|43)} \\ & = & \mathbb{C} & \oplus & [0,0,0,1] & \oplus & [1,0,0,0] & \oplus & \mathbb{C} \\ & = & (R_{G(2,5)})_{0,0} & \oplus & (R_{G(2,5)})_{1,2} & \oplus & (R_{G(2,5)})_{2,3} & \oplus & (R_{G(2,5)})_{3,5}, \end{array}$$

where $\pi_\emptyset$ denotes the empty semistandard Young tableau and the corresponding generator in $R_{G(2,5)}$ is given by c. The full list of other semistandard Young tableaux along with their generators in $R_{G(2,5)}$ are given by

$$\pi_{(0|3)}: \quad \tilde\Gamma_1 = \begin{array}{|c|}\hline 2\\\hline 3\\\hline 4\\\hline 5\\\hline\end{array} \quad \tilde\Gamma_2 = \begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline 5\\\hline\end{array} \quad \tilde\Gamma_3 = \begin{array}{|c|}\hline 1\\\hline 2\\\hline 4\\\hline 5\\\hline\end{array} \quad \tilde\Gamma_4 = \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline 5\\\hline\end{array} \quad \tilde\Gamma_5 = \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline 4\\\hline\end{array}$$

$$\pi_{(1|4)}: \quad \tilde\Lambda_i = \begin{array}{|c|c|}\hline 1 & i\\\hline 2 \\\cline{1-1} 3 \\\cline{1-1} 4 \\\cline{1-1} 5 \\\cline{1-1}\end{array} \qquad \pi_{(10|43)}: \quad c^* = \begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline 3&3\\\hline 4&4\\\hline 5&5\\\hline\end{array}$$

so that $\dim\pi_{(0|3)} = \dim\pi_{(1|4)} = 5$ and $\dim\pi_{(10|43)} = 1$. Further, the algebra structure is given by $\tilde{\Gamma}_i\tilde{\Lambda}_j = \tilde{\Lambda}_j\tilde{\Gamma}_i = (-1)^{i+1}\delta_{ij}c^*$, c is the unit of the algebra and all other multiplications are zero.

Example 4.5. We consider the syzygies of $G(2,6)$ given by $\mathsf{A}(6) = \mathsf{A}_0(6) \oplus \mathsf{A}_1(6) \oplus \mathsf{A}_2(6) \oplus \mathsf{A}_3(6)$ where

$$\begin{array}{rcccccc} \mathsf{A}_1(6) & = & \mathsf{A}_{1,2}(6) & \oplus & \mathsf{A}_{1,3}(6) & \oplus & \mathsf{A}_{1,4}(6) \\ & = & \pi_{(0|3)} & \oplus & \pi_{(1|4)} & \oplus & \pi_{(2|5)} \\ & = & [0,0,0,1,0] & \oplus & [1,0,0,0,1] & \oplus & [2,0,0,0,0] \\ & = & (R_{G(2,6)})_{1,2} & \oplus & (R_{G(2,6)})_{2,3} & \oplus & (R_{G(2,6)})_{3,4}, \end{array}$$

$$\begin{array}{rcccccc} \mathsf{A}_2(6) & = & \mathsf{A}_{2,5}(6) & \oplus & \mathsf{A}_{2,6}(6) & \oplus & \mathsf{A}_{2,7}(6) \\ & = & \pi_{(10|43)} & \oplus & \pi_{(20|53)} & \oplus & \pi_{(21|54)} \\ & = & [0,0,0,0,2] & \oplus & [1,0,0,0,1] & \oplus & [0,1,0,0,0] \\ & = & (R_{G(2,5)})_{3,5} & \oplus & (R_{G(2,6)})_{4,6} & \oplus & (R_{G(2,6)})_{5,7}, \end{array}$$

and $\mathsf{A}_0(6) = \mathsf{A}_3(6) = \mathbb{C}$.

