

# New invariants of $G_2$ –structures

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We define a  $\mathbb{Z}_{48}$ –valued homotopy invariant  $v(\varphi)$  of a  $G_2$ –structure  $\varphi$  on the tangent bundle of a closed 7–manifold in terms of the signature and Euler characteristic of a coboundary with a Spin(7)–structure. For manifolds of holonomy  $G_2$  obtained by the twisted connected sum construction, the associated torsion-free  $G_2$ –structure always has  $v(\varphi) = 24$ . Some holonomy  $G_2$  examples constructed by Joyce by desingularising orbifolds have odd  $v$ .

We define a further homotopy invariant  $\xi(\varphi)$  such that if  $M$  is 2–connected then the pair  $(v, \xi)$  determines a  $G_2$ –structure up to homotopy and diffeomorphism. The class of a  $G_2$ –structure is determined by  $v$  on its own when the greatest divisor of  $p_1(M)$  modulo torsion divides 224; this sufficient condition holds for many twisted connected sum  $G_2$ –manifolds.

We also prove that the parametric  $h$ –principle holds for coclosed  $G_2$ –structures.

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## 1 Introduction

In this paper we develop methods to determine when two  $G_2$ –structures on a closed 7–manifold are deformation equivalent, by which we mean related by homotopies and diffeomorphisms. The main motivation is to study the problem of deformation equivalence of metrics with holonomy  $G_2$ . Such metrics can be defined in terms of torsion-free  $G_2$ –structures. The torsion-free condition is a complicated PDE, but we ignore that and consider only the  $G_2$ –structure as a topological residue of the holonomy  $G_2$  metric: for a pair of  $G_2$  metrics to be deformation equivalent, it is certainly necessary that the associated  $G_2$ –structures are. One would not expect this necessary condition to be sufficient since the torsion-free constraint is quite rigid. A much weaker constraint on a  $G_2$ –structure is for it to be coclosed, and we find that the  $h$ –principle holds in this case: if two coclosed  $G_2$ –structures can be connected by a path of  $G_2$ –structures then they can also be connected by a path of coclosed  $G_2$ –structures.

## 1.1 The $v$ -invariant

A  $G_2$ -structure on a 7-manifold  $M$  is a reduction of the structure group of the frame bundle of  $M$  to the exceptional Lie group  $G_2$ . As we review in [Section 2.1](#), a  $G_2$ -structure on  $M$  is equivalent to a 3-form  $\varphi \in \Omega^3(M)$  of a certain type and we will therefore refer to such “positive” 3-forms as  $G_2$ -structures. A  $G_2$ -structure induces a Riemannian metric and spin structure on  $M$ . Throughout this introduction  $M$  shall be a closed connected spin 7-manifold and all  $G_2$ -structures  $\varphi$  will be compatible with the chosen spin structure. We denote the space of all such  $G_2$ -structures by  $\mathcal{G}_2(M)$ .

We say that two  $G_2$ -structures are homotopic if they can be connected by a continuous path of  $G_2$ -structures, so the set of homotopy classes of  $G_2$ -structures on  $M$  is  $\pi_0 \mathcal{G}_2(M)$ . The following observation is not new, but the closest statement we have found in the literature is Witt [\[34, Proposition 3.3\]](#). The proof is simple and provided in [Section 3.1](#).

**Lemma 1.1** *The group  $H^7(M; \pi_7(S^7)) \cong \mathbb{Z}$  acts freely and transitively on  $\pi_0 \mathcal{G}_2(M)$ .*

The group of spin diffeomorphisms of  $M$ ,  $\text{Diff}(M)$ , acts by pull-back on  $\mathcal{G}_2(M)$  with quotient  $\bar{\mathcal{G}}_2(M) := \mathcal{G}_2(M)/\text{Diff}(M)$ . Since  $\mathcal{G}_2(M)$  is locally path connected,

$$\pi_0 \bar{\mathcal{G}}_2(M) = \pi_0 \mathcal{G}_2(M)/\pi_0 \text{Diff}(M),$$

and we call  $\pi_0 \bar{\mathcal{G}}_2(M)$  the set of deformation classes of  $G_2$ -structures on  $M$ . Until now neither invariants of  $\pi_0 \bar{\mathcal{G}}_2(M)$  nor results about its cardinality have appeared in the literature.

Our starting point for studying both of these problems is the following characteristic class formula, valid for any closed spin 8-manifold  $X$  (see [Corollary 2.5](#)):

$$(1) \quad e_+(X) = 24\hat{A}(X) + \frac{\chi(X) - 3\sigma(X)}{2}.$$

Here the terms are the integral of the Euler class of the positive spinor bundle, the  $\hat{A}$ -genus, ordinary Euler characteristic and signature of  $X$  ( $\hat{A}(X)$  is an integer because  $X$  is spin, and  $\sigma(X) \equiv \chi(X) \pmod{2}$  for any closed oriented  $X$ ). Moving from  $\text{Spin}(8)$  to  $\text{Spin}(7)$ , if we use the (real dimension 8) spin representation of  $\text{Spin}(7)$  to regard  $\text{Spin}(7)$  as a subgroup of  $\text{GL}(8, \mathbb{R})$ , then a  $\text{Spin}(7)$ -structure on an 8-manifold  $X$  can be characterised by a certain kind of 4-form  $\psi \in \Omega^4(X)$ . A  $\text{Spin}(7)$ -structure defines a spin structure and Riemannian metric on  $X$ , and (up to a sign) a unit spinor field of positive chirality. In particular, if a closed 8-manifold  $X$  has a  $\text{Spin}(7)$ -structure then  $e_+(X) = 0$ , and (1) implies

$$(2) \quad 48\hat{A}(X) + \chi(X) - 3\sigma(X) = 0.$$

If  $W$  is a compact 8–manifold with boundary  $M$  then a Spin(7)–structure on  $W$  induces a  $G_2$ –structure on  $M$ . From (2) one deduces that the “ $\hat{A}$ –defect”  $\chi(W) - 3\sigma(W)$  mod 48 depends only on the induced  $G_2$ –structure on  $M$ . It turns out, see Lemma 3.4, that any  $G_2$ –structure  $\varphi$  on  $M$  bounds a Spin(7)–structure on some compact 8–manifold and this allows us to define an invariant  $v(\varphi)$ .

**Definition 1.2** Let  $(M, \varphi)$  be a closed spin 7–manifold with  $G_2$ –structure and Spin(7)–coboundary  $(W, \psi)$ . The  $v$ –invariant of  $\varphi$  is the residue

$$v(\varphi) := \chi(W) - 3\sigma(W) \pmod{48} \in \mathbb{Z}_{48}.$$

This definition makes sense even if  $M$  is not connected, and is additive under disjoint unions. Among the many analogous invariants in differential topology, perhaps the one best known to nontopologists is Milnor’s  $\mathbb{Z}_7$ –valued  $\lambda$ –invariant [28] of homotopy 7–spheres, defined as a “ $p_2$ –defect” of a spin coboundary. To distinguish all 28 smooth structures on a homotopy sphere one can use the Eells–Kuiper invariant  $\mu$  [14], which is another  $\hat{A}$ –defect (see (9)).

In Section 1.2 we describe how  $v$  is related to Lemma 1.1 by interpreting  $G_2$ –structures in terms of spinor fields, and we develop most of the theory in those terms. However, the definition above is sometimes useful when dealing with examples. It lets us compute  $v$  from a coboundary with the right type of 4–form, and finding such 4–forms can be easier than describing spinor fields directly, eg in the proof of Theorem 1.7 and Examples 1.14 and 1.15.

Theorem 1.3 below summarises the basic properties of  $v$ . Note that if  $\varphi$  is a  $G_2$ –structure on  $M$ , then the 3–form  $-\varphi$  is also a  $G_2$ –structure, but compatible with the *opposite* orientation;  $-\varphi$  is a  $G_2$ –structure on  $-M$ . In addition, if  $X$  is a closed  $(2n+1)$ –manifold, we define its rational semicharacteristic by

$$\chi_{\mathbb{Q}}(X) := \sum_{i=0}^n b_i(X) \pmod{2}.$$

**Theorem 1.3** For all  $G_2$ –structures  $\varphi$  on  $M$ ,  $v(\varphi) \in \mathbb{Z}_{48}$  is well-defined, and invariant under homotopies and diffeomorphisms. Hence  $v$  defines a function

$$(3) \quad v: \pi_0 \bar{\mathcal{G}}_2(M) \rightarrow \mathbb{Z}_{48}.$$

Moreover  $v(-\varphi) = -v(\varphi)$ , and  $v$  takes exactly the 24 values allowed by the parity constraint

$$(4) \quad v(\varphi) \equiv \chi_{\mathbb{Q}}(M) \pmod{2}.$$

**Theorem 1.3** entails that  $\pi_0 \bar{\mathcal{G}}_2(M)$  has at least 24 elements. Here are some related questions that motivate our investigations:

- What are the values of  $\nu$  for torsion-free  $G_2$ -structures, ie ones arising from  $G_2$  holonomy metrics? Are there  $G_2$  metrics on the same manifold that can be distinguished by  $\nu$ ?
- Do there exist  $G_2$  metrics that are not deformation equivalent, but whose associated torsion-free  $G_2$ -structures belong to the same class in  $\pi_0 \bar{\mathcal{G}}_2(M)$ ?
- What is the cardinality of  $\pi_0 \bar{\mathcal{G}}_2(M)$ ? For example, for which closed spin manifolds  $M$  is  $\nu$  a complete invariant of  $\pi_0 \bar{\mathcal{G}}_2(M)$ ?

We give partial answers to the first and third of these questions below, and discuss directions for further research in [Section 1.7](#).

