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This paper develops theory missing in the sizeable literature that uses data envelopment analysis to construct return : risk ratios for investment funds. It explores the production possibility set of the investment funds to identify an appropriate form of returns to scale. It discusses what risk and return measures can justifiably be combined and how to deal with negative risks, and identifies suitable sets of measures. It identifies the problems of failing to deal with diversification and develops an iterative approximation procedure to deal with it. It identifies relationships between diversification, coherent measures of risk and stochastic dominance. It shows how the iterative procedure makes a practical difference using monthly returns of 30 hedge funds over the same time period. It discusses possible shortcomings of the procedure and offers directions for future research.

DATA ENVELOPMENT ANALYSIS, INVESTMENT FUND, DIVERSIFICATION, COHERENT RISK MEASURE, RETURNS TO SCALE, STOCHASTIC DOMINANCE

1 Introduction

Data envelopment analysis (DEA) is a method for estimating the technical efficiency of several decision-making units (DMUs) given several inputs and several outputs. Typically the efficiency can be written in the form

$$\frac{\text{weighted sum of outputs}}{\text{weighted sum of inputs}} \quad (1)$$

In contrast to other methods of estimating technical efficiency, DEA does not use fixed weights. Instead it chooses the best nonnegative weights for each DMU subject to the constraint that, given those weights, no DMU has efficiency greater than 100%. Thus DEA gives nonsubjective estimates of efficiency.

Typically the DMUs are companies or public utilities that decide on their inputs and use them to produce the outputs. However, authors such as [Murthi et al. \(1997\)](#) and [Gregoriou and Zhu \(2005\)](#) observe the following analogy. If we use risk in place of input and return in place of output in expression (1), we get a ratio that generalizes return–risk ratios such as the Sharpe ([Sharpe, 1966](#)), Calmar and Sortino ([Lhabitant, 2004](#)) ratios, used to compare investment funds.

When we use return–risk ratios to compare investment funds, three issues arise. The first is investor preference. Different investors may take different attitudes to risk and return. So we should allow different investment funds to be measured against different criteria. We assume investors behave rationally. So our models should not prefer one fund to another if no rational investor would do so. The second issue is distributional shape. Given two funds with the same mean and standard deviation, investors usually prefer the one with greater positive skewness. The third is scope for diversification. Two funds can have identical distribution but one be better for diversifying a portfolio than the other. DEA can model different investor preferences and distributional shapes by using several measures of risk and return such as mean, median, standard deviation (Gregoriou et al., 2005a), lower and upper semivariance and semiskewness (Gregoriou et al., 2005b), skewness (Wilkins and Zhu, 2001), excess kurtosis (Nguyen-Thi-Thanh, 2006), time horizons (Galagadera and Silvapulle, 2002), percentage of periods with negative returns, skewness (Wilkins and Zhu, 2001), value at risk, conditional value at risk (Chen and Lin, 2006), downside absolute standard deviation, weighted absolute deviation from quantile, and tail value at risk (Lozano and Gutiérrez, 2008a). Eling (2006) reviews the measures used and concludes there is no single standard choice. Lozano and Gutiérrez (2008a,b) try to account for rational investor behavior using stochastic dominance (Levy, 1992). Basso and Funari (2001); Sengupta (2003) account for diversification indirectly using market β (Alexander, 2001) as a risk measure. And Morey and Morey (1999); Joro and Na (2006); Briec and Kerstens (2009) account for it directly using nonlinear versions of DEA.

The published literature on DEA for investment funds is not theoretically justified. In using the analogy between output–input and return–risk ratios it usually makes an implicit assumption that fund returns are perfectly correlated. And it often ignores the need for comparable measures of risk and return. We investigate theoretically when and how DEA may be used to compare investment funds, and develop a new method that models diversification directly. To do this we discuss four important issues.

1. What risk and return measures can be combined? A generalized return–risk ratio, for example of the form

$$\frac{\text{mean} + \text{skewness}}{\text{variance} + \text{kurtosis}}$$

is more problematic than a simple one like the Sharpe ratio. First, we must consider whether we can reasonably add and divide measures in different units. Second, some measures may take negative values. Conventionally, DEA requires positive or nonnegative values, and although some DEA models (Silva Portela et al., 2004; Sharp et al., 2007) allow negative values, these are not unproblematic. Third, we must choose measures that allow us to account reasonably for diversification.

2. Originally DEA (Charnes et al., 1978) assumed constant returns to scale (CRS) but later allowed for variable returns to scale (VRS) (Banker et al., 1984). Both are used for investment funds (Gregoriou and Zhu, 2005), but the analogy between output–input and risk–return suggests neither and many authors use one or other with limited or no justification. For example, Gregoriou et al. (2005a) argue for a VRS model only because it compensates for what Section 3 identifies as noncommensurable measures.
3. Usually DEA models use linear programs, though they can use nonlinear ones. The former are much easier to solve and make it easier to deal with investor preference and distributional

shape. But it is not obvious how to deal with diversification in them. The latter (Morey and Morey, 1999; Jurczenko et al., 2005; Joro and Na, 2006; Briec and Kerstens, 2009) model diversification using covariance, but are harder to solve and to use with the sets of measures we argue are appropriate.

4. DEA depends on technical assumptions of free disposability and convexity. The literature does not discuss whether these reasonably hold when modeling investment funds. We show that, in general, they do not.

We resolve these issues as follows. We show that a nonincreasing returns to scale (NRS) model is usually appropriate when modeling rational choice among investors. We show when multiple risk and return measures can justifiably be combined and identify some suitable measures. We show we need a nonlinear model to justify the assumption of convexity and to model diversification. We develop a method to approximate a solution to this model as accurately as needed using a sequence of linear models.

Coherent measures of risk come up again and again in our discussion. They relate to investor preference because they help model stochastic dominance. They allow us to account for distributional shape and also to use multiple risk measures. They also relate closely to what we need in order to deal with diversification. The best-known coherent measure of risk is conditional value at risk (CVAR).

The value at risk (VAR) of a security asset is the maximum loss that investors might suffer over a time horizon at a specified confidence level. For example, $\text{VAR}_{0.05}$ with a 1-month time horizon is the negative of the 5th percentile of the distribution of monthly returns. VAR is widely used as a risk measure in the literature on investment funds (Liang and Park, 2007) because the distribution of returns often shows significant skewness, which is captured by asymmetric measures like VAR but not by symmetric measures like standard deviation (Markovitz, 1952). Note that investors can lose more than VAR: it gives little information on the size of the loss. But CVAR measures this. It is the expected loss conditional on this loss exceeding VAR. It is sometimes called expected shortfall, tail conditional expectation, conditional loss or tail loss (Jorion, 2007). Acerbi and Tasche (2002) note some subtle differences among these terms, which we can ignore for continuous distribution functions, and show that CVAR is a coherent measure of risk. Acerbi (2007) shows how to estimate CVAR from a sample. We use this in Section 5.

Section 2 discusses the assumptions needed for a DEA model of investment funds and how to handle returns to scale. Section 3 introduces commensurability as a way of choosing a set of risk and return measures that can justifiably be combined. Section 4 explains coherent measures of risk and how to satisfy the assumption of convexity and model diversification through a series of linear approximations to an ideal nonlinear model. Section 5 illustrates our method using monthly returns from 30 hedge funds over the same time period. Section 6 discusses possible shortcomings of our method and some future directions for research.

2 Background

This section describes DEA models for investment funds and the assumptions needed for them and shows an NRS model is appropriate.

A DEA model compares n DMUs using m input and s output measures. An investment fund (or fund for short) produces returns that we can measure at regular time intervals. Section 5 uses examples with average monthly returns. We assume the regular returns are realizations of a real random variable and ignore, for now, the possibility that the random variable is a function of time. We suppose we have funds described by random variables f_1, \dots, f_n . We define a random variable $\sum_{j=1}^n \lambda_j f_j$ with $\lambda_j \geq 0$ ($j = 1, \dots, n$) and $\sum_{j=1}^n \lambda_j \leq 1$ to be a *portfolio* on f_1, \dots, f_n and we define the *portfolio possibility set* \mathcal{F} to be the set of all such portfolios. Then a *measure* is a function $g : \mathcal{F} \rightarrow \mathbb{R}$. Since a measure is a population statistic, we invariably must estimate it with a sample statistic. We call the sample statistic also a measure and use the same notation for both, distinguishing only when the difference is unclear. We classify some measures as return measures and some as risk measures. For example, the mean value is a return measure and the standard deviation a risk measure. Later we describe other risk and return measures and what properties they might have. We suppose we have risk measures x_1, \dots, x_m and return measures y_1, \dots, y_s . We write $x_{ij} = x_i(f_j)$, $y_{rj} = y_r(f_j)$, $\mathbf{x}_j = (x_{1j}, \dots, x_{mj})$ and $\mathbf{y}_j = (y_{1j}, \dots, y_{sj})$ ($i = 1, \dots, m$, $r = 1, \dots, s$, $j = 1, \dots, n$) so we can describe DEA compactly.

