CENTRALIZERS OF NORMAL SUBGROUPS AND THE Z^*-THEOREM

E. HENKE AND J. SEMERARO

Abstract. Glauberman's Z^*-theorem and analogous statements for odd primes show that, for any prime p and any finite group G with Sylow p-subgroup S, the centre of $G/O_{p'}(G)$ is determined by the fusion system $\mathcal{F}_S(G)$. Building on these results we show a statement that seems a priori more general: For any normal subgroup H of G with $O_{p'}(H) = 1$, the centralizer $C_S(H)$ is expressed in terms of the fusion system $\mathcal{F}_S(G)$ and its normal subsystem induced by H.

Keywords: Finite groups; fusion systems; Glauberman's Z^*-theorem.

Throughout p is a prime. Glauberman's Z^*-theorem [3] and its generalization to odd primes, which is shown using the classification of finite simple groups (see [7] and [4]), can be reformulated as follows:

Theorem A. *Let G be a finite group with $O_{p'}(G) = 1$, and $S \in \mathrm{Syl}_p(G)$. Then $Z(G) = Z(\mathcal{F}_S(G))$.*

We refer the reader here to [2] for basic definitions and results regarding fusion systems; see in particular Definitions I.4.1 and I.4.3 for the definition of central subgroups and the centre $Z(\mathcal{F})$. A more common formulation of the Z^*-theorem states that, assuming the hypothesis of Theorem A, we have $t \in Z(G)$ if and only if $t^G \cap S = \{t\}$ for every element $t \in S$ of order p. Given a normal subgroup H of a finite group G, a Sylow p-subgroup $S \in \mathrm{Syl}_p(G)$, and an element $t \in S$ of order p, one can apply the Z^*-theorem with $H\langle t\rangle$ in place of G to obtain the following corollary: Provided $O_{p'}(H) = 1$, we have $t^H \cap S = \{t\}$ if and only $t \in C_S(H)$. In this short note, we use Theorem A to give a less obvious characterization of $C_S(H)$.

Given a saturated fusion system $\mathcal{F}$ on a finite p-group S and a normal subsystem $\mathcal{E}$ of $\mathcal{F}$ on $T \leqslant S$, Aschbacher [1, (6.7)(1)] showed that the set of subgroups X of $C_S(T)$ with $\mathcal{E} \subseteq C_{\mathcal{F}}(X)$ has a largest member $C_S(\mathcal{E})$. He furthermore constructed a normal subsystem $C_{\mathcal{F}}(\mathcal{E})$ on $C_S(\mathcal{E})$, the centralizer of $\mathcal{E}$ in $\mathcal{F}$; see [1, Chapter 6]. Note that $C_S(\mathcal{E})$ depends not only on S and $\mathcal{E}$ but also on the fusion system $\mathcal{F}$ in which both S and $\mathcal{E}$ are contained.

The definition of $C_S(\mathcal{E})$ generalizes the definition of $Z(\mathcal{F})$ since $C_S(\mathcal{F}) = Z(\mathcal{F})$. Moreover, for every normal subgroup H of a finite group G with Sylow p-subgroup S, $\mathcal{F}_{S\cap H}(H)$ is a normal subsystem of $\mathcal{F}_S(G)$ by [2, I.6.2]. Thus, the following theorem, which we prove later on, can be seen as a generalization of Theorem A.

Theorem B. *Let G be a finite group and let S be a Sylow p-subgroup of G. Let $H \trianglelefteq G$ with $O_{p'}(H) = 1$. Then $C_S(\mathcal{F}_{S\cap H}(H)) = C_S(H)$.*

In the statement of Theorem B it is understood that $C_S(\mathcal{F}_{S\cap H}(H))$ is formed inside of $\mathcal{F}_S(G)$. The result says in other words that, under the hypothesis of Theorem B, for any $X \leqslant S$ with $\mathcal{F}_{S\cap H}(H) \subseteq C_{\mathcal{F}_S(G)}(X)$, we have $X \leqslant C_S(H)$.