## 1.2 The affine difference $D$ , spinors and the $\nu$ -invariant

An important feature of homotopy classes of  $G_2$ -structures is that the identification  $\pi_0 \mathcal{G}_2(M) \cong \mathbb{Z}$  from [Lemma 1.1](#) should be regarded as affine, or as a  $\mathbb{Z}$ -torsor: there is no preferred base point, but [Lemma 1.1](#) has the following consequence.

**Lemma 1.4** *For any pair of  $G_2$ -structures  $\varphi, \varphi'$  on  $M$  there is a  $\mathbb{Z}$ -valued difference  $D(\varphi, \varphi')$  such that  $(\pi_0 \mathcal{G}_2(M), D) \cong (\mathbb{Z}, \text{subtraction})$ , ie  $D(\varphi, \varphi') = 0$  if and only if  $\varphi$  is homotopic to  $\varphi'$ , and for all  $\varphi, \varphi'$  and  $\varphi''$ ,*

$$(5) \quad D(\varphi, \varphi') + D(\varphi', \varphi'') = D(\varphi, \varphi'').$$

To understand the relationship between  $D$  and  $\nu$ , we first explain the reasoning which goes into the proof of [Lemma 1.1](#). As we describe in [Section 2.2](#), a choice of Riemannian metric and unit spinor field on the spin manifold  $M$  defines a  $G_2$ -structure. Because any two Riemannian metrics are homotopic, this sets up a bijection between  $\pi_0 \mathcal{G}_2(M)$  and homotopy classes of sections of the unit spinor bundle. This is an  $S^7$ -bundle, and [Lemma 1.1](#) follows from obstruction theory for sections of sphere bundles.

We can both describe  $D$  in concrete terms and prove [Lemma 1.4](#) by counting zeros of homotopies of spinor fields (see [Section 3.1](#)). With this understanding of  $D$ , the next lemma is elementary. The intuitive notion of a  $\text{Spin}(7)$ -bordism is spelled out in [Section 3.3](#).

**Lemma 1.5** *Let  $\varphi, \varphi'$  be  $G_2$ -structures on  $M$ . Suppose  $(W, \psi)$  is a  $\text{Spin}(7)$ -bordism from  $(M, \varphi)$  to  $(M, \varphi')$ , and let  $\overline{W}$  be the closed spin 8-manifold formed by identifying the two boundary components (see (20)). Then*

$$(6) \quad D(\varphi, \varphi') = -e_+(\overline{W}).$$

Combining Lemma 1.5 with the characteristic class formula (1), the mod 24 residue of  $D(\varphi, \varphi')$  can be computed from just the signature and Euler characteristic of  $\overline{W}$ , which equal those of  $W$ . So while  $D$  only makes sense as an “affine” invariant, its mod 24 residue is related to the “absolute” invariant  $\nu$  (in particular,  $\nu$  is affine linear).

**Proposition 1.6** *Let  $\varphi$  and  $\varphi'$  be  $G_2$ -structures on  $M$ . Then*

$$(7) \quad \nu(\varphi') - \nu(\varphi) \equiv 2D(\varphi, \varphi') \pmod{48}.$$

### 1.3 The $\nu$ -invariant for manifolds with $G_2$ holonomy

The exceptional Lie group  $G_2$  also occurs as an exceptional case in the classification of Riemannian holonomy groups due to Berger [3]. It is immediate from the definitions that a metric on a 7-manifold  $M$  has holonomy contained in  $G_2$  if and only if it is induced by a  $G_2$ -structure  $\varphi \in \Omega^3(M)$  that is parallel. The covariant derivative  $\nabla\varphi$  of  $\varphi$  with respect to the Levi-Civita connection  $\nabla$  of its induced metric can be identified with the intrinsic torsion of the  $G_2$ -structure, so metrics with holonomy in  $G_2$  correspond to torsion-free  $G_2$ -structures; see Salamon [31, Corollary 2.2, Section 11].

One can define a moduli space of torsion-free  $G_2$ -structures on a fixed closed  $G_2$ -manifold  $M$ , which is an orbifold locally homeomorphic to finite quotients of  $H_{dR}^3(M)$ . But while the local structure is well understood, little is known about the global structure. One basic question is whether the moduli space is connected, ie whether any pair of torsion-free  $G_2$ -structures are equivalent up to homotopies through torsion-free  $G_2$ -structures and diffeomorphism. If one could find examples of diffeomorphic  $G_2$ -manifolds where the associated  $G_2$ -structures have different values of  $\nu$ , this would prove that the moduli space is disconnected.

Finding compact manifolds with holonomy  $G_2$  is a hard problem. The known constructions solve the nonlinear PDE  $\nabla\varphi = 0$  using gluing methods. Joyce [22] found the first examples by desingularising flat orbifolds, and later Kovalev [24] implemented a “twisted connected sum” construction. In [10], Corti, Haskins, Nordström, and Pacini used the classification theory of closed 2-connected 7-manifolds to find examples of twisted connected sum  $G_2$ -manifolds that are diffeomorphic, but without any evidence either way as to whether the torsion-free  $G_2$ -structures are in the same component of the moduli space.

The twisted connected sum  $G_2$ -manifolds are constructed by gluing a pair of pieces of the form  $S^1 \times V$ , where  $V$  are asymptotically cylindrical Calabi–Yau 3-folds with asymptotic ends  $\mathbb{R} \times S^1 \times K3$ . We review this construction in Section 4.3 and then compute  $\nu$  for all such  $G_2$ -structures.

**Theorem 1.7** *If  $(M, \varphi)$  is a twisted connected sum, then  $\nu(\varphi) = 24$ .*

We carry out this calculation by finding an explicit Spin(7)-bordism from a twisted connected sum  $G_2$ -structure  $\varphi$  to a  $G_2$ -structure that is a product of structures on lower-dimensional manifolds, for which  $\nu$  is easier to evaluate.

For all the explicit examples of pairs of diffeomorphic  $G_2$ -manifolds found in [10], Corollary 1.13 below implies that  $\nu$  classifies the homotopy classes of  $G_2$ -structures up to diffeomorphism. Thus diffeomorphisms between these  $G_2$ -manifolds can always be chosen so that the corresponding torsion-free  $G_2$ -structures are homotopic. Theorem 1.8 implies that they are then also homotopic as coclosed  $G_2$ -structures, but the question whether they can be connected by a path of torsion-free  $G_2$ -structures, so that they are in the same component of the moduli space of  $G_2$  metrics, remains open.

Theorem 1.7 does not necessarily apply to more general gluings of asymptotically cylindrical  $G_2$ -manifolds. For example, a small number of the  $G_2$ -manifolds  $M$  constructed by Joyce [23, Section 12.8.4] have  $\chi_{\mathbb{Q}}(M) = 1$ , so those torsion-free  $G_2$ -structures have odd  $\nu \neq 24$ ; yet they can be regarded at least topologically as a gluing of asymptotically cylindrical manifolds.

## 1.4 The $h$ -principle for coclosed $G_2$ -structures

We call a  $G_2$ -structure with defining 3-form  $\varphi$  *closed* if  $d\varphi = 0$  and *coclosed* if  $d^*\varphi = 0$ , where  $d^*$  is defined in terms of the metric induced by the  $G_2$ -structure. For  $\varphi$  to be torsion-free is equivalent to it being both closed and coclosed (Fernández and Gray [16]). Individually, the conditions of being closed or coclosed are much more flexible than the torsion-free condition, and we show that coclosed  $G_2$ -structures satisfy the  $h$ -principle. Let  $\mathcal{G}_2^{cc}(M) \subset \mathcal{G}_2(M)$  be the subspace of coclosed  $G_2$ -structures.

**Theorem 1.8** *The inclusion  $\mathcal{G}_2^{cc}(M) \hookrightarrow \mathcal{G}_2(M)$  is a homotopy equivalence.*

If  $M$  is an open manifold then Theorem 1.8 is a straight-forward application of Eliashberg and Mishachev [15, Theorem 10.2.1] (see Lê [27, Theorem–Remark 3.17]).  $h$ -principles are generally much harder to prove on closed manifolds, but for coclosed  $G_2$ -structures we can use a microextension trick to reduce the problem to an application of [15, Theorem 10.2.1] on  $M \times (-\epsilon, \epsilon)$ . (There is no apparent way to apply the same trick to closed  $G_2$ -structures, which seem closer to symplectic structures in this sense.)

One motivation for considering coclosed  $G_2$ -structures is that they are the structures induced on 7-manifolds immersed in 8-manifolds with holonomy Spin(7). One can attempt to construct Spin(7) metrics on  $M \times (-\epsilon, \epsilon)$  using the “Hitchin flow” of

coclosed  $G_2$ -structures; see Hitchin [21]. Bryant [5, Theorem 7] shows that this can be solved provided that the initial coclosed  $G_2$ -structure is real analytic.

**Theorem 1.8** implies that any spin 7-manifold  $M$  admits smooth coclosed  $G_2$ -structures. When  $M$  is closed, Grigorian [19] proves short-time existence of solutions  $\varphi_t$  for a version of the ‘‘Laplacian coflow’’ of coclosed  $G_2$ -structures. Even if the initial  $G_2$ -structure  $\varphi_0$  is merely smooth, the coclosed  $G_2$ -structures  $\varphi_t$  will be real analytic for  $t > 0$  (sufficiently small so that the solution exists). We deduce:

**Corollary 1.9** *For every closed spin 7-manifold  $M$ ,  $M \times (-\epsilon, \epsilon)$  admits torsion-free  $\text{Spin}(7)$ -structures.*

## 1.5 Counting deformation classes of $G_2$ -structures

We can think of the set of deformation equivalence classes of  $G_2$ -structures as the quotient (isomorphic to  $\pi_0 \bar{\mathcal{G}}_2(M)$ ) of  $\pi_0 \mathcal{G}_2(M)$  under the action

$$\pi_0 \mathcal{G}_2(M) \times \text{Diff}(M) \rightarrow \pi_0 \mathcal{G}_2(M), \quad ([\varphi], f) \mapsto [f^* \varphi].$$

The deformation invariance of  $\nu$  implies that this action on  $\pi_0 \mathcal{G}_2(M) \cong \mathbb{Z}$  is by translation by multiples of 24, so that  $\pi_0 \bar{\mathcal{G}}_2(M)$  has at least 24 elements. To determine to what extent  $\nu$  classifies elements of  $\pi_0 \bar{\mathcal{G}}_2(M)$  we need to understand precisely which multiples of 24 are realised as translations. Combining the characteristic class formula (1) with Lemma 1.5 we arrive at the following proposition.