The (input-oriented) DEA model is this. For each DMU $o \in \{1, \dots, n\}$ choose $\mathbf{u}_o \in \mathbb{R}^s$, $\mathbf{v}_o \in \mathbb{R}^m$, and $\alpha \in \mathbb{R}$ to

maximize

$$\phi_o = \frac{\alpha + \mathbf{y}_o \cdot \mathbf{u}_o}{\mathbf{x}_o \cdot \mathbf{v}_o} \quad (2)$$

subject to

$$\frac{\alpha + \mathbf{y}_j \cdot \mathbf{u}_o}{\mathbf{x}_j \cdot \mathbf{v}_o} \leq 1, \quad (j = 1, \dots, n) \quad (3)$$

and

$$\mathbf{u}_o \geq \mathbf{0}, \mathbf{v}_o \geq \mathbf{0}. \quad (4)$$

Note that we solve (2)–(4) separately for each o . The *efficiency* of DMU o is ϕ_o , which must be in $[0, 1]$ provided $\mathbf{x}_j > \mathbf{0}$ and $\mathbf{y}_j \geq \mathbf{0}$ for $j = 1, \dots, n$. We set $\alpha = 0$ for a VRS model and do not constrain it for a CRS model. We use an input-oriented model because funds are usually compared using return–risk ratios like expression (2). Gregoriou and Zhu (2005) show an output-oriented model is possible. A slacks-based model (Tone, 2001) is also possible, though we know of none used to model funds.

Model (2)–(4) is usually recast as a linear program. We present the dual form. Gregoriou and Zhu (2005) give details. This form lets us discuss the assumptions needed for DEA, relate them to portfolios and the assumptions needed to model funds, describe the efficient frontier, and discuss what form of returns to scale is appropriate. We find the CRS DEA efficiencies from the following (dual) linear programs for each $o = 1, \dots, n$. Choose $\phi_o, \lambda_{1o}, \dots, \lambda_{no}$ to

minimize

$$\phi_o \quad (5)$$

subject to

$$\sum_{j=1}^n y_{rj} \lambda_{jo} \geq y_{ro}, \quad (r = 1, \dots, s) \quad (6)$$

$$\sum_{j=1}^n x_{ij} \lambda_{jo} \leq x_{io} \phi_o, \quad (i = 1, \dots, m) \quad (7)$$

and

$$\lambda_{jo} \geq 0 \quad (j = 1, \dots, n). \quad (8)$$

To get the VRS model we add the following constraint to (5)–(8).

$$\sum_{j=1}^n \lambda_{jo} = 1. \quad (9)$$

Following [Cazals et al. \(2002\)](#), we define a VRS *production possibility set* as a set $\Psi = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}_+^m \times \mathbb{R}_+^s\}$ satisfying

Free disposability: (see, for example, [\(Shephard, 1970\)](#)): if $(\mathbf{x}, \mathbf{y}) \in \Psi$, then $(\mathbf{x}', \mathbf{y}') \in \Psi$ whenever $\mathbf{x}' \geq \mathbf{x}$ and $\mathbf{y}' \leq \mathbf{y}$.

Convexity: if $(\mathbf{x}, \mathbf{y}), (\mathbf{x}', \mathbf{y}') \in \Psi$ then $(t\mathbf{x} + (1-t)\mathbf{x}', t\mathbf{y} + (1-t)\mathbf{y}') \in \Psi$ for $t \in (0, 1)$.

Ψ is the set of (\mathbf{x}, \mathbf{y}) such that it is possible to produce \mathbf{y} from \mathbf{x} . Free disposability and convexity are usually assumed to be reasonable for DEA models in economics, though [Bogetoft \(1996\)](#); [Cazals et al. \(2002\)](#) discuss cases where convexity is questionable. The *input-oriented efficient frontier* of Ψ is the set $\{(\mathbf{x}, \mathbf{y}) \in \Psi : (\mathbf{x}', \mathbf{y}) \notin \Psi \text{ for } \mathbf{x}' < \mathbf{x}\}$. It is straightforward to show that if $\lambda_{j1}, \dots, \lambda_{jn}, \phi_o$ solves the VRS DEA model, then $\sum_{j=1}^n (\mathbf{x}_j, \mathbf{y}_j) \lambda_j$ is on the efficient frontier of the smallest production possibility set containing all the DMUs and so ϕ_o is a measure of the efficiency of DMU o . It is also straightforward to check that we get the corresponding result for the CRS model if we expand the definition of Ψ so that $(t\mathbf{x}, t\mathbf{y}) \in \Psi$ whenever $(\mathbf{x}, \mathbf{y}) \in \Psi$ and $t > 0$. Figure 1 shows the mean (return) and standard deviation (risk) in the monthly returns of 30 funds described in Section 5. The darker shaded area is the production possibility set for the VRS model. Its frontier is shown as a line on the left of this area that is dashed then solid. It represents the possibility, common in economic models, that the marginal return on output decreases as input increases. The total shaded area is the production possibility set for the CRS model. Its frontier is shown as a line on the left of this area that is solid then dashed.

We know of no literature that checks if the standard assumptions about production possibility sets and returns to scale are valid for models of funds. They are not. There is a minor problem with free disposability. Suppose we use $\text{var}_{0.2}$ and $\text{var}_{0.05}$ as risk measures and have $\text{var}_{0.2}(f) = 0.1$ and $\text{var}_{0.05}(f) = 0.2$ for fund f . Then the free disposability assumption suggests an investment f' with $\text{var}_{0.2}(f') = 0.3$, $\text{var}_{0.05}(f') = 0.2$ is possible because it exceeds f in one risk measure. But $\text{var}_{0.05} \geq \text{var}_{0.2}$ by definition. This does not invalidate DEA because the efficiencies do not depend on free disposability. So we assume from here on that free disposability holds for any production possibility set we consider.

There is a much bigger problem with convexity. Suppose $\mathbf{x}(g)$ is the risk measure and $\mathbf{y}(g)$ the return measure for each fund g . And suppose f and f' are funds. Then, for $t \in (0, 1)$ we

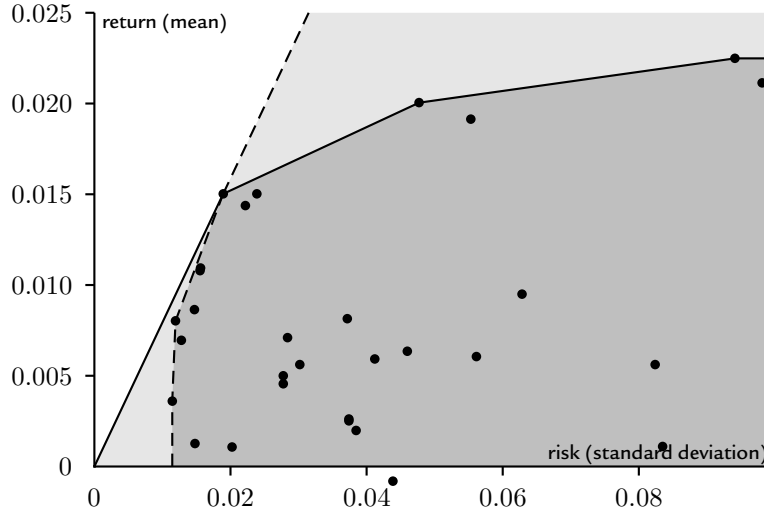


Figure 1: Comparison of different returns to scale

can assume that $(\mathbf{x}(tf + (1-t)f'), \mathbf{y}(tf + (1-t)f')) \in \Psi$ because $tf + (1-t)f'$ is a possible portfolio. But the equalities

$$\begin{aligned} t\mathbf{x}(f) + (1-t)\mathbf{x}(f') &= \mathbf{x}(tf + (1-t)f'), \\ t\mathbf{y}(f) + (1-t)\mathbf{y}(f') &= \mathbf{y}(tf + (1-t)f') \end{aligned} \quad (10)$$

only hold for perfectly correlated funds for all measures we know of other than mean value. So convexity is at best an approximation. Section 4 develops a method to deal with this approximation. An alternative approach, adopted by [Morey and Morey \(1999\)](#); [Jurczenko et al. \(2005\)](#); [Joro and Na \(2006\)](#); [Briec and Kerstens \(2009\)](#) handles convexity using nonlinear models. Section 3 discusses the problems of this approach.

We now consider returns to scale in models of funds. A *vrs* model is implausible because it allows for a minimum level of risk before a positive return is possible. A *crs* model is implausible because it allows returns greater than any portfolio can produce. The portfolio possibility set allows us to choose a portfolio that has some proportion uninvested: that is, invested with zero risk and zero return. Assuming equalities (10) are reasonable, this is equivalent to defining the production possibility set Ψ to satisfy

NRS convexity: if $(\mathbf{x}, \mathbf{y}), (\mathbf{x}', \mathbf{y}') \in \Psi$ then $(t\mathbf{x} + u\mathbf{x}', t\mathbf{y} + u\mathbf{y}') \in \Psi$ for $t, u \geq 0, t + u \leq 1$.