We suggest two reasons why this is not a Koszul algebra. The first is that it is not a hook algebra in the sense of [8, Section 5.1]: it fails to satisfy $x \cdot y = 0$ for all $x, y \in \pi_T$ where T is a partition. Indeed by the Littlewood-Richardson rule $\pi_{(1|4)} \otimes \pi_{(1|4)} \supset \pi_{(20|53)}$ is not zero as claimed.

Further the syzygies appear to have a nontrivial A_∞-algebra structure. Certainly one can see there is a ternary product $m_3 : \mathsf{A}_{1,2}(6) \otimes \mathsf{A}_{1,2}(6) \otimes \mathsf{A}_{1,2}(6) \to \mathsf{A}_{2,6}(6)$ directly by multiplying the Schur functions[1]

$${s_{(0|3)}}^3 = s_{(10|54)} + 2s_{(20|53)} + 3s_{(21|52)} + s_{(21|43)} + s_{(210|510)} + 2s_{(210|420)} + s_{(210|321)}.$$

The second factor is precisely $\mathsf{A}_{2,6}(6)$. If the algebra were Koszul then there could be no such nontrivial map of homological degree 1 and internal degree 0.

4.2. **The Complex.** We now construct the complex that calculates the Berkovits homology for $A_{G(2,5)}$ with $A^!_{G(2,5)} = U(L)$. Initially, the idea was to construct this complex for a general commutative Koszul algebra whose algebra of syzygies is also Koszul. However, as pointed out to us by the referee, this is a rare situation in the real world. We borrow the name 'bv_μ' from Movshev–Schwarz. Choose a basis $\{y_1, \ldots, y_5\}$ of L_2 inside the symmetric algebra $S(L_2)$ and consider the complex

$$bv_\mu \quad = \quad S(L_2) \otimes_{\mathsf{A}(5)^!} K(\mathsf{A}(5)^!) \quad \cong \quad S(L_2) \otimes_{\mathsf{A}(5)^!} \mathsf{A}(5)^! \otimes_{\mathbb{C}} \mathsf{A}(5)^*.$$

Here L_2 is concentrated in degree 0, and as usual for the Koszul complex we have $\mathsf{A}(5)^!$ concentrated in degree 0 and $\mathsf{A}(5)^*$ has grading induced by having its generators concentrated in degree 1. In the more general situation, one would have to replace the Koszul resolution by a suitable resolution of the A_∞-algebra of syzygies R, such as the bar resolution.

Consider the degree zero map of quadratic algebras

$$\bigwedge L_2^* \to \mathsf{A}(5), \qquad y_k^* \mapsto \tilde{\Gamma}_k.$$

The dual map induces a map $\mathsf{A}(5)^! \to S(L_2)$ which gives an action of $\mathsf{A}(5)^!$ on $S(L_2)$ by multiplication. Again, in the more general situation we would have to consider $S(L_2)$ as an R-module and this action may be more complicated.

[1]The calculation was performed using SAGE.

The differential is $1 \otimes d$ where d is the Koszul differential of $K(\mathsf{A}(5)^!)$. Thus we may write

$$bv_\mu = \left(S(L_2) \otimes \mathsf{A}(5)^*, \ \sum_{k=1}^{5} y_k \frac{\partial}{\partial \tilde{\Gamma}_k^*} \right).$$

In more general cases the differential would have a more complicated form due to the presence of higher multiplications and may contain terms of higher order in $\partial/\partial\tilde{\Gamma}_k^*$.

Taking into account the natural grading on the symmetric algebra we see that bv_μ is filtered by the complexes $S^{\geq p}(L_2) \otimes \mathsf{A}(5)^*$ and the associated graded is isomorphic to the bigraded vector space

$$\mathrm{Gr}(bv_\mu) = S(L_2) \otimes \mathsf{A}(5)^*. \tag{7}$$

In the more general case there is no guarantee that this associated graded object has no differential.