**Proposition 1.10** *Let  $f: M \cong M$  be a spin diffeomorphism with mapping torus  $T_f$ . Then*

$$D(\varphi, f^* \varphi) = 24 \hat{A}(T_f) \in \mathbb{Z}.$$

The possible values of  $\hat{A}(T_f)$  are closely related to the spin characteristic class  $p_M := \frac{p_1}{2}(M)$  (see Section 6.1). More precisely, the theory developed by the authors in [11] identifies the two key quantities

$$d_o(M) := \begin{cases} 0 & \text{if } p_M \text{ is torsion,} \\ \max\{s \mid s, m \in \mathbb{Z}, m^2 s \text{ divides } mp_M\} & \text{otherwise,} \end{cases}$$

and a certain value  $r \in \{0, 1, 2\}$  that depends on the properties of the automorphisms of  $H^4(M)$  preserving  $p_M$  and the torsion linking form. If  $H^4(M)$  is torsion-free then  $d_o(M)$  is simply the greatest integer dividing  $p_M$ , and  $r = 1$  whenever  $H^4(M)$  is 2-torsion-free.  $d_o(M)$  is always even by Lemma 6.1.

The following theorem gives lower bounds on  $|\pi_0 \bar{\mathcal{G}}_2(M)|$ . For a fraction  $a/b$  without common factors, denote  $\text{Num}(a/b) = a$ .

**Theorem 1.11** If  $p_M = 0 \in H^4(M; \mathbb{Q})$ , then  $\pi_0 \bar{\mathcal{G}}_2(M) \cong \pi_0 \mathcal{G}_2(M) \cong \mathbb{Z}$ . In general

$$|\pi_0 \bar{\mathcal{G}}_2(M)| \geq 24 \cdot \text{Num} \left( \frac{2^r d_o(M)}{224} \right).$$

So, in particular, if  $H^4(M)$  has no 2-torsion then  $|\pi_0 \bar{\mathcal{G}}_2(M)| \geq 24 \cdot \text{Num} \left( \frac{d_o(M)}{112} \right)$ .

For upper bounds on  $|\pi_0 \bar{\mathcal{G}}_2(M)|$  we need spin diffeomorphisms  $f: M \cong M$  with  $D(\varphi, f^* \varphi) \neq 0$ . When  $M$  is 2-connected and  $p_M$  is not torsion, these are provided by [11].

**Theorem 1.12** If  $M$  is 2-connected and  $p_M \neq 0 \in H^4(M; \mathbb{Q})$ , then

$$|\pi_0 \bar{\mathcal{G}}_2(M)| = 24 \cdot \text{Num} \left( \frac{2^r d_o(M)}{224} \right).$$

Also  $|\pi_0 \bar{\mathcal{G}}_2(N \# M)| \leq 24 \cdot \text{Num} \left( \frac{2^r d_o(M)}{224} \right)$  for any connected spin 7-manifold  $N$ .

**Theorem 1.12** helps identify certain manifolds  $M$  for which  $\nu$  is a complete invariant of  $\pi_0 \bar{\mathcal{G}}_2(M)$ .

**Corollary 1.13** If  $2^r d_o(M_0)$  divides 224 for some 2-connected  $M_0$  such that  $M \cong N \# M_0$ , then  $|\pi_0 \bar{\mathcal{G}}_2(M)| = 24$ . In this case two  $G_2$ -structures  $\varphi$  and  $\varphi'$  on  $M$  are deformation equivalent if and only if  $\nu(\varphi) = \nu(\varphi')$ .

## 1.6 The $\xi$ -invariant

We now describe a further invariant that, depending on the topology of  $M$ , can distinguish more classes of  $\pi_0 \bar{\mathcal{G}}_2(M)$ . For the moment we restrict to the special case when  $p_M$  is rationally trivial, and postpone the full definition to Section 6.4.

In dimension 7, the Eells–Kuiper invariant  $\mu$  [14, Section 6] arises from considering the following characteristic class formula: if  $X$  is a closed spin 8-manifold, then

$$(8) \quad 224 \hat{A}(X) = p_X^2 - \sigma(X).$$

If  $M$  is closed spin with  $p_M$  a torsion class and  $W$  is a spin coboundary, then  $p_W \in H^4(W; \mathbb{Q})$  is in the image of  $H^4(W, M; \mathbb{Q})$ , the cohomology relative to the boundary, and  $p_W^2 \in \mathbb{Q}$  is well defined. Then (8) implies that the  $\hat{A}$ -defect

$$(9) \quad \mu(M) := \frac{p_W^2 - \sigma(W)}{8} \in \mathbb{Q}/28\mathbb{Z}$$

is independent of the choice of  $W$ . (This differs from the definition in [14] by a factor of 28. The mod  $\mathbb{Z}$  residue of  $\mu(M)$  is determined by the almost-smooth structure of

$M$  because  $p_W$  is a characteristic element for the intersection form; therefore  $\mu(M)$  can take 28 different values if the underlying almost-smooth manifold is fixed.)

If we consider a  $G_2$ -structure  $\varphi$  on a spin manifold  $M$  such that  $p_M$  is torsion, then we can in a sense cancel the ambiguities in the definitions of the  $\hat{A}$ -defects  $\nu$  and  $\mu$  to obtain a stronger invariant. A linear combination of (2) and (8) gives that

$$7\chi(X) + \frac{3p_X^2 - 45\sigma(X)}{2} = 0$$

for any closed  $X^8$  with Spin(7)-structure. Hence

$$(10) \quad \xi(\varphi) := 7\chi(W) + \frac{3p_W^2 - 45\sigma(W)}{2} \in \mathbb{Q}$$

is independent of choice of Spin(7)-coboundary  $W$ . If we consider  $G_2$ -structures on a fixed smooth  $M$  with  $p_M$  torsion, then the relation

$$\xi(\varphi) = 7\nu(\varphi) + 12\mu(M) \pmod{336\mathbb{Z}}$$

means that  $\nu(\varphi)$  can be determined from  $\xi(\varphi)$  and  $\mu(M)$ . The  $\xi$ -invariant takes precisely the values allowed by the constraint

$$\xi(\varphi) = 7\chi_{\mathbb{Q}}(M) + 12\mu(M) \pmod{14\mathbb{Z}}.$$

Similarly to Proposition 1.6,  $14D(\varphi, \varphi') = \xi(\varphi') - \xi(\varphi)$ , so  $\xi$  distinguishes between all elements of  $\pi_0 \mathcal{G}_2(M)$ . Since  $\xi$  is patently invariant under diffeomorphisms, this entails the claim from Theorem 1.12 that  $\pi_0 \mathcal{G}_2(M) = \pi_0 \bar{\mathcal{G}}_2(M)$  when  $p_M$  is torsion.

**Example 1.14**  $S^7$  has a standard  $G_2$ -structure  $\varphi_{\text{rd}}$ , induced as the boundary of  $B^8$  with a flat Spin(7)-structure. Clearly  $\nu(\varphi_{\text{rd}}) \equiv \chi(B^8) - 3\sigma(B^8) \equiv 1$ . Meanwhile  $p_{B^8} = 0$ , so  $\xi(\varphi_{\text{rd}}) = 7$ .

On the other hand, the flat Spin(7)-structure on the complement of  $B^8 \subset \mathbb{R}^8$  induces the  $G_2$ -structure  $-\varphi_{\text{rd}}$  on  $S^7$  (with the orientation reversed). If  $r$  is a reflection of  $S^7$  then  $\widehat{\varphi}_{\text{rd}} = r^*(-\varphi_{\text{rd}})$  is a different  $G_2$ -structure on  $S^7$  inducing the same orientation as  $\varphi_{\text{rd}}$ . Since  $\nu(\widehat{\varphi}_{\text{rd}}) = \nu(-\varphi_{\text{rd}}) = -\nu(\varphi_{\text{rd}}) = -1$  (and  $\xi(\widehat{\varphi}_{\text{rd}}) = \xi(-\varphi_{\text{rd}}) = -\xi(\varphi_{\text{rd}}) = -7$ ) there can be no homotopy between  $\varphi_{\text{rd}}$  and  $\widehat{\varphi}_{\text{rd}}$ .

**Example 1.15** The sphere  $S^7$  has a “squashed”  $G_2$ -structure  $\varphi_{\text{sq}}$  that is invariant under  $\text{Sp}(2)\text{Sp}(1)$  and nearly parallel (ie the corresponding cone metric on  $\mathbb{R} \times S^7$  has exceptional holonomy Spin(7)). This  $G_2$ -structure is the asymptotic link of the asymptotically conical Spin(7)-manifold constructed by Bryant and Salamon [6] on the total space  $W$  of the positive spinor bundle of  $S^4$ . This bundle is  $\mathcal{O}(-1)$  over

$\mathbb{H}\mathbb{P}^1$  with the orientation reversed. Since this space has  $\sigma = 1$  and  $\chi = 2$ , it follows that  $v(\varphi_{\text{sq}}) = 2 - 3 = -1$ . Further  $p_W^2 = 1$ , so  $\xi(\varphi_{\text{sq}}) = -7$ . In particular,  $\varphi_{\text{sq}}$  is homotopic to  $\widehat{\varphi}_{\text{rd}}$ ; if we glue  $W$  and  $B^8$  to form  $\mathbb{H}\mathbb{P}^2$  then we can interpolate to define a Spin(7)-structure on  $\mathbb{H}\mathbb{P}^2$ .