The solid line in Figure 1 shows the efficient frontier of the smallest production possibility set containing all 30 funds and satisfying this. The corresponding DEA model is (5)–(8) with the extra constraint

$$\sum_{j=1}^n \lambda_{jo} \leq 1. \quad (11)$$

We call it a *nonincreasing returns to scale* (NRS) model. We have not seen it used for linear models though [Joro and Na \(2006\)](#) use NRS with limited explanation in a nonlinear mean–variance–

skewness model. It is straightforward to check, using linear programming duality that the NRS model is equivalent to (2)–(4) with $\alpha \leq 0$. We consider only NRS models from here on.

3 Commensurable sets of measures

We call a set of inputs and outputs *commensurable* if each input or output is measured in a positive constant multiple of some common unit. This section argues that any DEA model should ideally use only commensurable sets of inputs and outputs. Later it argues that an input-oriented model of funds should use nonnegative risk measures and shows there are reasonable sets of measures that are commensurable and contain only nonnegative risk measures.

Lovell and Pastor (1995) note that the technical efficiency estimated by a DEA model should be dimensionless and that most models, including all the models we discuss, have the property that the efficiency is units invariant: that is, multiplying the values of any input or output by a positive constant does not change the efficiencies. This property is immediately clear in model (2)–(4) and implies that a commensurable set of inputs and outputs is necessary and sufficient for a dimensionless efficiency measure. Typically in economic models we can assume that both the inputs and the outputs in are measured approximately in some positive constant multiple of currency. However, we note below that the assumption of a commensurable set of inputs and outputs is violated in many models of funds. The following example shows why this is a problem. Suppose we have three DMUs whose inputs are $x_1 = 1.1$ units, $x_2 = 1.3$ units and $x_3 = 1.4$ units and whose outputs are $y_1 = 0.5$ units, $y_2 = 1.0$ units and $y_3 = 1.5$ units. Table 1 show the efficiencies from two sets of DEA models. The commensurable models use the inputs and outputs in the original units. The noncommensurable models use the square of the input values together with the outputs. In all three cases the set of efficient DMUs changes and so the incommensurable models fail even to satisfy the property of preserving the rank order of the efficiencies. The effect illustrates the problems of replacing standard deviation with variance.

RETURNS TO SCALE	MODEL TYPE	DMU 1	DMU 2	DMU 3
VRS	commensurable	1.000	0.958	1.000
	incommensurable	1.000	1.000	1.000
NRS	commensurable	0.762	0.889	1.000
	incommensurable	1.000	1.000	1.000
CRS	commensurable	0.762	0.889	1.000
	incommensurable	1.000	0.681	0.574

Table 1: Efficiencies for two sets of models

Risk and return measures for models of funds are typically statistics such as mean and standard deviation. A straightforward way to get a commensurable set of measures is to choose only statistics measured in the same units as the data. Often the literature uses statistics such as variance, semivariance, skewness and kurtosis not in the same units (Wilkins and Zhu, 2001; Gregoriou et al., 2005b; Nguyen-Thi-Thanh, 2006). There are several plausible explanations. First, funds returns are usually expressed as proportions and so it may not be immediately obvious that the efficiencies are not dimensionless. Second, DEA models are often expressed in dual form (like model (5)–(9)) in which the units of the efficiencies are less obvious. Third, the

nonlinear models (Morey and Morey, 1999; Joro and Na, 2006; Bricc and Kerstens, 2009) must use noncommensurable measures to be practically solvable. For example, Morey and Morey (1999) use mean, variance and covariance as measures in models with quadratic constraints. Their efficiencies are not dimensionless and their models have the problem illustrated in Table 1.

Initially DEA did not allow negative values for inputs and outputs. This is usually reasonable for economic models but less so for models of funds. Even the mean can be negative, as Figure 1 illustrates. If $y_{ro} \leq 0$ for some r and o then $\phi_o = 0$, $\lambda_{1o} = \dots = \lambda_{no} = 0$ satisfies constraints (6)–(8), (II). But $\phi_o \geq 0$ to satisfy constraints (7) provided $x_{io} > 0$ for some i . So negative return measures are largely unproblematic, and a NRS model with positive inputs and outputs unrestricted in sign is equivalent to one with positive inputs in which any negative output value is replaced by zero.

Negative or zero risk measures are more problematic. If $x_{io} < 0$ for some i it is possible that $\phi_o > 1$. And if $\mathbf{x}_o = \mathbf{0}$, $\phi_o = -\infty$. So we need positive risk measures for a reasonable definition of efficiency. However, to allow risk-free investments, ideally we would like at least the possibility that the risk measure could be zero. One approach to the problem of nonpositive input values is to add a constant to each value. If doing this does not change the efficiencies, we can call the input *translation invariant*. While some DEA models exhibit (at least limited) translation invariance (Lovell and Pastor, 1995), it is easy to check that the inputs in the NRS model are not translation invariant. So ideally we should avoid this approach. Another approach deals with negative values by measuring efficiency relative to an ideal point (Silva Portela et al., 2004). We eschew it because we have found no way to make it compatible with the production possibility set assumptions for a set of funds. Instead we define the *complete* DEA model as follows. Let $\mathbf{x}_j \geq \mathbf{0}$ and $\mathbf{y}_j \geq \mathbf{0}$ for $j = 1, \dots, n$. If $\mathbf{x}_o > \mathbf{0}$ define ϕ_o as in model (5)–(8), (II); otherwise put $\phi_o = 1$ or 0 according as $\mathbf{y}_o \geq \mathbf{0}$ or $\mathbf{y}_o < \mathbf{0}$. The complete model has the desirable property that a risk-free fund is efficient whenever the value of at least one return measure is nonnegative. We wish to establish further the consistency in the definition of ϕ_o . To do this we first show how the NRS and CRS models are related.

Suppose that $\mathbf{x}_j > \mathbf{0}$ and $\mathbf{y}_j > \mathbf{0}$ for $j = 1, \dots, n-1$, and $\mathbf{x}_n = \mathbf{y}_n = \mathbf{0}$. Then for $j = 1, \dots, n-1$, we can write the programs of the VRS model (5)–(9) as minimize ϕ_o subject to $\sum_{j=1}^{n-1} y_{rj} \lambda_{jo} \geq y_{ro}$ ($r = 1, \dots, s$), $\sum_{j=1}^{n-1} x_{rj} \lambda_{jo} \leq x_{io} \phi_o$ ($i = 1, \dots, m$), $\sum_{j=1}^{n-1} \lambda_{jo} \leq 1$, and $\lambda_{jo} \geq 0$ ($j = 1, \dots, n-1$). This is just the NRS model with DMUs $1, \dots, n-1$. It follows that we can find the efficiencies of an NRS model by solving all but one of the programs of the VRS model gotten by adding an extra DMU with all inputs and outputs equal to zero.

Ideally we would like to show that ϕ_o , considered as a function of $\mathbf{x}_1, \dots, \mathbf{x}_n, \mathbf{y}_1, \dots, \mathbf{y}_n$, is continuous at $\mathbf{x}_o = \mathbf{0}$. This cannot be done. Scheel and Scholtes (2003) show for the VRS model that, although ϕ_o is continuous almost everywhere, it may have discontinuities on the frontier and at points where $x_{ij} = 0$. While continuity of efficiency measures is desirable (Russell, 1990), we deliberately allow a discontinuity at $\mathbf{y}_o = \mathbf{0}$ because it is unlikely that a rational investor would consider a fund with no risk and negative return to be efficient. The following result shows ϕ_o is continuous at $\mathbf{x}_o = \mathbf{0}$ if $\mathbf{y}_o \geq \mathbf{0}$ and $x_{ij} > 0$ for $j \neq o$ and for all i . We write ϕ_o for the efficiency of DMU o in the complete VRS model and ϕ'_o for the efficiency of the o th DMU in the complete VRS model with DMUs given by inputs \mathbf{x}'_j and outputs \mathbf{y}'_j ($j = 1, \dots, n$).

THEOREM 1 Suppose $\mathbf{x}_o = \mathbf{0}$, $x_{ij} > 0$ ($i = 1, \dots, n$, $j \in \{1, \dots, n\} \setminus \{i\}$) and $\mathbf{y}_o \geq \mathbf{0}$. Then, for some $\delta > 0$, $\phi_o = \phi'_o = 1$ for all $\mathbf{x}'_1, \mathbf{y}'_1, \dots, \mathbf{x}'_n, \mathbf{y}'_n$ such that $\|\mathbf{x}_1 - \mathbf{x}'_1, \mathbf{y}_1 - \mathbf{y}'_1, \dots, \mathbf{x}_n - \mathbf{x}'_n, \mathbf{y}_n - \mathbf{y}'_n\|_2 < \delta$, $\mathbf{x}'_o \geq \mathbf{0}$ and $\mathbf{y}'_o \geq \mathbf{0}$.

