Arch. Math. 106 (2016), 323–325

0003-889X/16/040323-3
published online February 23, 2016
DOI 10.1007/s00013-016-0880-8

Archiv der Mathematik

A counterexample for lightning flash modules over $E(e_1, e_2)$

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Abstract. We give a counterexample to Theorem 5 in Section 18.2 of Margolis' book, "Spectra and the Steenrod Algebra" and make remarks about the proofs of some later theorems in the book that depend on it. The counterexample is a module which does not split as a sum of lightning flash modules and free modules.

Mathematics Subject Classification. 55S10, 16W50.

Keywords. Steenrod algebra, Lightning flash, Graded exterior algebra.

1. Introduction. Let k be a field and $E(e_1, e_2)$ be a graded exterior algebra on generators e_1 and e_2 with degrees satisfying $0 < |e_1| < |e_2|$. Theorem 5 in Section 18.2 of Margolis [2] states that every graded $E(e_1, e_2)$-module is a coproduct of free modules and lightning flashes. In this note, we give a simple counterexample to this statement.

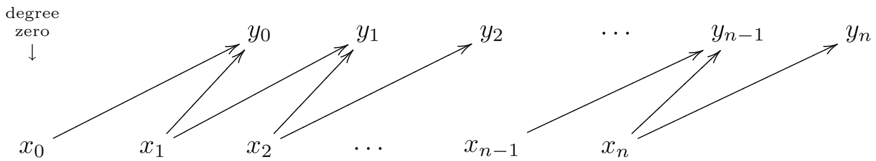
Statement (c) following Proposition 7 of the same section is true, but not because of Theorem 5. The proof of Theorem 8 in Section 18.3 depends on this statement. The proofs of Proposition 9 and Lemma 10 of the same section also depend on Theorem 5, and are used in Chapter 20. Fortunately, the paper of Adams and Margolis [1] provides correct proofs of these statements that do not rely on Theorem 5.

2. The counterexample. In this section, we display a bounded below module M for $E(e_1, e_2)$ which is not isomorphic to a coproduct of free modules and lightning flashes.

First, we note that every module for $E(e_1, e_2)$ can be written as a direct sum of a free module and a module on which e_1e_2 acts as zero. So we may as well work with modules for $E(e_1, e_2)/(e_1e_2)$.

This work was partially supported by the Simons Foundation and by the Mathematisches Forschungsinstitut Oberwolfach.

We use the notation of Section 18.2 of Margolis. Let $M(n)$ be the lightning flash module $L(n,0,1)$ of dimension $2n+2$. Here is a picture of $M(n)$:



The shorter arrows represent the action of e_1, and the longer ones e_2. Thus, a presentation of the module is given by $e_1x_{i+1} = e_2x_i = y_i$ $(0 \le i \le n-1)$, $e_1x_0 = 0$, $e_2x_n = y_n$. We arrange that the element x_0 in $M(n)$ is in degree zero, so that x_i has degree $i(|e_2| - |e_1|)$ and y_i has degree $|x_i| + |e_2|$. Similarly, $L(\infty, 0)$ is the infinite lightning flash obtained by letting this diagram continue to the right indefinitely.

Our counterexample is the module

$$M = \prod_{n=0}^{\infty} M(n).$$

To see that it is a counterexample, first note that $e_1M(n)$ is the linear span of $y_0, \dots, y_{n-1}$, so $e_2^{-1}e_1M(n)$ is the linear span of all the basis elements except x_n. Here, if U is a linear subspace of a module, we write $e_2^{-1}U$ for the linear subspace consisting of the vectors whose image under e_2 is in U.

Inductively, we see that for $j > 0$, $(e_2^{-1}e_1)^jM(n)$ is the linear span of the basis elements $y_0, \dots, y_n, x_0, \dots, x_{n-j}$. Thus, x_0 is in $(e_2^{-1}e_1)^jM(n)$ if and only if $j \le n$.

Taking degree zero parts, we have

$$((e_2^{-1}e_1)^jM)_0 = \prod_{n=j}^{\infty} M(n)_0.$$

Thus,

$$\bigcap_{j\ge 0}((e_2^{-1}e_1)^jM)_0 = 0, \tag{2.1}$$

and

$$((e_2^{-1}e_1)^jM)_0/((e_2^{-1}e_1)^{j+1}M)_0$$

is one dimensional. On the other hand, x_0 is in $(e_2^{-1}e_1)^jL(\infty,0)$ for all $j > 0$, so

$$\bigcap_{j\ge 0}((e_2^{-1}e_1)^jL(\infty,0))_0 \neq 0.$$

Since a finite sum is always a direct summand of the product, it follows that M has exactly one copy of each $M(n)$ as a summand, and no summand isomorphic to $L(\infty,0)$. Since $e_1M_0 = 0$, no summand of the form $L(\infty,1)$ (the module with generators $x_0, x_1, \dots$ and relations $e_2x_i = e_1x_{i+1}$), $L(n,1,1)$ (generators $x_0, \dots, x_n$ and relations $e_2x_i = e_1x_{i+1}$) ,or $L(n,1,0)$ (the same as $L(n,1,1)$ with one more relation $e_2x_n = 0$) can contribute to M_0; and finally (2.1) shows that no summand of the form $L(n,0,0)$ (the same as $L(n,1,0)$ with one more relation $e_1x_0 = 0$) can contribute to M_0, since that intersection is non-zero for such a module. These are the lightning flash modules which are zero in sufficiently negative degrees. The summands we have identified do not exhaust M_0, and hence M cannot be a direct sum of lightning flash modules.

On the other hand, modules of finite type for $E(e_1, e_2)$ can be shown to be direct sums of lightning flashes, by the method of filtrations of the forgetful functor to graded vector spaces. The proof is similar to but easier than the functorial filtration proof given in Ringel [3].

References

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Received: 23 December 2015