The definition of  $\xi$  becomes more involved when  $p_M$  is rationally nontrivial. In general, let  $d_\pi$  denote the greatest integer dividing  $p_M$  modulo torsion (which is even by Lemma 6.1), and  $\tilde{d}_\pi := \gcd(d_\pi, 4)$ . One can then replace the  $p_W^2 \in \mathbb{Q}$  that appears in (10) with a  $\mathbb{Q}/2\tilde{d}_\pi\mathbb{Z}$ -valued function on

$$S_{d_\pi} := \{k \in H^4(M) \mid p_M - d_\pi k \text{ is torsion}\}.$$

Hence one can define  $\xi(\varphi)$  as a function  $S_{d_\pi} \rightarrow \mathbb{Q}/3\tilde{d}_\pi\mathbb{Z}$ , see Definition 6.8. It is invariant in the sense that if  $f: M' \rightarrow M$  is a diffeomorphism, then

$$f^*: H^4(M) \rightarrow H^4(M')$$

restricts to a bijection  $S_{d_\pi} \rightarrow S'_{d_\pi}$ , and  $\xi(f^*\varphi) \circ f^* = \xi(\varphi)$  for any  $G_2$ -structure  $\varphi$  on  $M$ .

### Lemma 1.16

$$(11) \quad \xi(\varphi') - \xi(\varphi) = 14D(\varphi, \varphi') \mod 3\tilde{d}_\pi.$$

Together with Proposition 1.6, this means that the values of  $(v, \xi)$  determine  $D(\varphi, \varphi')$  modulo  $\text{lcm}(24, \text{Num}(3\tilde{d}_\pi/14)) = 24 \text{Num}(d_\pi/112)$ . However, this does not mean that the pair  $(v, \xi)$  distinguishes between  $24 \text{Num}(d_\pi/112)$  classes in  $\pi_0 \bar{\mathcal{G}}_2(M)$ , but only that it distinguishes that many classes modulo homotopies and diffeomorphisms *acting trivially on cohomology*. The reason is that for a general diffeomorphism  $f$  of  $M$ ,  $\xi(\varphi) \circ f^* - \xi(\varphi)$  can be a nonzero constant in  $\mathbb{Q}/3\tilde{d}_\pi\mathbb{Z}$ . Understanding the action of  $f$  on  $\xi$  reduces to the same technical problem as for the action on  $\pi_0 \mathcal{G}_2(M)$ , and we find that in general  $(v, \xi)$  can distinguish between  $24 \text{Num}(2^r d_o(M)/224)$  elements of  $\pi_0 \bar{\mathcal{G}}_2(M)$ , which in a sense is a more precise version of Theorem 1.11. In particular, combining with Theorem 1.12 we find the following result.

**Theorem 1.17** *If  $M$  is 2-connected then  $(v, \xi)$  is a complete invariant of  $\pi_0 \bar{\mathcal{G}}_2(M)$ .*

In combination with the diffeomorphism classification of closed 2-connected 7-manifolds from [11], we obtain a classification result for 2-connected 7-manifolds with  $G_2$ -structures, stated in Theorem 6.9.

## 1.7 Further problems

The main motivation for this work is to help distinguish between connected components of the moduli space of  $G_2$  metrics on a fixed  $M$ . One supply of candidates comes from 2-connected twisted connected sums, but [Theorem 1.7](#) shows that  $\nu$  is not enough to distinguish between those. All twisted connected sum  $G_2$ -manifolds  $M$  have  $d_o(M)$  a divisor of  $d_o(K3) = 24$ , so when  $M$  is 2-connected, the only remaining chance of using the homotopy theory to distinguish between different twisted connected sums  $G_2$  metrics is when  $d_o$  is divisible by 3: by [Theorem 1.11](#) there are in this case 3 different homotopy classes of  $G_2$ -structures with  $\nu = 24$ , and they are distinguished by  $\xi$ . A number of examples of diffeomorphic pairs of twisted connected sums with  $d_o(M) = d_\pi(M) = 6$  are exhibited by the authors in [\[12, Remark 5.7\]](#), but we do not currently have any way to compute  $\xi$  in this situation.

The examples of Joyce with odd  $\nu$  mentioned above can be viewed as a kind of twisted connected sums, gluing asymptotically cylindrical manifolds with holonomy a proper subgroup of  $G_2$  and cross-section  $K3 \times T^2$ , but where the torus factor is not rectangular (as for usual twisted connected sums) but hexagonal. Such “extra-twisted connected sums” provide candidates of 2-connected  $G_2$ -manifolds with fewer restrictions on the possible values of  $\nu$ , and we will return to this elsewhere.

The definition of  $\nu$  in terms of a coboundary is not always amenable to explicit computations. A common theme in differential topology is to find ways to express “extrinsic” invariants (defined in terms of a coboundary) intrinsically, eg Donnelly [\[13\]](#) expresses the classical Eells–Kuiper invariant in terms of eta invariants. Sebastian Goette informs us that it is possible to express  $\nu$  analytically, and we plan to study this and applications to extra-twisted connected sums further in future work.

Some necessary conditions are known for a closed spin 7-manifold  $M$  to admit a metric with holonomy  $G_2$  (see eg Joyce [\[23, Section 10.2\]](#)), but there is currently no conjecture as to what the right sufficient conditions would be. A refinement of this already very hard problem would be to ask: Which deformation classes of  $G_2$ -structures on  $M$  contain torsion-free  $G_2$ -structures? This is of course related to the problem of whether there is any  $M$  with torsion-free  $G_2$ -structures that are not deformation equivalent, which was one of our motivations for introducing  $\nu$ . If one attempts to find torsion-free  $G_2$ -structures as limits of a flow of  $G_2$ -structures as in Bryant and Xu [\[7\]](#), Grigorian [\[19\]](#), Weïß and Witt [\[33\]](#) and Xu and Ye [\[35\]](#), does the homotopy class of the initial  $G_2$ -structures affect the long-term behaviour of the flow?

**Organisation** The rest of the paper is organised as follows. In [Section 2](#) we establish preliminary results needed to define and compute  $\nu$ . In [Section 3](#) we define the

affine difference  $D(\varphi, \varphi')$  and the  $\nu$ -invariant, establish the existence of Spin(7)-coboundaries for  $G_2$ -structures and hence prove [Theorem 1.3](#). We also describe examples of  $G_2$ -structures on  $S^7$  in more detail. In [Section 4](#) we compute the  $\nu$ -invariant for twisted connected sum  $G_2$ -manifolds, proving [Theorem 1.7](#). [Section 5](#) establishes the  $h$ -principle for coclosed  $G_2$ -structures stated in [Theorem 1.8](#). In [Section 6](#) we describe the action of spin diffeomorphisms on  $\pi_0 \mathcal{G}_2(M)$ , give the general definition of the  $\xi$ -invariant and prove the results from [Section 1.5-1.6](#).

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## 2 Preliminaries

In this section we describe  $G_2$ -structures and Spin(7)-structures on 7 and 8-manifolds, and their relationships to spinors. We also establish some basic facts about the characteristic classes of spin manifolds in dimensions 7 and 8.

### 2.1 The Lie groups Spin(7) and $G_2$

We give a brief review of how Spin(7) and  $G_2$ -structures can be characterised in terms of forms. For more detail on the differential geometry of such structures, and how they can be used in the study metrics with exceptional holonomy, see eg Salamon [31] or Joyce [23]. We defer the analogous discussion of SU(3) and SU(2)-structures until we use it in [Section 4](#).

The stabiliser in  $GL(8, \mathbb{R})$  of the 4-form

$$(12) \quad \psi_0 = dx^{1234} + dx^{1256} + dx^{1278} + dx^{1357} - dx^{1368} - dx^{1458} - dx^{1467} \\ - dx^{2358} - dx^{2367} - dx^{2457} + dx^{2468} + dx^{3456} + dx^{3478} + dx^{5678} \in \Lambda^4(\mathbb{R}^8)^*$$

is Spin(7) (identified with a subgroup of SO(8) by the spin representation). Here and elsewhere,  $dx^{1234}$  abbreviates  $dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4$  etc. On an 8-dimensional manifold  $X$ , a 4-form  $\psi \in \Omega^4(X)$  which is pointwise equivalent to  $\psi_0$  defines a Spin(7)-structure, and induces a metric and orientation (the orientation form is  $\psi^2$ ).

The exceptional Lie group  $G_2$  can be defined as the automorphism group of  $\mathbb{O}$ , the normed division algebra of octonions. Equivalently,  $G_2$  is the stabiliser in  $\mathrm{GL}(7, \mathbb{R})$  of the 3-form

$$(13) \quad \varphi_0 = dx^{123} + dx^{145} + dx^{167} + dx^{246} - dx^{257} - dx^{347} - dx^{356} \in \Lambda^3(\mathbb{R}^7)^*.$$

On a 7-dimensional manifold  $M$ , a 3-form  $\varphi \in \Omega^3(M)$  that is pointwise equivalent to  $\varphi_0$  defines a  $G_2$ -structure, which induces a Riemannian metric and orientation. Note that

$$(14) \quad dt \wedge \varphi_0 + * \varphi_0 \cong \psi_0$$

on  $\mathbb{R} \oplus \mathbb{R}^7$ , so the stabiliser in  $\mathrm{Spin}(7)$  of a nonzero vector in  $\mathbb{R}^8$  is exactly  $G_2$ . Therefore the product of a 7-manifold with a  $G_2$ -structure and  $S^1$  or  $\mathbb{R}$  has a natural product  $\mathrm{Spin}(7)$ -structure, while a  $\mathrm{Spin}(7)$ -structure  $\psi$  on  $W^8$  induces a  $G_2$ -structure on  $\partial W$  by contracting  $\psi$  with an outward pointing normal vector field.