Proof. We have $\phi_o = 1$ by definition. Choose $\delta = 0.5 \min \{x_{ij} : i \in \{1, \dots, m\}, j \in \{1, \dots, n\} \setminus \{i\}\}$. Then $\delta > 0$ and $x'_{ij} > 0$ for all i and for $j \neq 0$. If $\mathbf{x}'_o = \mathbf{0}$ then $\phi'_o = 1$ by definition. Otherwise, let h satisfy $x'_{hj} \geq x'_{ij}$ for $j = 1, \dots, n$. Then constraint (7) for $i = h$ gives

$$\phi'_o x'_{ho} \geq \sum_{j=1}^n x'_{ij} \lambda_{jo} \geq \sum_{j=1}^n \lambda_{jo} x'_{hj} = x'_{hj}$$

because $\sum_{j=1}^n \lambda_{jo} = 1$. Since $\mathbf{x}' > \mathbf{0}$, $x'_{hj} > 0$, and so $\phi'_o \geq 1$. The remaining constraints imply $\phi'_o \leq 1$, and so $\phi'_o = 1$ as required. \blacksquare

It follows from the relation between the NRS and VRS models that the result holds *mutatis mutandis* for the complete NRS model.

From here on we consider only DEA models of investment funds and use only models based on the complete NRS model. It remains to show there are commensurable sets of measures suitable for a complete NRS model. Return measures are straightforward since they can be negative: the mean, upper quantiles and upper tail means are in the same units as the data. The k th central moment with k odd is in the wrong units, but with suitable choice of sign, its k th root is a plausible return measure. Risk measures need to be nonnegative. The standard deviation, absolute deviation and lower semideviation are all reasonable. So is the positive k th root of the k th central moment with k even. And var and cvar are measured in the same units as the data. In practice they are usually positive, but if they have any negative values we can reasonably replace them with zero. This corresponds to saying there is no risk in the measure and is consistent with how the complete NRS model treats negative return measures.

4 Diversification, convexity, and coherent measures of risk

Section 2 discusses the production possibility set Ψ for a set of funds. It is the set of values of $((x_1(f), \dots, x_m(f)), (y_1(f), \dots, y_s(f)))$ such that f is in the portfolio possibility set \mathcal{F} . If the measures are nonnegative and form a commensurable set, the efficiency ϕ_o with respect to Ψ is given by the complete NRS model if $\mathbf{x}_o = \mathbf{0}$ and by the following mathematical program otherwise. Choose $\phi_o, \lambda_{1o}, \dots, \lambda_{no}$ to

minimize ϕ_o subject to

$$y_r \left(\sum_{j=1}^n \lambda_{jo} f_j \right) \geq y_r(f_o), \quad (r = 1, \dots, s) \quad (12)$$

$$x_i \left(\sum_{j=1}^n \lambda_{jo} f_j \right) \leq x_i(f_o) \phi_o, \quad (i = 1, \dots, m) \quad (13)$$

and constraints (ii) and (8). We call this model the *diversification-consistent (input-oriented complete NRS DEA) model* because it deals fully with diversification. We believe it is new. Its main drawback is that it is not generally solvable by any known practical method. We see existing models as simplifications or approximations of it. The nonlinear models (Morey and Morey, 1999; Joro and Na, 2006; Briec and Kerstens, 2009) are simple special cases that are solvable because they are so restricted in the choice of measures that they fail to form a commensurable set. The remaining models are at best linear approximations because they assume

$$\kappa \left(\sum_{j=1}^n \lambda_j f_j \right) = \sum_{j=1}^n \lambda_j \kappa(f_j) \quad (14)$$

for each $\kappa \in \{x_1, \dots, x_m, y_1, \dots, y_s\}$. Equation (14) holds for the mean value but it does not hold in general. Worse, a model in which it holds cannot deal with diversification.

The diversification-consistent model is a convex programming problem – and so more tractable – whenever Ψ satisfies the assumptions of free disposability and NRS convexity. It is easy to check that NRS convexity is equivalent to requiring

$$x_i \left(\sum_{j=1}^n \lambda_j f_j \right) \leq \sum_{j=1}^n \lambda_j x_i(f_j) \quad \text{and} \quad y_r \left(\sum_{j=1}^n \lambda_j f_j \right) \leq \sum_{j=1}^n \lambda_j y_r(f_j) \quad (15)$$

for $i = 1, \dots, m$ and $r = 1, \dots, s$ in the diversification-consistent model. We need measures satisfying inequalities (15).

Artzner et al. (1999) call a measure $\kappa : \mathcal{F} \rightarrow \mathbb{R}$ a *coherent measure of risk* if it satisfies the following four properties.

Translation invariance: for $f \in \mathcal{F}$ and $r \in \mathbb{R}^+$, $\kappa(f + r) = \kappa(f) - r$.

Subadditivity: for $f, g \in \mathcal{F}$, $\kappa(f + g) \leq \kappa(f) + \kappa(g)$.

Positive homogeneity: for $\lambda \geq 0$ and $f \in \mathcal{F}$, $\kappa(\lambda f) = \lambda \kappa(f)$.

Monotonicity: for $f, g \in \mathcal{F}$ with $f \leq g$, $\kappa(g) \leq \kappa(f)$.

Here r represents the total rate of return on a reference instrument. Translation invariance ensures that if we add $r\kappa(f)$ to f we get a random variable with risk measured as zero. Subadditivity ensures a sum of random variables has no greater risk than the sum of the risks of the individual random variables. Positive homogeneity ensures the risk is proportional to the size of investment. And monotonicity ensures that coherent risk measures are consistent with a general stochastic ordering. We define an additional property that is useful for return measures.

Superadditivity: for $f, g \in \mathcal{F}$, $\kappa(f + g) \geq \kappa(f) + \kappa(g)$.

The following theorem shows how some of these properties are useful. We define a measure to be *convexity consistent* if it is positively homogeneous and either: (i) a risk measure and subadditive; or (ii) a return measure and superadditive.

THEOREM 2 *Suppose that each measure in a diversification-consistent model is convexity consistent. Then its production possibility set Ψ satisfies NRS convexity.*

Proof. Suppose $(\mathbf{x}_1, \mathbf{y}_1), (\mathbf{x}_2, \mathbf{y}_2) \in \Psi$ and $t, u \in (0, 1)$ with $t + u \leq 1$. Then, for some $f_1, f_2 \in \mathcal{F}$ we have $\mathbf{x}(f_1) \leq \mathbf{x}_1, \mathbf{y}(f_1) \geq \mathbf{y}_1, \mathbf{x}(f_2) \leq \mathbf{x}_2$ and $\mathbf{y}(f_2) \geq \mathbf{y}_2$.

Put $\mathbf{x}' = (x'_1, \dots, x'_m) = \mathbf{x}(tf_1 + uf_2)$ and for $j = 1, 2$ write $\mathbf{x}_j = (x_{1j}, \dots, x_{mj})$. Then, for $i = 1, \dots, m$,

$$\begin{aligned} x'_i &= x_i(tf_1 + uf_2) \\ &\leq x_i(tf_1) + x_i(uf_2) \quad (\text{subadditivity}) \\ &= tx_i(f_1) + ux_i(f_2) \quad (\text{positive homogeneity}) \\ &\leq tx_{i1} + ux_{i2}. \end{aligned}$$

Hence $t\mathbf{x}_1 + u\mathbf{x}_2 \geq \mathbf{x}'$.

Similarly, put $\mathbf{y}' = (y'_1, \dots, y'_s) = \mathbf{y}(tf_1 + uf_2)$ and for $j = 1, 2$ write $\mathbf{y}_j = (y_{1j}, \dots, y_{sj})$. Then, for $r = 1, \dots, s$,

$$\begin{aligned} y'_r &= y_r(tf_1 + uf_2) \\ &\geq y_r(tf_1) + y_r(uf_2) \quad (\text{superadditivity}) \\ &= ty_r(f_1) + uy_r(f_2) \quad (\text{positive homogeneity}) \\ &\geq ty_{r1} + uy_{r2}. \end{aligned}$$

Hence $t\mathbf{y}_1 + u\mathbf{y}_2 \leq \mathbf{y}'$.