Choose a basis $\{q_1, \ldots, q_m\}$ of L_2 inside $U(L_{\geq 2})$. Now let

$$E_\mu = \left(\bigwedge L_2 \otimes U(L_{\geq 2}), \ \sum_{k=1}^{m} y_k \frac{\partial}{\partial q_k} \right)$$

with the standard augmented algebra structure. This has a filtration $\bigwedge^{\geq p} L_2 \otimes U(L_{\geq 2})$ whose associated graded $\mathrm{Gr}(E_\mu)$ is the algebra itself but with zero differential. The reduced bar complex B^+E_μ has an induced filtration, and we observe that the corresponding associated graded complex is isomorphic to the bar complex of the associated graded,

$$\mathrm{Gr}(B^+E_\mu) \ = \ B^+(\mathrm{Gr}(E_\mu)) \ = \ B^+\left(\bigwedge L_2 \otimes U(L_{\geq 2}) \right). \tag{8}$$

We show that bv_μ calculates the cohomology of $L_{\geq 3}$ by adapting the argument from [11, Section 3] which studies the case where $R^! \cong U(L_{\geq 2})$ is the Yang-Mills algebra, and which is also inspired by [15].

The idea is to use the complex B^+E_μ as an intermediary whose homology is the same as bv_μ and $\mathrm{CE}(L_{\geq 3})$:

$$bv_\mu \xrightarrow{\sim} B^+E_\mu \xleftarrow{\sim} \mathrm{CE}(L_{\geq 3}).$$

The map $p : bv_\mu \to B^+E_\mu$ is constructed as the composition of a map $bv_\mu \to E_\mu$ with the quasi-isomorphism $E_\mu \to B^+E_\mu$. If we map $y_k \mapsto y_k$, $\tilde{\Gamma}_k \mapsto q_k$ and $\tilde{\Lambda}_k$ to the corresponding generator in L_3 inside $U(L_{\geq 2})$, then this map clearly respects the filtrations.

Theorem 4.6. *Homologies of the complexes bv_μ and B^+E_μ coincide.*

Proof. Consider the spectral sequences associated to the filtrations on bv_μ and B^+E_μ defined above, which converge to their respective homologies and which have zero pages the associated graded objects (7), (8).

By Theorem 3.11 we can say that there exists a chain map

$$\mathsf{A}(5)^* \cong \mathsf{A}(5) \xrightarrow{\sim} \mathrm{CE}(L_{\geq 2}) \xrightarrow{\sim} B^+(U(L_{\geq 2}))$$

that induces isomorphism of homology groups, given by some choice of homology classes in the Chevalley–Eilenberg or bar complexes. Using this, together with classical Koszul duality for symmetric and exterior algebras, and the Künneth theorem, we can construct a chain map

$$S(L_2) \otimes \mathsf{A}(5)^* \xrightarrow{\sim} B^+(\bigwedge L_2) \otimes B^+(U(L_{\geq 2})) \xrightarrow{\sim} B^+\left(\bigwedge L_2 \otimes U(L_{\geq 2}) \right)$$

inducing isomorphism in homology. In other words, we have a chain map inducing isomorphism between the homologies of the associated graded objects above,

$$\mathrm{Gr}(bv_\mu) \xrightarrow{\sim} \mathrm{Gr}(B^+E_\mu).$$

This is a weak equivalence between the zero pages of the spectral sequences associated to the filtrations on bv_μ and B^+E_μ. Hence we have an isomorphism between the first pages. The result follows from the comparison theorem of spectral sequences. □

For the second quasi-isomorphism we adapt the proof in [11].

Proposition 4.7. *The algebras E_μ and $U(L_{\geq 3})$ are quasi-isomorphic, or the homologies of $B^+(E_\mu)$ and $\mathrm{CE}(L_{\geq 3})$ coincide. The quasi-isomorphism is given by the inclusion $z \in U(L_{\geq 3}) \mapsto 1 \otimes z \in E_\mu$.*

Proof. We must prove that the cohomology of E_μ is $U(L_{\geq 3})$. Consider the following filtration of $U(L_{\geq 2})$:

$$F^j := \begin{cases} 0 & j = 0 \\ \left\{ z \in U(L_{\geq 2}) \mid \frac{\partial}{\partial q_i}(z) \in F^{j-1}\ \forall i \right\} & j > 0 \end{cases}$$