**Remark 2.1** If  $\varphi$  is a  $G_2$ -structure on  $M^7$ , then  $-\varphi$  is a  $G_2$ -structure too, inducing the same metric and opposite orientation (because  $\varphi_0$  is equivalent to  $-\varphi_0$  under the orientation-reversing isomorphism  $-1 \in O(7)$ ). The product  $\mathrm{Spin}(7)$ -structure  $dt \wedge \varphi + * \varphi$  on  $M \times [0, 1]$  induces  $\varphi$  on the boundary component  $M \times \{1\} \cong M$ , and  $-\varphi$  on  $M \times \{0\} \cong -M$ .

## 2.2 $G_2$ -structures and spinors

In this paper we are concerned with  $G_2$ -structures on a manifold  $M^7$  up to homotopy. Since there is an obvious way to reverse the orientation of a  $G_2$ -structure, while any two Riemannian metrics are homotopic, we may as well consider  $G_2$ -structures compatible with a fixed orientation and metric. Because  $G_2$  is simply-connected, the inclusion  $G_2 \hookrightarrow \mathrm{SO}(7)$  lifts to  $G_2 \hookrightarrow \mathrm{Spin}(7)$ . Therefore a  $G_2$ -structure on  $M$  also induces a spin structure, and we focus on studying  $G_2$ -structures compatible also with a fixed spin structure. As in [Section 1](#), we let  $\pi_0 \mathcal{G}_2(M)$  denote the homotopy classes of  $G_2$ -structures on  $M$  with a choice of spin structure.

As we already saw,  $G_2$  is exactly the stabiliser of a nonzero vector in the spin representation  $\Delta$  of  $\mathrm{Spin}(7)$ ; as a representation of  $G_2$ ,  $\Delta$  splits as the sum of a 1-dimensional trivial part and the standard 7-dimensional representation.  $\mathrm{Spin}(7)$  acts transitively on the unit sphere in  $\Delta$  with stabiliser  $G_2$ , so  $\mathrm{Spin}(7)/G_2 \cong S^7$ .

From the above, we deduce that given a spin structure on  $M$ , a compatible  $G_2$ -structure  $\varphi$  induces an isomorphism  $SM \cong \underline{\mathbb{R}} \oplus TM$  for the spinor bundle  $SM$ : here  $\underline{\mathbb{R}}$  denotes the trivial line bundle. Hence we can associate to  $\varphi$  a unit section of  $SM$ , well-defined

up to sign. Conversely, any unit section of  $SM$  defines a compatible  $G_2$ -structure. A transverse section  $s$  of the spinor bundle  $SM$  of a spin 7-manifold has no zeros, so defines a  $G_2$ -structure; thus a 7-manifold admits  $G_2$ -structures if and only if it is spin (see Gray [17] and Lawson and Michelsohn [26, Theorem IV.10.6]).

Note that  $s$  and  $-s$  are always homotopic, because they correspond to sections of the trivial part in a splitting  $SM \cong \mathbb{R} \oplus TM$  and the Euler class of an oriented 7-manifold vanishes. It follows that  $SM$  contains a trivial 2-plane field  $K \supset \mathbb{R}$  which accommodates a homotopy from  $s$  to  $-s$ . Therefore  $\pi_0 \mathcal{G}_2(M)$  can be identified with homotopy classes of unit sections of the spinor bundle. As stated in [Section 1](#), [Lemma 1.1](#) now follows by a standard application of obstruction theory, but we will describe the bijection  $\pi_0 \mathcal{G}_2(M) \cong \mathbb{Z}$  in elementary terms in [Section 3.1](#).

**Remark 2.2** Let us make some further comments on the signs of the spinors. Given a principal  $\text{Spin}(7)$  lift  $\tilde{F}$  of the frame bundle  $F$  of  $M$ , the principal  $G_2$ -subbundles of  $\tilde{F}$  are in bijective correspondence with sections of the associated unit spinor bundle. The  $G_2$ -subbundles corresponding to spinors  $s$  and  $-s$  have the same image in  $F$ , hence they define the same  $G_2$ -structure on  $M$  (they have the same 3-form  $\varphi$ ).

While  $\text{SO}(7)$  does not itself act on  $\Delta$ , the action of  $\text{Spin}(7)$  on  $(\Delta \setminus \{0\})/\mathbb{R}^*$   $\cong \mathbb{RP}^7$  does descend to an action of  $\text{SO}(7)$ . Therefore the orbit  $\text{SO}(7)\varphi_0$ , the set of  $G_2$ -structures on  $\mathbb{R}^7$  defining the same orientation and metric as  $\varphi_0$ , is  $\text{SO}(7)/G_2 \cong \mathbb{RP}^7$ .  $G_2$ -structures compatible with a fixed orientation and metric on  $M$  but without any constraint on the spin structure therefore correspond to sections of an  $\mathbb{RP}^7$  bundle. If  $M$  is not spin then this bundle has no sections. Given a spin structure, the unit sphere bundle in the associated spinor bundle is an  $S^7$  lift of the  $\mathbb{RP}^7$ -bundle, and two  $G_2$ -structures induce the same spin structure if they can both be lifted to the same  $S^7$  bundle.

## 2.3 Spin(7)-structures and characteristic classes of Spin(8)-bundles

The spin representation of  $\text{Spin}(7)$  is faithful, so defines an inclusion homomorphism  $\text{Spin}(7) \hookrightarrow \text{SO}(8)$ , which has a lift  $i_\Delta: \text{Spin}(7) \hookrightarrow \text{Spin}(8)$ . The restriction of the positive half-spin representation  $\Delta^+$  of  $\text{Spin}(8)$  to  $\text{Spin}(7)$  is a sum of a trivial rank 1 part and the 7-dimensional vector representation (factoring through  $\text{Spin}(7) \rightarrow \text{SO}(7)$ ). Therefore  $i_\Delta(\text{Spin}(7)) \subset \text{Spin}(8)$  can be characterised as the stabiliser of a unit positive spinor  $s_0 \in \Delta^+$ , and  $\text{Spin}(7)$ -structures on a spin 8-manifold are equivalent to unit positive spinor fields (up to sign, in the same sense as  $G_2$ -structures). Hence there is an obvious obstruction to the existence of  $\text{Spin}(7)$ -structures on an 8-manifold  $X$ : it must be spin, and the Euler class in  $H^8(X)$  of the positive half-spinor bundle on  $X$  must vanish.

**Remark 2.3** One can of course also define an embedding  $i_0: \text{Spin}(7) \hookrightarrow \text{Spin}(8)$  as the stabiliser of the coordinate vector  $e_8$  in the vector representation  $\mathbb{R}^8$  of  $\text{Spin}(8)$ . The restrictions to this copy of  $\text{Spin}(7)$  of the half-spin representation  $\Delta^\pm$  of  $\text{Spin}(8)$  are both isomorphic to the spin representation of  $\text{Spin}(7)$ . Therefore, if  $W^8$  is a spin manifold then the restrictions of the half-spinor bundles  $S^\pm W$  to  $\partial W$  are naturally isomorphic to the spinor bundle  $S(\partial W)$ .

In particular, a positive spinor field on  $W^8$  can be restricted to a spinor field on  $\partial W$ , so the restriction of a  $\text{Spin}(7)$ -structure on  $W$  to a  $G_2$ -structure on  $\partial W$  can be described in terms of the spinorial picture. Of course, this gives exactly the same result as if we describe the restriction in terms of differential forms. This is because the image of the composition of the inclusions

$$G_2 \hookrightarrow \text{Spin}(7) \xrightarrow{i_0} \text{Spin}(8)$$

is equally well described as the stabiliser in  $\text{Spin}(8)$  of  $(s_0, e^8) \in \Delta^+ \times \mathbb{R}^8$  and as the lift of the stabiliser in  $\text{GL}(\mathbb{R}, 8)$  of  $(\psi_0, e_8) \in \Lambda^4 \mathbb{R}^8 \times \mathbb{R}^8$ .

Let us describe briefly our conventions for orientations on the half-spin representations of  $\text{Spin}(8)$ . For each fixed nonzero  $v \in \mathbb{R}^8$ , the Clifford multiplication  $\mathbb{R}^8 \times \Delta^\pm \rightarrow \Delta^\mp$  defines orientation-preserving isomorphisms  $c_v^\pm: \Delta^\pm \rightarrow \Delta^\mp$ . A feature of the “triality” in dimension 8 is that the map  $\hat{c}_{s_\pm}: \mathbb{R}^8 \rightarrow \Delta^\mp$  induced by Clifford multiplication with a fixed nonzero spinor  $s_\pm \in \Delta^\pm$  is an isomorphism too. The Clifford relations imply that, for  $s_+ = vs_-$ ,

$$c_v^+ \circ \hat{c}_{s_-} = \hat{c}_{s_+} \circ r_v: \mathbb{R}^8 \rightarrow \Delta^-,$$

where  $r_v: \mathbb{R}^8 \rightarrow \mathbb{R}^8$  is reflection in the hyperplane orthogonal to  $v$ . Thus  $\hat{c}_{s_+}$  and  $\hat{c}_{s_-}$  have opposite orientability. Our convention is that  $\hat{c}_{s_-}$  is orientation-preserving, while  $\hat{c}_{s_+}$  is not.