Since we defined \mathbf{x}' and \mathbf{y}' so that $(\mathbf{x}', \mathbf{y}') \in \Psi$, it follows from the free disposability of Ψ that $(t\mathbf{x}_1 + u\mathbf{x}_2, t\mathbf{y}_1 + u\mathbf{y}_2) \in \Psi$. It follows that Ψ satisfies NRS convexity. \blacksquare

The production possibility set of the complete NRS model is the smallest set containing $(\mathbf{x}_j, \mathbf{y}_j)$ ($j = 1, \dots, n$) and satisfying free disposability and NRS convexity. So it is contained in the production possibility set Ψ of the diversification-consistent model whenever Ψ is convex. This happens whenever all the measures are convexity consistent. The efficiencies of the complete NRS model estimate the efficiencies of the diversification-consistent model, but may do so poorly. To see how we might improve these estimates, choose $\mu_{jk} \geq 0$ ($j = 1, \dots, n$) with $\sum_{j=1}^n \mu_{jk} \leq 1$ for $k = 1, \dots, \hat{n}$ and write $\hat{f}_k = \sum_{j=1}^n \mu_{jk} f_j$. Put $\hat{x}_{ij} = x_i(\hat{f}_j)$, ($i = 1, \dots, m, j = 1, \dots, \hat{n}$) and $\hat{y}_{rj} = y_r(\hat{f}_j)$, ($r = 1, \dots, s, j = 1, \dots, \hat{n}$). If $\mathbf{x}_o \leq \mathbf{0}$ put $\phi_o = 1$ or 0 according as $\mathbf{y}_o \geq \mathbf{0}$ or $\mathbf{y}_o < \mathbf{0}$. Otherwise, for $o = 1, \dots, n$, choose $\phi_o, \lambda_{1o}, \dots, \lambda_{no}, \hat{\lambda}_{1o}, \dots, \hat{\lambda}_{\hat{n}o}$ to

minimize ϕ_o subject to

$$\sum_{j=1}^n y_{rj} \lambda_{jo} + \sum_{j=1}^{\hat{n}} \hat{y}_{rj} \hat{\lambda}_{jo} \geq y_{ro}, \quad (r = 1, \dots, s) \quad (16)$$

$$\sum_{j=1}^n x_{ij} \lambda_{jo} + \sum_{j=1}^{\hat{n}} \hat{x}_{ij} \hat{\lambda}_{jo} \leq x_{io} \phi_o, \quad (i = 1, \dots, m) \quad (17)$$

put $S = \{f_1, \dots, f_n\}$;
 repeat:
 if there is no pair of distinct funds f and g in S not considered before: stop;
 choose a pair of distinct funds f and g in S not considered before;
 if f or g is notional and inefficient: remove it from S
 else:
 if h is efficient, $\mathbf{x}(h) \neq 0.5(\mathbf{x}(f) + \mathbf{x}(g))$ and $\mathbf{y}(h) \neq 0.5(\mathbf{y}(f) + \mathbf{y}(g))$: add $h = 0.5(f + g)$
 to S ;

Figure 2: Iterative procedure to generate expanded models

$$\sum_{j=1}^n \lambda_{jo} + \sum_{j=1}^{\hat{n}} \hat{\lambda}_{jo} \leq 1, \quad (18)$$

and

$$\lambda_{jo} \geq 0 \quad (j = 1, \dots, n), \quad \hat{\lambda}_{jo} \geq 0 \quad (j = 1, \dots, \hat{n}). \quad (19)$$

We call this model the *expanded (complete NRS) model*. It is just the complete NRS model with some extra portfolios, which we call *notional funds*. If Ψ_c , Ψ_e , and Ψ are the production possibility sets of the complete, expanded and diversification-consistent models then we have $\Psi_c \subseteq \Psi_e \subseteq \Psi$ because each notional unit must be contained in Ψ but need not be contained in Ψ_c . So if we can find notional funds in $\Psi \setminus \Psi_c$ we can use them to construct an expanded model that better estimates the efficiencies of the diversification-consistent model.

Two observations help identify notional funds that might improve efficiency estimates. The first is a standard result, which is easy to show holds for the complete and expanded models. Put $\hat{\mathbf{x}}_j = (\hat{x}_1, \dots, \hat{x}_m)$ and $\hat{\mathbf{y}}_j = (\hat{y}_1, \dots, \hat{y}_s)$ ($j = 1, \dots, \hat{n}$). Suppose $\sum_{j=1}^n \lambda_{jo}(\mathbf{x}_j, \mathbf{y}_j) + \sum_{j=1}^{\hat{n}} \lambda_{jo}(\hat{\mathbf{x}}_j, \hat{\mathbf{y}}_j)$ is on the efficient frontier. Then, $\lambda_{jo} = 0$ unless $(\mathbf{x}_j, \mathbf{y}_j)$ is on the frontier ($j = 1, \dots, n$) and $\hat{\lambda}_{jo} = 0$ unless $(\hat{\mathbf{x}}_j, \hat{\mathbf{y}}_j)$ is on the frontier ($j = 1, \dots, \hat{n}$). The second observation is that we expect inequalities (15) to get further from equality as the correlations $\rho(f_j, f_k)$ between the funds get further from -1 . We use these observations in the iterative procedure of Figure 2. Clearly it improves the efficiency estimates each time it adds a notional fund. We want to choose f and g so that h is reasonably likely to be added to S . To achieve this we maintain a priority queue (see [Austern \(1999\)](#)) of untested pairs of funds. The observations above suggest $\phi_j^2 \phi_k^2 (1 - \rho(f_j, f_k))$ should be a reasonable priority measure and we find it works well in practice. When we add a notional fund h , we add all pairs of distinct funds containing h to the priority queue.

To see why we might reasonably expect the iterative procedure to give efficiencies that converge to the values in the diversification-consistent model, consider the following result.

THEOREM 3 *Suppose $\tilde{f}_1, \dots, \tilde{f}_{m+s}$ are funds in an expanded model with convexity-consistent measures. Suppose also $\lambda_1 \geq 0, \dots, \lambda_{m+s} \geq 0$ satisfy $\sum_{j=1}^{m+s} \lambda_j$. Then*

1. if $x_i \left(\sum_{j=1}^{m+s} \lambda_j \tilde{f}_j \right) < \sum_{j=1}^{m+s} \lambda_j x_i \left(\tilde{f}_j \right)$ for some $i \in \{1, \dots, m\}$, $x_i \left(\sum_{j=1}^{m+s} \mu_j \tilde{f}_j \right) < \sum_{j=1}^{m+s} \mu_j x_i \left(\tilde{f}_j \right)$ whenever $\mu_j > 0$ ($j = 1, \dots, n$) and $\sum_{j=1}^{m+s} \mu_j = 1$;
2. if $y_r \left(\sum_{j=1}^{m+s} \lambda_j \tilde{f}_j \right) > \sum_{j=1}^{m+s} \lambda_j y_r \left(\tilde{f}_j \right)$ for some $r \in \{1, \dots, s\}$, $y_r \left(\sum_{j=1}^{m+s} \mu_j \tilde{f}_j \right) > \sum_{j=1}^{m+s} \mu_j y_r \left(\tilde{f}_j \right)$ whenever $\mu_j > 0$ ($j = 1, \dots, n$) and $\sum_{j=1}^{m+s} \mu_j = 1$.

Proof. Put $\lambda = \min \{ \lambda_j / \mu_j : j = 1, \dots, n \}$ and $\lambda_j > 0$. Then $\lambda > 0$ and $\mu_j - \lambda \lambda_j \geq 0$ for $j = 1, \dots, n$. If $x_i \left(\sum_{j=1}^{m+s} \lambda_j \tilde{f}_j \right) < \sum_{j=1}^{m+s} \lambda_j x_i \left(\tilde{f}_j \right)$ for some $i \in \{1, \dots, m\}$, $\mu_j > 0$ ($j = 1, \dots, n$) and $\sum_{j=1}^{m+s} \mu_j = 1$ then

$$\begin{aligned}
x_i \left(\sum_{j=1}^{m+s} \mu_j \tilde{f}_j \right) &= x_i \left(\lambda \sum_{j=1}^{m+s} \lambda_j \tilde{f}_j + \sum_{j=1}^{m+s} (\mu_j - \lambda \lambda_j) \tilde{f}_j \right) \\
&\leq \lambda x_i \left(\sum_{j=1}^{m+s} \lambda_j \tilde{f}_j \right) + \sum_{j=1}^{m+s} (\mu_j - \lambda \lambda_j) x_i \left(\tilde{f}_j \right) \\
&< \lambda \sum_{j=1}^{m+s} \lambda_j x_i \left(\tilde{f}_j \right) + \sum_{j=1}^{m+s} (\mu_j - \lambda \lambda_j) x_i \left(\tilde{f}_j \right) \\
&= \sum_{j=1}^{m+s} \mu_j x_i \left(\tilde{f}_j \right).
\end{aligned}$$

The second part follows by a similar argument. ■

Since the production possibility set Ψ_e of an expanded model with convexity-consistent measures is an unbounded convex polytope, the practical consequence of Theorem 3 is that we need only check one point in the interior of each facet on the frontier of Ψ_e to find a new notional unit or show none exists. For $m + s > 2$ the iterative procedure of Figure 2 only checks points on the boundary of each facet. However, it is reasonable to expect that if the interior points of a facet give new notional units, then so should the boundary.