This is not true if one drops the assumption that H is normal in G as the following example shows: Let $G := G_1 \times G_2$ with $G_1 \cong G_2 \cong S_3$. Set $p = 3$, $S = O_3(G)$, $S_i := O_3(G_i)$ and let R be a subgroup of G of order 2 which acts fixed point freely on S. Set $H := S_1 \rtimes R$. Then $S_1 = S \cap H \in \mathrm{Syl}_3(H)$ and $\mathcal{F}_{S_1}(H) = \mathcal{F}_{S_1}(G_1) \subseteq C_{\mathcal{F}_S(G)}(S_2)$ as $S_2 = C_S(G_1)$. However, $S_2 \not\leqslant C_S(H)$ by the choice of R.

Theorem B was conjectured by the second author of this paper in [6]. Our proof of Theorem B builds on Theorem A and the reduction uses only elementary group theoretical results. Essential is the following lemma, whose proof is self-contained apart from using the conjugacy of Hall-subgroups in solvable groups.

Lemma 1. *Let G be a finite group with Sylow p-subgroup S and a normal subgroup H. Let $P \leqslant S$ such that $P \cap H$ is centric in $\mathcal{F}_{S\cap H}(H)$. Then for every p'-element $\varphi \in \mathrm{Aut}_G(P)$ with $[P, \varphi] \leqslant P \cap H$ and $\varphi|_{P\cap H} \in \mathrm{Aut}_H(P \cap H)$, we have $\varphi \in \mathrm{Aut}_H(P)$.*

Proof. This is [5, Proposition 3.1]. □

Proof of Theorem B. We assume the hypothesis of Theorem B. Furthermore, we set $\mathcal{F} := \mathcal{F}_S(G)$, $T := S \cap H$ and $\mathcal{E} := \mathcal{F}_T(H)$. If a homomorphism φ between subgroups A and B of T is induced by conjugation with an element $h \in H$, then φ extends to $c_h : AC_S(H) \to BC_S(H)$ and c_h restricts to the identity on $C_S(H)$. Thus $\mathcal{E} \subseteq C_{\mathcal{F}}(C_S(H))$, so by the definition of $C_S(\mathcal{E})$, we have $C_S(H) \leqslant C_S(\mathcal{E})$. To prove the converse inclusion, choose $t \in C_S(\mathcal{E})$. Define:

$$G_0 := H\langle t\rangle \qquad \text{and} \qquad S_0 := T\langle t\rangle,$$

so that plainly S_0 is a Sylow p-subgroup of G_0 and $\mathcal{F}_0 := \mathcal{F}_{S_0}(G_0)$ is a saturated fusion system on S_0. Note also that $O_{p'}(G_0) = 1$ as $O^p(G_0) = O^p(H)$ and $O_{p'}(H) = 1$ by assumption.

By Theorem A, $Z(\mathcal{F}_0) = Z(G_0) \leqslant C_S(H)$. It thus suffices to prove $t \in Z(\mathcal{F}_0)$. As $t \in C_S(\mathcal{E}) \leqslant C_S(T)$, $t \in Z(S_0)$. Let P be a subgroup of S_0 which is centric radical and fully normalized in $\mathcal{F}_0$. Then $t \in Z(S_0) \leqslant C_{S_0}(P) \leqslant P$. It is sufficient to prove $[t, \mathrm{Aut}_{\mathcal{F}_0}(P)] = 1$. For as P is arbitrary, Alperin's fusion theorem [2, Theorem 3.6] implies then $t \in Z(\mathcal{F}_0)$. As P is fully $\mathcal{F}_0$-normalized, $\mathrm{Aut}_{S_0}(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}_0}(P))$ and thus $\mathrm{Aut}_{\mathcal{F}_0}(P) = \mathrm{Aut}_{S_0}(P)O^p(\mathrm{Aut}_{\mathcal{F}_0}(P))$. Note that $[t, \mathrm{Aut}_{S_0}(P)] = 1$ as $t \in Z(S_0)$. Hence, it is enough to prove