More explicitly,  $\mathbb{R}^8$ ,  $\Delta^+$  and  $\Delta^-$  can each be identified with the octonions  $\mathbb{O}$  so that the Clifford multiplication  $\mathbb{R}^8 \times \Delta^- \rightarrow \Delta^+$  corresponds to the octonionic multiplication  $(x, y) \mapsto xy$ ; see Baez [2, page 162 above (5)]. Then, to satisfy the Clifford relations,  $\mathbb{R}^8 \times \Delta^+ \rightarrow \Delta^-$  must correspond to  $(x, y) \mapsto -\bar{x}y$ , where  $\bar{x}$  is the octonion conjugate of  $x$ . This map is orientation-reversing on the first factor.

Let  $X$  be a spin 8-manifold,  $e \in H^8(X)$  the Euler class of  $TX$ , and  $e_\pm \in H^8(X)$  the Euler classes of the half-spinor bundles  $S^\pm X$ . More generally, for any principal  $\text{Spin}(8)$ -bundle on any  $X$ , let  $e, e_\pm$  denote the Euler classes of the vector bundles associated to the vector and half-spin representations of  $\text{Spin}(8)$ . With our orientation conventions, the nondegeneracy of the Clifford product implies

$$(15) \quad e_+ = e + e_-.$$

The following statement can be found for instance in Gray and Green [18, page 89].

**Proposition 2.4** *For any principal  $\text{Spin}(8)$ -bundle*

$$e_{\pm} = \frac{1}{16}(p_1^2 - 4p_2 \pm 8e).$$

In degree 8, the  $\hat{A}$  and  $L$  genera are given by

$$(16) \quad \begin{aligned} 45 \cdot 2^7 \hat{A} &= 7p_1^2 - 4p_2, \\ 45L &= 7p_2 - p_1^2, \end{aligned}$$

so [Proposition 2.4](#) can be rewritten as  $e_{\pm} = 24\hat{A} + \frac{1}{2}(\pm e - 3L)$ . If  $X$  is closed and orientable then the integral of the  $L$  genus of  $TX$  is the signature of  $X$  by the Hirzebruch signature theorem, while the integral of the Euler class is just the ordinary Euler characteristic.

**Corollary 2.5** *If  $X$  is a closed spin 8-manifold then*

$$e_{\pm}(X) = 24\hat{A}(X) + \frac{\pm\chi(X) - 3\sigma(X)}{2}.$$

**Remark 2.6** Modulo torsion, the group of integral characteristic classes of a principal  $\text{Spin}(8)$ -bundle in dimension 8 is generated by  $p_1^2$ ,  $p_2$  and  $e$ , so we could prove [Corollary 2.5](#) (and hence [Proposition 2.4](#)) by checking that the formula holds for the following spin 8-manifolds.

- $S^8$ :  $\chi = 2$ ,  $\hat{A} = \sigma = 0$ ,  $e_{\pm} = \pm 1$ .
- $K3 \times K3$ :  $\chi = 24^2$ ,  $\sigma = (-16)^2$ .  $\hat{A} = 4$  because the holonomy is  $\text{SU}(2) \times \text{SU}(2)$ . Because this also defines a  $\text{Spin}(7)$ -structure (see [\(22\)](#)),  $e_+ = 0$  and  $e_- = -\chi$ .
- $\mathbb{HP}^2$ :  $\chi = 3$ ,  $\sigma = 1$ .  $\hat{A} = 0$  by the Lichnerowicz formula since there is a metric with positive scalar curvature.  $e_- = -\chi$  because  $S^- X \cong -TX$  for any spin 8-manifold  $X$  with  $\text{Sp}(2)\text{Sp}(1)$ -structure. This structure also splits  $S^+ X$  into a sum of a rank 5 and a rank 3 part, so  $e_+ = 0$ . (Alternatively, we can identify a quaternionic line subbundle of  $T\mathbb{HP}^2$ , like that spanned by the projection of the vector field  $(q_1, q_2, q_3) \mapsto (0, q_1, q_2)$  on  $\mathbb{H}^3$ , with a nonvanishing section of the rank-5 part of  $S^+ X$ .)

### 3 The $v$ -invariant

In this section we study the set  $\pi_0 \mathcal{G}_2(M)$  of homotopy classes of  $G_2$ -structures on a closed spin 7-manifold  $M$ , and prove the basic properties of the invariants  $D$  and  $v$ . We conclude the section with some concrete examples.

### 3.1 The affine difference

Let  $M$  be a closed connected spin 7-manifold, and  $\varphi, \varphi'$  a pair of  $G_2$ -structures on  $M$ . We describe how to define the difference  $D(\varphi, \varphi') \in \mathbb{Z}$  from [Lemma 1.4](#).

A homotopy of  $G_2$ -structures is equivalent to a path of nonvanishing spinor fields. Any path of spinor fields on  $M$  can be identified with a positive spinor field  $s$  on  $M \times [0, 1]$ . We can always find  $s$  such that the restrictions to  $M \times \{1\}$  and  $M \times \{0\}$  are the nonvanishing spinor fields corresponding to  $\varphi$  and  $-\varphi'$ , respectively. Then the pull-back by  $s$  of the Thom class of the positive spinor bundle defines a relative Euler class in  $H^8(W, M)$ , independent of the choice of  $s$ , and we define  $D(\varphi, \varphi')$  to be its integral  $n_+(M \times [0, 1], \varphi, \varphi')$ . If we take  $s$  to have transverse zeroes then we can interpret this geometrically as the intersection number of the graph of  $s$  with the zero section.

The affine relation (5) is immediate from this definition. If  $n_+(M \times [0, 1], \varphi, \varphi') = 0$ , then  $s$  can be chosen to be nonvanishing, so  $\varphi$  and  $\varphi'$  are homotopic if and only if  $D(\varphi, \varphi') = 0$ . Given  $\varphi$  we can construct  $\varphi'$  such that  $D(\varphi, \varphi') = 1$  by modifying the defining spinor of  $\varphi$  in a 7-disc  $B^7$ : in a local trivialisation we change it from a constant map  $B^7 \rightarrow S^7$  to a degree 1 map. Thus  $D$  can take any integer value, so  $D$  really corresponds to the difference function under a bijection  $\mathbb{Z} \cong \pi_0 \mathcal{G}_2(M)$ , completing the proof of [Lemma 1.4](#).

To compute  $D(\varphi, \varphi')$ , we can consider more general spin 8-manifolds  $W$  with boundary  $M \sqcup -M$ . Generalising the above, let  $n_+(W, \varphi, \varphi')$  be the intersection number with the zero section of a positive spinor whose restriction to the two boundary components correspond to  $\varphi$  and  $-\varphi'$ . Form a closed spin 8-manifold  $\overline{W}$  by gluing the  $M$  piece of the boundary of  $W$  to the  $-M$  piece. We can define a continuous positive spinor field on  $\overline{W}$  by modifying the spinor field from  $W$  in an  $M \times [0, 1]$  neighbourhood of the former boundary, to interpolate between  $\varphi'$  on  $M \times \{1\}$  and  $-\varphi$  on  $M \times \{0\}$ . Its intersection number with the zero section is  $n_+(W, \varphi, \varphi') - D(\varphi, \varphi')$ , so we can compute  $D$  as

$$(17) \quad D(\varphi, \varphi') = n_+(W, \varphi, \varphi') - e_+(\overline{W}).$$

### 3.2 The definition of $\nu$

Let  $M$  be a closed spin 7-manifold (not necessarily connected) with  $G_2$ -structure  $\varphi$ , and  $W$  a compact spin 8-manifold with  $\partial W = M$ . Such  $W$  always exist since the bordism group  $\Omega_7^{\text{Spin}}$  is trivial [\[29\]](#). The restrictions of the half-spinor bundles  $S^\pm W$  of  $W$  to  $M$  are isomorphic to the spinor bundle on  $M$  ([Remark 2.3](#)), and the

composition  $S^+W|_M \rightarrow S^-W|_M$  of these isomorphisms is Clifford multiplication by a unit normal vector field to the boundary. Let  $n_{\pm}(W, \varphi)$  be the intersection number with the zero section of a section of  $S^{\pm}W$  whose restriction to  $M$  is the nonvanishing spinor field defining  $\varphi$ . Let

$$(18) \quad \bar{v}(W, \varphi) := -2n_+(W, \varphi) + \chi(W) - 3\sigma(W) \in \mathbb{Z}.$$

Reversing the orientations,  $-W$  is a spin 8–manifold whose boundary  $-M$  is equipped with a  $G_2$ –structure  $-\varphi$ .