Section 5 shows how the iterative procedure works in practice. We now show there are practical convexity-consistent measures. It is easy to check that standard deviation is convexity consistent, and it is nonnegative by definition. Theorem 2 shows any nonnegative coherent measure of risk is convexity consistent. Krause (2002) and Acerbi and Tasche (2002) describe a range of coherent risk measures including lower semideviation and cvar: the negative of the expectation of the lower α tail of a distribution. We can use several values of α to get several risk measures and so model much of the shape of the distribution. It is easy to check the mean and $-cvar$ are convexity-consistent return measures. Typically we might use $cvar_\alpha$ as a risk measure for $\alpha \leq 0.4$ and $-cvar_\alpha$ as a return measure for $\alpha \geq 0.6$.

Coherent measures of risk may have negative values, but we need nonnegative risk measures. The following result shows we can construct a convexity-consistent risk measure from any coherent measure.

THEOREM 4 *Suppose \mathcal{F} is a class of random variables and $x : \mathcal{F} \rightarrow \mathbb{R}$ a coherent measure of risk. Then $\hat{x} = \max(x, 0)$ is positively homogeneous, subadditive and monotonic.*

Proof. Let f and $g \in \mathcal{F}$. Then

$$\hat{x}(\lambda f) = \max(x(\lambda f), 0) \leq \max(\lambda x(f), 0) = \lambda \hat{x}(f).$$

So \hat{x} is positively homogeneous. And

$$\begin{aligned} \hat{x}(f + g) &= \max(x(f + g), 0) \leq \max(x(f) + x(g), 0) \\ &\leq \max(x(f), 0) + \max(x(g), 0) = \hat{x}(f) + \hat{x}(g). \end{aligned}$$

So \hat{x} is subadditive. It follows that \hat{x} is convexity consistent. Finally,

$$\begin{aligned} f \leq g &\Rightarrow x(g) \leq x(f) \Rightarrow \max(x(g), 0) \leq \max(x(f), 0) \\ &\Rightarrow \hat{x}(g) \leq \hat{x}(f). \end{aligned}$$

So \hat{x} is monotonic. ■

It is easy to check that we lose translation invariance. The main benefit of translation invariance is that it lets us to include a reference instrument in the measure. This does not matter when we can include it in the model.

Theorem 4 shows we can choose risk measures that are monotonic. We do not require this for convexity-consistent measures. The following result shows how monotonicity might be useful.

THEOREM 5 *Suppose f_p and f_q are funds in a diversification-consistent model with nonnegative convexity-consistent monotonic measures. Then $f_p \geq f_q \implies \phi_p \geq \phi_q$.*

Proof. Suppose $f_p \geq f_q$. If $x_i(f_p) = 0$ for $i = 1, \dots, m$ then $\phi_p = 1 \geq \phi_q$ and so the result holds. Otherwise suppose $x_i(f_p) > 0$ for some i . Then, for $r = 1, \dots, s$,

$$y_r \left(\sum_{j=1}^n \lambda_{jp} f_j \right) \geq y_r(f_p) \geq y_r(f_q).$$

Similarly, for $i = 1, \dots, m$,

$$x_i \left(\sum_{j=1}^n \lambda_{jp} f_j \right) \geq x_i(f_p) \phi_p \geq x_i(f_q) \phi_p.$$

So $\lambda_{jq} = \lambda_{jp}$, ($j = 1, \dots, n$), $\phi_q = \phi_p$ satisfies constraints (I2)–(I3), (II) and (8) and so $\phi_p \geq \phi_q$ as required. ■

Theorem 5 shows we can construct DEA models consistent with a general stochastic ordering. We note that, for any stochastic ordering \preceq , we can replace monotonicity with the condition that $\kappa(g) \leq \kappa(f)$ for $f, g \in \mathcal{F}$ with $f \preceq g$ for each measure κ . So, for example, we can construct

diversification-consistent models that are consistent with second or third degree stochastic dominance (Levy, 1992), as Lozano and Gutiérrez (2008a,b) do for DEA models than are not diversification consistent. Some caution is needed. The definitions of consistency with stochastic ordering (Ogryczak and Ruszczyński, 2002; Lozano and Gutiérrez, 2008a) give implications of the form

$$f \preceq g \implies \text{some conditions on the measures} \implies \phi_f \leq \phi_g$$

while we typically want

$$\phi_f \leq \phi_g \implies f \preceq g.$$

We may be able to do no better than choose measures that make this reverse implication reasonably likely.

5 A practical illustration

We show here that the procedure of Figure 2 works well in practice. We show how we implement it. We also show it converges reasonably quickly to good estimates of the efficiencies of the diversification-consistent model. We test whether these efficiencies change enough to make the procedure useful.

We need some data. We use 60 monthly returns from 30 hedge funds between 2000 and 2004, taken from Center for International Securities and Derivatives Markets (2010). The monthly returns are computed from the net asset values of each fund. The data set contains ten of each of three classes of fund. These are market neutral (MN), global macro (GM) and long/short equity (LS). Global macro funds usually adopt a riskier strategy than long/short or market neutral ones. They are more exposed to losses while long/short and market neutral funds tend to focus on hedging exposed positions. So we expect diverse risk-return characteristics in different fund types. We also expect substantial asymmetry in the distributions of returns, justifying a model that incorporates more than mean and standard deviation. The columns on the right of Table 2 summarize the main features of the distributions of returns. A normal distribution would have zero skewness and kurtosis: we see substantial departure from normality. Table 4 shows the correlations between eight funds that perform well. These reflect the pattern in the 870 correlations between pairs of funds: at 5% significance level, 38 are significantly negatively correlated and only 137 significantly positively correlated.

We use C++ code to maintain the data structures and implement the iterative procedure, and use CPLEX (Ilog, 2008) to solve the linear programs. We compile the code with gcc and optimize it for the processor. We use a 2.66 GHz Intel Core 2 duo processor and a Gnu/Linux operating system. We use parallel CPLEX but not parallel C++ because most of the work is in solving the linear programs. The iterative procedure can run indefinitely. So we introduce a stopping criterion. We check the average reduction in efficiency after a small fixed number of improving iterations and stop when this average reduction falls below a prespecified tolerance. In practice we find that 10 improving iterations and a tolerance level of 10^{-6} works well and gives a solution in under ten seconds for the examples we consider.

We consider two examples. Both use a commensurable set of convexity-consistent measures including only nonnegative risk measures. Our first example illustrates both the possibility of

zero risk and the improvement available from diversification. It uses the mean return of the 30 funds as a return measure and $\max(\text{cvar}_{0.4}, 0)$ as a risk measure. Figure 3 shows the production possibility set and efficient frontier from our approximation to a diversification-consistent NRS model. The dashed line shows the frontier from the conventional NRS model. There is clear evidence that the conventional model substantially overestimates the efficiency of many funds.

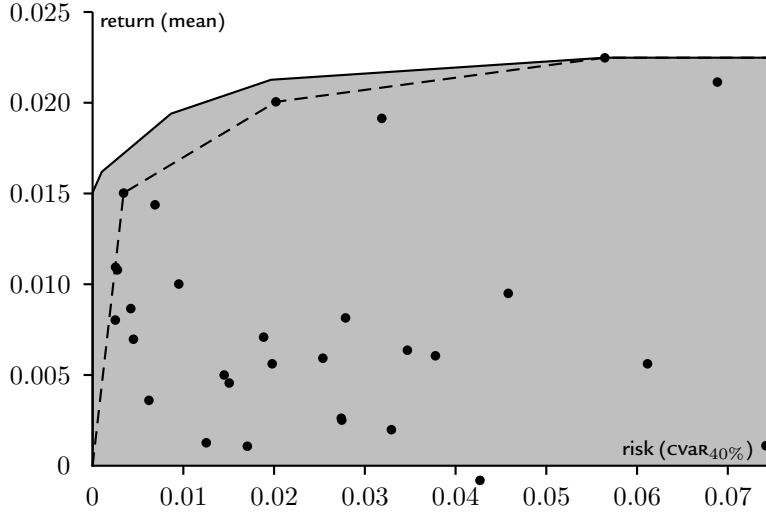


Figure 3: Approximating a diversification-consistent NRS model

Our second example uses risk measures $\max(\text{cvar}_{0.05}, 0)$, $\max(\text{cvar}_{0.1}, 0)$, $\max(\text{cvar}_{0.2}, 0)$ and $\max(s_2 - \bar{y}, 0)$, where s_2 is the lower semideviation and \bar{y} the mean, and return measures \bar{y} and $-\text{cvar}_{0.9}$. (Ogryczak and Ruszczyński (2002) show $\max(s_2 - \bar{y}, 0)$ is coherent and consistent with second degree stochastic dominance.) Six performance measures is typical for a DEA model of investment funds (Gregoriou et al., 2005a). They are enough to model most of the features of the data: much of the variance, skewness and kurtosis. They give $f_p \leq f_q \implies \phi_p \leq \phi_q$ for $p, q \in \{1, \dots, n\}$ in the diversification-consistent model. Column NRS shows the efficiencies from the NRS model described at the end of Section 2. Column DC shows the efficiencies found by the iterative procedure of Figure 2, which approximate closely those of the diversification-consistent model. Column RATIO shows the mean : standard deviation ratio for each fund. The diversification-consistent model has lower efficiencies than the NRS model for all but one fund and changes the rank order of the efficiencies. The rank order of the mean : standard deviation ratio shows greater difference, indicating the weakness of a measure that ignores the shape of funds' distributions.