$$[t, O^p(\mathrm{Aut}_{\mathcal{F}_0}(P))] = 1.$$

Let $\varphi \in \mathrm{Aut}_{\mathcal{F}_0}(P)$ be a p'-element. Since $O^p(H) = O^p(G_0)$, we have $O^p(\mathrm{Aut}_{\mathcal{F}_0}(P)) = O^p(\mathrm{Aut}_H(P))$. In particular, $\varphi \in \mathrm{Aut}_H(P)$ and thus $\varphi|_{P\cap T} \in \mathrm{Aut}_H(P \cap T) = \mathrm{Aut}_{\mathcal{E}}(P\cap T)$. As $t \in P \leqslant S_0 = T\langle t\rangle$, we have $P = (P\cap T)\langle t\rangle$. Moreover, $t \in C_S(\mathcal{E})$ implies that $\mathcal{E} \subseteq C_{\mathcal{F}}(\langle t\rangle)$. Hence, $\varphi|_{P\cap T}$ extends to $\psi \in \mathrm{Aut}_{\mathcal{F}}(P)$ with the property that $t\psi = t$. Note that $o(\psi) = o(\varphi|_{P\cap T})$ and thus ψ is a p'-element as φ has order prime to p. Moreover, plainly $[P, \psi] \leqslant P \cap T$ and $\psi|_{P\cap T} = \varphi|_{P\cap T} \in \mathrm{Aut}_H(P \cap T)$. Since $\mathcal{E} \trianglelefteq \mathcal{F}_0$, $P \cap T$ is $\mathcal{E}$-centric by [1, 7.18]. Now it follows from Lemma 1 that $\psi \in \mathrm{Aut}_H(P)$. Thus, $\chi := \varphi \circ \psi^{-1} \in \mathrm{Aut}_H(P) \leqslant \mathrm{Aut}_{\mathcal{F}_0}(P)$. Clearly $\chi|_{P\cap T} = \mathrm{Id}$ as ψ extends $\varphi|_{P\cap T}$. Moreover, using that H is normal in G, we obtain $[P, \chi] \leqslant [P, \mathrm{Aut}_H(P)] = [P, N_H(P)] \leqslant P \cap H = P \cap T$. Hence, by [2, Lemma A.2], $\chi \in C_{\mathrm{Aut}_{\mathcal{F}_0}(P)}(P/(P \cap T)) \cap C_{\mathrm{Aut}_{\mathcal{F}_0}(P)}(P \cap T) = O_p(\mathrm{Aut}_{\mathcal{F}_0}(P)) = \mathrm{Inn}(P)$ as P is radical in $\mathcal{F}_0$. As $\mathrm{Inn}(P) \leqslant \mathrm{Aut}_{S_0}(P)$ and $[t, \mathrm{Aut}_{S_0}(P)] = 1$, it follows $t\chi = t$. By the choice of ψ, also $t\psi = t$ and consequently $t\varphi = t$. Since φ was chosen to be an arbitrary p'-element in $\mathrm{Aut}_{\mathcal{F}_0}(P)$ and $O^p(\mathrm{Aut}_{\mathcal{F}_0}(P))$ is the subgroup generated by these elements, it follows that $[t, O^p(\mathrm{Aut}_{\mathcal{F}_0}(P))] = 1$. As argued above, this yields the assertion. $\square$

References

1. M. Aschbacher, *The generalized Fitting subsystem of a fusion system*, Mem. Amer. Math. Soc. **209** (2011), no. 986, vi+110.
2. M. Aschbacher, R. Kessar, and B. Oliver, *Fusion systems in algebra and topology*, London Math. Soc. Lecture Note Series, vol. 391, Cambridge University Press, 2011.
3. G. Glauberman, *Central elements in core-free groups*, J. Algebra **4** (1966), 403–420.
4. R. M. Guralnick and G. R. Robinson, *On extensions of the Baer-Suzuki theorem*, Israel J. Math. **82** (1993), no. 1-3, 281–297.
5. E. Henke, *Products in fusion systems*, J. Algebra **376** (2013), 300–319.
6. J. Semeraro, *Centralizers of subsystems of fusion systems*, J. Group Theory **18** (2015), no. 5, 393–405.
7. W. J. Xiao, *Glauberman's conjecture, Mazurov's problem and Peng's problem*, Sci. China Ser. A **34**, no. 9.

Institute of Mathematics, University of Aberdeen, U.K.
E-mail address: ellen.henke@abdn.ac.uk

Heilbronn Institute for Mathematical Research, Department of Mathematics, University of Bristol, U.K.
E-mail address: js13525@bristol.ac.uk