**Lemma 3.1** *Let  $W$  be a compact spin 8–manifold, and  $\varphi$  a  $G_2$ –structure on  $M = \partial W$ .*

- (i) *If  $\varphi'$  is another  $G_2$ –structure on  $M$  then  $\bar{v}(W, \varphi') - \bar{v}(W, \varphi) = 2D(\varphi, \varphi')$ .*
- (ii)  $\bar{v}(W, \varphi) \equiv \chi_{\mathbb{Q}}(M) \bmod 2$ .
- (iii)  $\bar{v}(-W, -\varphi) = -\bar{v}(W, \varphi)$ .
- (iv) *If  $W'$  is another compact spin 8–manifold with  $\partial W' = M$  then the closed spin 8–manifold  $X = W \cup_{\text{Id}_M} (-W')$  has*

$$48\hat{A}(X) = \bar{v}(W', \varphi) - \bar{v}(W, \varphi).$$

**Proof** (i) Clearly  $n_+(W, \varphi) = n_+(M \times I, \varphi, \varphi') + n_+(W, \varphi')$ .

(ii) For  $W^{4n}$  any compact oriented manifold with boundary,  $\sigma(W)$  is by definition the signature the intersection form, a nondegenerate symmetric form on the image  $H_0^{2n}(W)$  of  $H^{2n}(W, M) \rightarrow H^{2n}(W)$ . In particular,  $\sigma(W) \equiv \dim H_0^{2n}(W) \bmod 2$ . Writing

$$\chi(W) = \sum_{i=0}^{2n-1} b_i(W) + \sum_{i=0}^{2n} b_{4n-i}(W)$$

and using  $b_{4n-i}(W) = b_i(W, M)$  and the definition that  $\chi_{\mathbb{Q}}(W) = \sum_{i=0}^{2n-1} b_i(\partial W) \bmod 2$ , the exactness of the sequence

$$0 \rightarrow H^0(W, M) \rightarrow H^0(W) \rightarrow \cdots \rightarrow H^{2n-1}(\partial W) \rightarrow H^{2n}(W, M) \rightarrow H_0^{2n}(W) \rightarrow 0$$

implies

$$(19) \quad \sigma(W) + \chi(W) \equiv \chi_{\mathbb{Q}}(\partial W) \bmod 2.$$

(iii) Let  $v$  be a vector field on  $W$  that is a unit outward-pointing normal field along  $M$ , and  $s \in \Gamma(S^+W)$  a spinor field whose restriction to  $M$  induces  $\varphi$ . Then the restriction of the Clifford product  $v \cdot s \in \Gamma(S^-W)$  also induces  $\varphi$ . By the Poincaré–Hopf index theorem, the number of zeros of  $v$  is  $\chi(W)$ , so  $n_-(W, \varphi) = n_+(W, \varphi) - \chi(W)$  (these signs are compatible with (15)).

Reversing the orientations swaps sections of  $S^+W$  and  $S^-W$ , and reverses the signs assigned to the zeros, so  $n_+(-W, -\varphi) = -n_-(W, \varphi)$ . It also reverses the signature, but preserves the Euler characteristic. Thus

$$\begin{aligned}\bar{v}(-W, -\varphi) &= 2n_-(W, \varphi) + \chi(W) + 3\sigma(W) \\ &= 2n_+(W, \varphi) - 2\chi(W) + \chi(W) + 3\sigma(W) = -\bar{v}(W, \varphi).\end{aligned}$$

(iv)  $\sigma(W) + \sigma(-W') = \sigma(X)$  by Novikov additivity [1, 7.1],  $\chi(W) + \chi(-W') = \chi(X)$  because  $\chi(M) = 0$ , and  $X$  has a transverse positive spinor field whose intersection number with the zero section is  $n_+(W, \varphi) + n_+(-W', -\varphi)$ . Hence

$$\bar{v}(W', \varphi) - \bar{v}(W, \varphi) = -\bar{v}(-W', -\varphi) - \bar{v}(W, \varphi) = 2e_+(X) - \chi(X) + 3\sigma(X) = 48\hat{A}(X)$$

by Corollary 2.5.  $\square$

**Corollary 3.2**  $v(\varphi) := \bar{v}(W, \varphi) \bmod 48 \in \mathbb{Z}_{48}$  is independent of the choice of  $W$ , and for all  $\varphi$  and  $\varphi'$ ,

$$v(\varphi') - v(\varphi) \equiv 2D(\varphi, \varphi') \bmod 48.$$

This gives the majority of the proofs of Theorem 1.3 and Proposition 1.6. To complete the proofs it remains only to show the existence of Spin(7)-coboundaries, since Definition 1.2 is phrased in terms of those. We show the existence of the required Spin(7)-coboundaries in the following subsection.

### 3.3 Spin(7)-bordisms

Let  $\varphi, \varphi'$  be  $G_2$ -structures on closed 7-manifolds  $M, M'$ . A Spin(7)-bordism from  $(M, \varphi)$  to  $(M', \varphi')$  is a compact 8-manifold with boundary  $M \sqcup -M'$  and a Spin(7)-structure  $\psi$  inducing the respective  $G_2$ -structures on the boundary. More formally, we require that  $\partial W = f(M) \sqcup f'(M')$  for embeddings  $f: M \hookrightarrow \partial W$ ,  $f': M' \hookrightarrow \partial W$  that pull back the contraction of  $\psi$  with the outward normal field to  $\varphi$  and  $-\varphi'$ , respectively. If  $M = M'$  then we can form a closed spin 8-manifold by identifying the boundary components,

$$(20) \quad \overline{W} := W / (f' \circ f^{-1}).$$

Clearly, there is a topologically trivial Spin(7)-bordism  $W$  (ie there is a diffeomorphism  $W \cong M \times [0, 1]$ , but it does not have to preserve the Spin(7)-structure) from  $\varphi$  to  $\varphi'$  if and only if they are deformation equivalent, ie  $f^*\varphi'$  is homotopic to  $\varphi$  for some diffeomorphism  $f: M \cong M$ .

**Remark 3.3** If  $(W, \psi, f, f')$  is a Spin(7)-bordism from  $(M, \varphi)$  to  $(M', \varphi')$  then  $(W, \psi, f', f)$  is Spin(7)-bordism from  $(-M', -\varphi')$  to  $(-M, -\varphi)$ . However, it does not follow in general that  $-W$  has a Spin(7)-structure making it a Spin(7)-bordism from  $(M', \varphi')$  to  $(M, \varphi)$  (because the orientation of a Spin(7)-structure cannot be reversed). In particular, if  $W$  is a Spin(7)-coboundary for  $(M, \varphi)$ , then  $-W$  is not necessarily a Spin(7)-coboundary for  $(-M, -\varphi)$ , unless  $\chi(W) = 0$ , see proof of Lemma 3.1(iii).

The Spin(7)-structure  $\psi$  induces a nonvanishing positive spinor field  $s$  on  $W$ . By Remark 2.3 the restriction of  $s$  to  $\partial W$  is the spinor defining the  $G_2$ -structures  $\varphi$  and  $-\varphi'$ , so  $n_+(W, \varphi, \varphi') = 0$ . In particular, when  $\varphi$  and  $\varphi'$  are  $G_2$ -structures on the same manifold  $M = M'$ , Lemma 1.5 follows from (17). Similarly, if  $W$  is a Spin(7)-coboundary for  $(M, \varphi)$  then  $\bar{v}(W, \varphi) = \chi(W) - 3\sigma(W)$ , so Corollary 3.2 and Lemma 3.4(ii) imply Theorem 1.3.

**Lemma 3.4** (i) For a connected compact spin 8-manifold  $W$  with connected boundary  $M$ , there is a unique homotopy class of  $G_2$ -structures on  $M$  that bound Spin(7)-structures on  $W$ .  
(ii) Any  $G_2$ -structure has a Spin(7) coboundary (that is, any two  $G_2$ -structures are Spin(7)-bordant).

**Proof** If  $W$  is connected with nonempty boundary then there is no obstruction to defining a nonvanishing positive spinor field on  $W$ , so there is some  $G_2$ -structure  $\varphi$  on  $M$  that bounds a Spin(7)-structure on  $W$ . If  $\varphi'$  is another  $G_2$ -structure bounding a Spin(7)-structure on  $W$ , consider an arbitrary spin filling  $W'$  of  $-M$ , and let  $-\varphi''$  be a  $G_2$ -structure on  $-M$  that bounds a Spin(7)-structure on  $W'$ . Then  $W \sqcup W'$  admits two Spin(7)-structures that define bordisms from  $\varphi$  and  $\varphi'$ , respectively, to  $\varphi''$ . Hence

$$D(\varphi, \varphi') = D(\varphi, \varphi'') - D(\varphi', \varphi'') = 0,$$

and  $\varphi$  and  $\varphi'$  must be homotopic.

For (ii), take any spin filling  $W$  of  $M$ , and let  $\varphi$  be a  $G_2$ -structure on  $M$  that bounds a Spin(7)-structure. In order to find a Spin(7)-coboundary for some other  $\varphi'$  with  $D(\varphi, \varphi') = \pm k$ , we use that if  $X$  and  $X'$  are closed spin 8-manifolds then, since  $\hat{A}$  and  $\sigma$  are bordism-invariants, and in particular additive under connected sums, Corollary 2.5 implies that

$$e_+(X \# X') = e_+(X) + e_+(X') - 1.$$

(We could also see that for any pair of positive spinor fields  $s, s'$  on  $X, X'$  one can define a spinor field on  $X \# X'$  that equals  $s$  and  $s'$  outside the connecting neck, and

with a single zero on the neck.) Therefore  $\varphi'$  will bound a Spin(7)-structure on  $W'$  the connected sum of  $W$  with  $k$  copies of a manifold with  $e_+ = 2$  or 0, eg  $S^4 \times S^4$  or  $T^8$ .  $\square$

### 3.4 Examples of $G_2$ -structures on $S^7$

To make the discussion more concrete, we elaborate on some examples on  $S^7$ , where  $D$  can be described in the following direct way. The spinor bundle of  $S^7$  can be trivialised by identifying it with the restriction of the positive half-spinor bundle on  $B^8$ , thus up to homotopy, a  $G_2$ -structure  $\varphi$  on  $S^7$  can be identified with a map  $f$  from  $S^7$  to the unit sphere in  $\Delta^+$ . The difference  $D$  between two  $G_2$ -structures on  $S^7$  equals the difference of the degrees of the corresponding maps  $S^7 \rightarrow S^7$ :  $D(\varphi, \varphi') = \deg f - \deg f'$ .

**Example 3.5** We first illustrate how this description works for the standard round  $G_2$ -structure  $\varphi_{\text{rd}}$  and its reverse  $\widehat{\varphi}_{\text{rd}}$ , which we already understand from [Example 1.14](#). By definition,  $\varphi_{\text{rd}}$  corresponds to a constant map  $f_{\text{rd}}: x \mapsto s_0$ . The  $G_2$ -structure  $\varphi_{\text{rd}}$  is invariant under the action of Spin(7), and so is  $f_{\text{rd}}$ , in the sense that  $f_{\text{rd}}(gx) = s_0 = gs_0 = g f_{\text{rd}}(x)$  for any  $g \in \text{Spin}(7)$ .