A DEA efficiency is gotten by comparing a DMU o with a point on the efficient frontier. For the diversification-consistent model, the frontier point is the (efficient) portfolio $\sum_{j=1}^n \lambda_{jo} f_j$, where $\lambda_{1o}, \dots, \lambda_{no}, \phi_o$ minimizes ϕ_o subject to constraints (11)–(12) and (8). Table 3 shows the efficient portfolios. For example a portfolio comprising 46.4% GM2640, 23.2% LS90 and 23.2% GM2695 with the remainder not invested (no risk or return) produces at least as much return on each measure as LS40 does. It has at most 55.3% of the risk of LS40 in each measure because (see Table 2) LS40 has efficiency 0.553. Note that no portfolio outperforms the efficient GM2640. Note also that only six funds are allocated to the efficient portfolios and these are among the ten most efficient. Four of these are GM funds, which use a riskier strategy than LS or MN funds. The higher expected return and lower correlation with other funds makes them likely to be allocated

FUND	NRS	DC	RATIO	MEAN	SD	SKEW	KURT	AC
GM2640	1.000	1.000	0.239	0.022	0.094	0.487	1.982	0.364
LS90	1.000	0.912	0.420	0.020	0.048	0.458	1.641	0.211
GM2695	1.000	0.635	0.346	0.019	0.055	0.606	-0.329	0.013
GM45	1.000	0.626	0.794	0.015	0.019	0.308	-0.594	0.320
LS40	0.668	0.553	0.215	0.021	0.098	0.206	1.249	0.087
MN2681	0.535	0.247	0.648	0.014	0.022	-0.214	-0.241	-0.121
GM366	0.860	0.234	0.695	0.011	0.016	0.725	2.219	0.157
GM2693	0.881	0.231	0.700	0.011	0.016	0.703	2.170	0.166
MN629	0.720	0.186	0.673	0.008	0.012	0.053	0.899	0.310
MN2540	0.618	0.182	0.589	0.009	0.015	0.455	0.872	0.393
MN2547	0.470	0.178	0.545	0.007	0.013	0.690	1.870	0.043
MN2704	0.262	0.102	0.315	0.004	0.011	1.094	2.155	0.330
GM2889	0.315	0.088	0.419	0.010	0.024	0.000	1.624	0.066
LS78	0.171	0.065	0.250	0.007	0.028	0.192	0.379	0.098
LS57	0.141	0.058	0.219	0.008	0.037	0.001	-0.646	-0.221
MN299	0.131	0.045	0.180	0.005	0.028	2.786	16.492	-0.306
LS6	0.101	0.042	0.151	0.009	0.063	0.941	1.906	-0.099
MN301	0.117	0.041	0.164	0.005	0.028	2.760	16.290	-0.312
LS64	0.102	0.037	0.144	0.006	0.041	2.355	11.882	-0.181
GM6876	0.010	0.033	0.186	0.006	0.030	-0.081	2.067	0.195
GM59	0.079	0.026	0.139	0.006	0.046	0.095	1.901	0.020
GM3434	0.069	0.024	0.108	0.006	0.056	1.469	7.007	-0.099
LS32	0.046	0.019	0.070	0.003	0.037	1.752	7.259	-0.140
LS20	0.044	0.018	0.067	0.003	0.037	1.759	7.297	-0.142
MNI47	0.043	0.017	0.085	0.001	0.015	0.139	1.268	-0.072
LS79	0.037	0.012	0.068	0.006	0.082	-1.146	6.987	-0.088
MN2639	0.027	0.010	0.053	0.001	0.020	0.063	1.378	-0.168
GM2068	0.023	0.008	0.051	0.002	0.038	-1.417	3.012	0.124
LS39	0.007	0.003	0.013	0.001	0.083	0.493	0.550	0.020
MN2777	0.000	0.000	-0.018	-0.001	0.044	-0.381	0.297	0.010

NRS: efficiency from NRS model; DC: efficiency from expanded model approximating diversification-consistent model; RATIO: mean-standard-deviation ratio; MEAN: mean return; SD: standard deviation in return; SKEW: skewness in return; KURT: kurtosis in return; AC: serial autocorrelation in return series. Boldface indicates differences from zero significant at 5% level.

Table 2: Efficiencies and summary statistics for 30 funds

to a portfolio. Table 4 shows the correlations between the allocated funds. These are far from the perfect correlation required for the NRS model to estimate efficiency accurately.

6 Discussion

We have identified the returns to scale and measures needed for a DEA model of investment funds and shown how to handle scope for diversification. However, a number of issues remain. Although the procedure of Figure 2 works well in practice, there may be a much more efficient method to estimate a set of frontier portfolios. This would help for large data sets or (see below)

<i>Fund portfolio outperforms</i>	FUNDS ALLOCATED TO PORTFOLIO						
	GM2640	LS90	GM2695	GM45	GM2693	MN2540	cash
<i>GM2640</i>	1.000						
<i>LS90</i>	0.333	0.278	0.278	0.111			
<i>GM2695</i>	0.073	0.391	0.391	0.146			
<i>GM45</i>	0.081	0.052	0.052	0.606	0.104	0.104	
<i>LS40</i>	0.464	0.232	0.232				0.071
<i>MN2681</i>	0.062	0.048	0.048	0.545	0.148	0.148	
<i>GM366</i>	0.040	0.020	0.020	0.235	0.288	0.279	0.117
<i>GM2693</i>	0.041	0.020	0.020	0.239	0.292	0.283	0.105
<i>MN629</i>	0.030	0.015	0.015	0.175	0.214	0.208	0.344
<i>MN2540</i>	0.032	0.016	0.016	0.189	0.231	0.224	0.292
<i>MN2547</i>	0.026	0.013	0.013	0.152	0.186	0.180	0.430
<i>MN2704</i>	0.013	0.007	0.007	0.079	0.096	0.093	0.705
<i>GM2889</i>	0.037	0.019	0.019	0.218	0.267	0.259	0.181
<i>LS78</i>	0.026	0.013	0.013	0.155	0.189	0.184	0.420
<i>LS57</i>	0.030	0.015	0.015	0.178	0.217	0.211	0.334
<i>MN299</i>	0.019	0.009	0.009	0.109	0.133	0.129	0.591
<i>LS6</i>	0.035	0.018	0.018	0.207	0.253	0.246	0.223
<i>MN301</i>	0.017	0.008	0.008	0.099	0.122	0.118	0.627
<i>LS64</i>	0.022	0.011	0.011	0.129	0.158	0.153	0.515
<i>GM6876</i>	0.021	0.010	0.010	0.122	0.150	0.145	0.541
<i>GM59</i>	0.024	0.012	0.012	0.139	0.170	0.165	0.479
<i>GM3434</i>	0.023	0.011	0.011	0.132	0.162	0.157	0.504
<i>LS32</i>	0.010	0.005	0.005	0.057	0.070	0.068	0.786
<i>LS20</i>	0.009	0.005	0.005	0.055	0.067	0.065	0.794
<i>MN147</i>	0.005	0.002	0.002	0.027	0.034	0.033	0.897
<i>LS79</i>	0.021	0.010	0.010	0.123	0.150	0.145	0.540
<i>MN2639</i>	0.004	0.002	0.002	0.023	0.029	0.028	0.912
<i>GM2068</i>	0.007	0.004	0.004	0.043	0.053	0.051	0.838
<i>LS39</i>	0.004	0.002	0.002	0.024	0.029	0.029	0.910
<i>MN2777</i>							1.000

Table 3: Proportions allocated to an efficient portfolio outperforming each fund

FUND	LS90	GM2695	GM45	GM2693	MN2540
GM2640	<i>-0.144</i>	0.303	<i>-0.306</i>	<i>-0.084</i>	<i>-0.136</i>
LS90		<i>0.063</i>	0.295	<i>0.121</i>	<i>0.390</i>
GM2695			0.387	<i>0.252</i>	<i>-0.074</i>
GM45				0.357	<i>0.160</i>
GM2693					<i>0.030</i>

Italics indicates correlation not significantly different from zero at 5% level; bold italics indicates correlation negative and significantly different from zero at 5% level.

Table 4: Correlation coefficients for allocated funds

if we want to repeat the procedure many thousands of times.

We show that convexity-consistent nonnegative measures are sufficient to approximate the diversification-consistent model but not that they are necessary. It would be nice if we could use measures such as var that are more widely used (Liang and Park, 2007) but not convexity-consistent. We suspect convexity-consistency is necessary but it may be, for example, that, for practical purposes, var satisfies or nearly satisfies its assumptions. Then we could approximate a diversification-consistent model using var as a risk measure.