Poincaré-Birkhoff-Witt implies that $F^1 = U(L_{\geq 3})$ (see [11, Proposition 3.1] for more details) and by [15, Lemma 28] the filtration is multiplicative, exhaustive, Hausdorff ($\bigcap F^j = 0$) and $q_i \in F^2$. We define a filtration of E_μ as in [11, Proposition 3.7]:

$$F_p E_{\mu,q} := F^{p+5-q} \otimes \bigwedge^{5-q} L_2.$$

It is clear that $d(F_p E_{\mu,q}) \subseteq F_{p-2} E_{\mu,q-1}$, where d is the differential on the complex E_μ, and so the differentials on the E^0 and E^1-pages of the spectral sequence associated to this filtration are zero.

Since [15, Lemma 29] is true in this context, we have that $\mathrm{Gr}_F(U(L_{\geq 2})) \cong U(L_{\geq 3}) \otimes \mathbb{C}[\hat{q}_1, \ldots, \hat{q}_5]$, where $\hat{q}_1, \ldots, \hat{q}_5$ are the images of $q_1, \ldots, q_5$ in $\mathrm{Gr}_F(U(L_{\geq 2}))$. The E^2-page is given by

$$E^2_{pq} = F_p E_{\mu,p+q} / F_{p-1} E_{\mu,p+q} \cong U(L_{\geq 3}) \otimes \mathbb{C}[\hat{q}_1, \ldots, \hat{q}_5] \otimes \bigwedge^{5-p-q} L_2.$$

The differential on the E^2-page is nothing but the Koszul differential for $\bigwedge L_2$ (see [11, Proposition 3.7] for more details), which is a resolution of $\mathbb{C}$. Hence, the spectral sequence collapses on the E^3-page to $E^3_{0,5} = U(L_{\geq 3})$. □

We have the following immediate corollary.

Corollary 4.8. *The complex bv_μ calculates the Berkovits homology. Further, we have $H^2(L_{\geq 3}, \mathbb{C}) = 0$ and so $L_{\geq 3}$ is free.*

Proof. The first statement is clear. For the second statement, we use a result from [4, Ex II.2.9] that if $L_{\geq 3}$ is positively graded then $L_{\geq 3}$ is a free Lie (super)algebra if and only if $H^2(L_{\geq 3}, \mathbb{C}) = 0$. The differential in the complex bv_μ is:

$$d(c) = 0, \qquad d(\tilde{\Gamma}_i) = y_i c, \qquad d(\tilde{\Lambda}_i) = 0, \qquad d(c^*) = \sum_{i=1}^{5} (-1)^{i+1} y_i \tilde{\Lambda}_i.$$

Since the only contribution to $H^2(L_{\geq 3}, \mathbb{C})$ could be given by c^*, and this is not a cycle, we obtain $H^2(L_{\geq 3}, \mathbb{C}) = 0$. □

4.3. **Further generalization.** We presented a simple construction of a complex bv_μ quasi-isomorphic to the Berkovits homology and to the Lie algebra $L_{\geq 3}$, which made certain calculations very straightforward. Although it is only constructed for $G(2,5)$, we would like to investigate whether the construction may be further extended to cases where the algebra of syzygies has higher structure.

We suspect that one may enrich the homology isomorphism between the complexes bv_μ and B^+E_μ. Namely, one would like quasi-isomorphisms of differential graded *algebras*:

between the cobar construction Ωbv_μ, the algebra E_μ, and the algebra $U(L_{\geq 3})$. For this, however, it would be necessary to define an appropriate coalgebra structure on bv_μ. In the case that the algebra of syzygies R is quadratic, bv_μ will will be a 'strict' differential graded coalgebra, and other cases it will be an A_∞-coalgebra. An explicit definition of such an A_∞ structure was given in the paper [15] of Movshev–Schwarz for the example they considered.

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