Our measures and models could be developed further. We discuss mainly risk measures because they typically concern investors more than return measures do. It might be of value to investigate return measures further and investigate the effect different sets of plausible measures has on the efficiencies. One issue here is that we cannot use the upper tail mean as a return measure in the same way we use cvar (the lower tail mean) as a risk measure. Superadditivity fails to hold because diversification can reduce return and not just risk.

It might also be of value to investigate sets of measures likely to give $\phi_p > \phi_q$ when, for example, second degree stochastic dominance (Levy, 1992; Lozano and Gutiérrez, 2008a) prefers f_p to f_q but first degree stochastic dominance does not.

We consider input-oriented models because these are most similar to ratios such as the Sharpe ratio. In the simpler model of Figure 3 most funds have efficiency zero and arguably it might be better to use an output-oriented model (Gregoriou and Zhu, 2005), which might distinguish better among these funds. We expect that our measures and methods can be adapted for this and possibly even for a slacks-based model (Tone, 2001).

Like all applications of DEA to modeling funds we know of, ours finds efficiencies that describe past rather than predict future performance. We treat measures as fixed quantities rather than random variables. This issue needs further investigation. We suggest an approach based on the bootstrap, which Simar and Wilson (2000); Dyson and Shale (2010) discuss for general DEA models. The multiple observations for each fund allow another bootstrap approach and we have some promising preliminary results. We need to repeat the iterative procedure many times but find mean efficiencies and confidence intervals. We also find the bootstrap resolves a problem with deterministic DEA: the fund with highest mean return (GM2640) is given efficiency 1 no matter how great its risk measures are.

Our data come from a time series. We have ignored autocorrelation, which may affect any estimate of future efficiency. We have also ignored the time horizon of investors (Galagadera and Silvapulle, 2002). Different funds may be at their most efficient at different time horizons, and both autocorrelation and time horizons could be investigated further.

References

- Acerbi, C. (2007). Coherent measures of risk in everyday market practice. *Quantitative Finance*, 7(4), 359–364.
- Acerbi, C. and Tasche, D. (2002). On the coherence of expected shortfall. *Journal of Banking and Finance*, 26, 1487–1503.
- Alexander, C. (2001). *Market Models*. John Wiley & Sons, Chichester. ISBN 978-0471-89975-4.

- Artzner, P., Delbaen, F., Eber, J.M., and Heath, D. (1999). Coherent measures of risk. *Mathematical Finance*, 9, 203–228.
- Austern, M.H. (1999). *Generic Programming and the STL*. Addison–Wesley.
- Banker, R.D., Charnes, A., and Cooper, W.W. (1984). Some models for estimating technical and scale efficiencies in data envelopment analysis. *Management Science*, 30, 1078–1092.
- Basso, A. and Funari, S. (2001). A data envelopment analysis approach to measure the mutual fund performance. *European Journal of Operational Research*, 135, 477–492.
- Bogetoft, P. (1996). DEA on relaxed convexity assumptions. *Management Science*, 42(3), 457–465.
- Briec, W. and Kerstens, K. (2009). Multi-horizon Markowitz portfolio performance appraisals: A general approach. *Omega*, 37, 50–62.
- Cazals, C., Florens, J.P., and Simar, L. (2002). Nonparametric frontier estimation: a robust approach. *Journal of Econometrics*, 106, 1–25.
- Center for International Securities and Derivatives Markets (2010). URL <http://www.isenberg.umass.edu/CISDM/>. Accessed March 2010.
- Charnes, A., Cooper, W.W., and Rhodes, E. (1978). Measuring the efficiency of decision making units. *European Journal of Operational Research*, 2, 429–444.
- Chen, Z. and Lin, R. (2006). Mutual fund performance evaluation using data envelopment analysis with new risk measures. *OR Spectrum*, 28, 375–398.
- Dyson, R.G. and Shale, E. (2010). Data envelopment analysis, operational research and uncertainty. *Journal of the Operational Research Society*, 61(1), 25–34.
- Eling, M. (2006). Performance management of hedge funds using data envelopment analysis. *Financial Markets and Portfolio Management*, 20(4), 442–472.
- Galagadera, U.A. and Silvapulle, P. (2002). Australian mutual fund performance appraisal using data envelopment analysis. *Managerial Finance*, 28, 60–73.
- Gregoriou, G.N., Rouah, F., Satchell, S., and Diz, F. (2005). Simple and cross-efficiency of CTAs using data envelopment analysis. *The European Journal of Finance*, 11(5), 393–409.
- Gregoriou, G.N., Sedzro, K., and Zhu, J. (2005). Hedge fund performance appraisal using data envelopment analysis. *European Journal of Operational Research*, 164(4), 555–571.
- Gregoriou, G.N. and Zhu, J. (2005). *Evaluating Hedge Fund and CTA Performance: Data Envelopment Analysis Approach*. John Wiley & Sons, Inc., Hoboken, New Jersey. ISBN 978-0-471-68185-4.
- Ilog (2008). CPLEX. <http://www.ilog.com/products/cplex/>. Accessed July 2009.
- Jorion, P. (2007). *Value at Risk*. McGraw–Hill, 3rd edition. ISBN 978-0-07-146495-6.
- Joro, T. and Na, P. (2006). Portfolio performance evaluation in a mean–variance–skewness framework. *European Journal of Operational Research*, 175, 446–461.

- Jurczenko, E., Maillet, B., and Merlin, P. (2005). Hedge funds portfolio selection with higher-order moments: a non-parametric mean-variance-skewness-*kurtosis* efficient frontier. Unpublished.
- Krause, A. (2002). Coherent risk measurement: an introduction. *Balance Sheet*, 10, 13–17.
- Levy, H. (1992). Stochastic dominance and expected utility. *Management Science*, 38, 555–593.
- Lhabitant, F.S. (2004). *Hedge Funds: Quantitative Insights*. John Wiley & Sons, Chichester.
- Liang, B. and Park, H. (2007). Risk measures for hedge fund: a cross-sectional approach. *European Financial Management*, 13(2), 333–370.
- Lovell, C.A.K. and Pastor, J.T. (1995). Units invariant and translation invariant DEA models. *Operations Research Letters*, 18, 147–151.
- Lozano, S. and Gutiérrez, E. (2008). Data envelopment analysis of mutual funds based on second-order stochastic dominance. *European Journal of Operational Research*, 189, 230–244.
- Lozano, S. and Gutiérrez, E. (2008). TSD-consistent performance assessment of mutual funds. *Journal of the Operational Research Society*, 59, 1352–1362.
- Markovitz, H. (1952). Portfolio selection. *Journal of Finance*, 8, 77–91.
- Morey, M.R. and Morey, R.C. (1999). Mutual fund performance appraisals: a multi-horizon perspective with endogenous benchmarking. *Omega*, 27, 241–288.
- Murthi, B.P.S., Choi, Y.K., and Desai, P. (1997). Efficiency of mutual funds and portfolio performance measurement: A non-parametric approach. *European Journal of Operational Research*, 98, 408–418.
- Nguyen-Thi-Thanh, H. (2006). On the use of data envelopment analysis in hedge fund selection. Working Paper, Université d'Orléans.
- Ogryczak, W. and Ruszczyński, A. (2002). Dual stochastic dominance and related mean-risk models. *SIAM Journal on Optimization*, 13(1), 60–78.
- Russell, R.R. (1990). Continuity of measures of technical efficiency. *Journal of Economic Theory*, 51, 255–267.
- Scheel, H. and Scholtes, S. (2003). Continuity of DEA efficiency measures. *Operations Research*, 51(1), 149–159.
- Sengupta, J.K. (2003). Efficiency tests for mutual fund portfolios. *Applied Financial Economics*, 13, 869–876.
- Sharp, J.A., Meng, W., and Liu, W. (2007). A modified slacks-based measure model for data envelopment analysis with 'natural' negative outputs and inputs. *Journal of the Operational Research Society*, 58(12), 1672–1677. ISSN 0160-5682/07.
- Sharpe, W. (1966). Mutual fund performance. *Journal of Business*, 39, 119–138.
- Shephard, R.W. (1970). *Theory of Cost and Production Function*. Princeton University Press, Princeton NJ.

- Silva Portela, M.C.A., Thanassoulis, E., and Simpson, G. (2004). Negative data in DEA: a directional distance approach applied to bank branches. *Journal of the Operational Research Society*, 55, 1111–1121. ISSN 0160-5682/04.
- Simar, L. and Wilson, P.W. (2000). Statistical inference in nonparametric frontier models: the state of the art. *Journal of Productivity Analysis*, 13, 49–78.
- Tone, K. (2001). A slacks-based measure of efficiency in data envelopment analysis. *European Journal of Operational Research*, 130, 498–509.
- Wilkins, K. and Zhu, J. (2001). Portfolio evaluation and benchmark selection: a mathematical programming approach. *Journal of Alternative Investments*, 4(1), 9–20.