Let  $r$  be a reflection of  $S^7$ , and  $\widehat{\varphi}_{\text{rd}} = r^*(-\varphi_{\text{rd}})$  as before. Then  $\widehat{\varphi}_{\text{rd}}$  is invariant under the action of the conjugate subgroup  $r \text{Spin}(7)r \subset \text{Spin}(8)$ . If  $x_0 \in S^7$  is a vector orthogonal to the hyperplane of the reflection, then  $\varphi_{\text{rd}}$  and  $\widehat{\varphi}_{\text{rd}}$  take the same value at  $x_0$ . Thus  $\widehat{f}_{\text{rd}}(x_0) = s_0$ , and  $\widehat{f}_{\text{rd}}(rgrx_0) = (rgr)s_0$  for any  $g \in \text{Spin}(7)$ . The outer automorphism on Spin(8) of conjugating by  $r$  swaps the positive and negative spin representations via Clifford multiplication by  $x_0$ , so  $(rgr)s_0 = x_0 \cdot (g(x_0 \cdot s_0)) = x_0 \cdot (g(x_0) \cdot s_0)$  for  $g \in \text{Spin}(7)$ . Hence  $\widehat{f}_{\text{rd}}: S^7 \rightarrow S^7$  equals the orientation-preserving diffeomorphism  $c_{x_0}^- \circ \widehat{c}_{s_0} \circ (-r)$ , and  $D(\widehat{\varphi}_{\text{rd}}, \varphi_{\text{rd}}) = \deg \widehat{f}_{\text{rd}} - \deg f_{\text{rd}} = 1$ .

**Example 3.6** Consider the octonionic left-multiplication parallelism on  $S^7$ , ie the trivialisation of  $TS^7$  obtained by considering  $u \in S^7$  as a unit octonion and defining  $L_u: \text{Im } \mathbb{O} \cong T_u S^7$  as left multiplication by  $u$ . Its associated  $G_2$ -structure  $\varphi_{\mathbb{O}}$  has  $\varphi_{\mathbb{O}}(u) = L_u \varphi_0$  for a fixed  $G_2$ -structure  $\varphi_0$ . The associated map  $f_{\mathbb{O}}: S^7 \rightarrow S^7$  is  $u \mapsto \tilde{L}_u s_0$ , where  $S^7 \rightarrow \text{Spin}(8)$ ,  $u \mapsto \tilde{L}_u$  is the continuous lift of  $S^7 \rightarrow \text{SO}(8)$ ,  $u \mapsto L_u$  (with  $\tilde{L}_1 = \text{Id}$ ) which acts on  $s_0 \in \Delta^+$ .

Here is one way to understand  $\tilde{L}_u$ . The Moufang identity  $u(xy)u = (ux)(yu)$  holds for any  $u, x, y \in \mathbb{O}$  [20, Lemma A.16(c)], so  $(L_u, R_u, L_u \circ R_u) \in \text{SO}(8)^3$  preserves the Cayley multiplication. As mentioned before, the Cayley multiplication on  $\mathbb{O}$  can be identified with Clifford multiplication  $\mathbb{R}^8 \times \Delta^- \rightarrow \Delta^+$ , whose stabiliser in the group

$\mathrm{SO}(\mathbb{R}^8) \times \mathrm{SO}(\Delta^-) \times \mathrm{SO}(\Delta^+)$  is precisely  $\mathrm{Spin}(8)$  [2, (5)]. Hence a copy of  $S^7$  in  $\mathrm{Spin}(8)$  whose action on  $\mathbb{R}^8$  is by  $L_u$  must act on  $\Delta^+$  by  $L_u \circ R_u$ . If we choose the identification  $\Delta^+ \cong \mathbb{O}$  so that  $s_0$  corresponds to 1, then  $f_{\mathbb{O}}(u) = \tilde{L}_u s_0$  corresponds to  $u^2$ , so  $\deg f_{\mathbb{O}} = 2$ . Hence  $D(\varphi_{\mathbb{O}}, \varphi_{\mathrm{rd}}) = 2$ , and  $v(\varphi_{\mathbb{O}}) = -3$ .

**Example 3.7** The  $G_2$ -structure  $\varphi_{\mathrm{rd}}$  is invariant under the order-4 diffeomorphism given by scalar multiplication by  $i$  on  $S^7 \subset \mathbb{C}^4$  (since  $i\mathrm{Id} \in \mathrm{SU}(4) \subset \mathrm{Spin}(7)$ ) so descends to a  $G_2$ -structure  $\varphi_{\mathrm{rd}}/\mathbb{Z}_4$  on the quotient  $S^7/\mathbb{Z}_4$ . This is the boundary of the unit disc bundle of  $\mathcal{O}(-4)$  on  $\mathbb{CP}^3$  (the canonical bundle of  $\mathbb{CP}^3$ ), which has an  $\mathrm{SU}(4)$ -structure restricting to  $\varphi_{\mathrm{rd}}/\mathbb{Z}_4$  (indeed, the total space admits a Calabi–Yau metric asymptotic to  $\mathbb{C}^4/\mathbb{Z}_4$ ; see Calabi [9, Section 4]). The self-intersection number of a hyperplane in the zero section is  $-4$ , so  $\sigma = -1$ , and  $v(\varphi_{\mathrm{rd}}/\mathbb{Z}_4) = 4 + 3 = 7$ .

**Remark 3.8** While Example 3.7 illustrates that  $v$  itself is not multiplicative under covers, if  $\varphi$  and  $\varphi'$  are  $G_2$ -structures on the same closed spin 7-manifold  $M$  and  $p: \tilde{M} \rightarrow M$  is a degree  $k$  covering map then  $D(p^*\varphi, p^*\varphi') = kD(\varphi, \varphi')$ .

**Remark 3.9** The fact that  $\varphi_{\mathrm{rd}}$  and  $\widehat{\varphi}_{\mathrm{rd}}$  are both invariant under the antipodal map on  $S^7$  is not incompatible with  $D(\varphi_{\mathrm{rd}}, \widehat{\varphi}_{\mathrm{rd}})$  being odd, because the  $G_2$ -structures they define on  $\mathbb{RP}^7 = S^7/\pm 1$  induce different spin structures. The actions of  $\mathrm{Spin}(7)$  and the conjugate  $r\mathrm{Spin}(7)r$  on  $\mathbb{RP}^7$  can both be lifted to the spinor bundle. Since  $-1$  acts trivially on  $\mathbb{RP}^7$ , its image under either lift will be  $\pm\mathrm{Id}$ , and the two spin structures can be distinguished by which of the two lifts acts as  $+\mathrm{Id}$ .

Similarly,  $\varphi_{\mathrm{rd}}$  defines the same spin structure on  $\mathbb{RP}^7$  as the octonionic left multiplication parallelism of  $\mathbb{RP}^7$ , but not the right multiplication one. This is related to the fact that  $\mathrm{Spin}(7)$  can be described as the subgroup of  $\mathrm{SO}(8)$  generated by left multiplication by unit imaginary octonions, while the subgroup generated by right multiplications is a conjugate of  $\mathrm{Spin}(7)$  by a reflection.

## 4 $v$ of twisted connected sum $G_2$ -manifolds

Our motivation for introducing the invariant  $v$  is to give a tool for studying the homotopy classes of  $G_2$ -structures. We now show how the definition of  $v$  in terms of  $\mathrm{Spin}(7)$ -bordisms allows us to compute it for the large class of “twisted connected sum” manifolds with holonomy  $G_2$ . Before describing the twisted connected sums, we explain how to compute  $v$  of  $G_2$ -structures defined as products of structures on lower-dimensional manifolds. This is then used in the proof of Theorem 1.7, that the torsion-free  $G_2$ -structures of twisted connected sum  $G_2$ -manifolds always have  $v = 24$ .

## 4.1 $SU(3)$ and $SU(2)$ -structures

Let us first describe  $SU(3)$  and  $SU(2)$ -structures in terms of forms, along the lines of Section 2.1.

Let  $z^k = x^{2k-1} + ix^{2k}$  be complex coordinates on  $\mathbb{R}^6$ . Then the stabiliser in  $GL(6, \mathbb{R})$  of the pair of forms

$$\begin{aligned}\Omega_0 &= dz^1 \wedge dz^2 \wedge dz^3 \in \Lambda^3(\mathbb{R}^6)^* \otimes \mathbb{C}, \\ \omega_0 &= \frac{i}{2}(dz^1 \wedge d\bar{z}^1 + dz^2 \wedge d\bar{z}^2 + dz^3 \wedge d\bar{z}^3) \in \Lambda^2(\mathbb{R}^6)^*,\end{aligned}$$

is  $SU(3)$ . An  $SU(3)$ -structure  $(\Omega, \omega)$  on a 6-manifold induces a Riemannian metric, almost complex structure and orientation (the volume form is  $-\frac{i}{8}\Omega \wedge \overline{\Omega} = \frac{1}{6}\omega^3$ ). On  $\mathbb{R} \oplus \mathbb{R}^6$ ,

$$(21) \quad dt \wedge \omega_0 + \operatorname{Re} \Omega_0 \cong \varphi_0,$$

and  $SU(3)$  is exactly the stabiliser in  $G_2$  of a nonzero vector in  $\mathbb{R}^7$ . The product of a 6-manifold with  $SU(3)$ -structure and  $S^1$  or  $\mathbb{R}$  has a product  $G_2$ -structure, while the boundary of a 7-manifold with  $G_2$ -structure has an induced  $SU(3)$ -structure.

The stabiliser in  $GL(4, \mathbb{R})$  of the triple of forms

$$\omega_0^I = dx^{12} + dx^{34}, \quad \omega_0^J = dx^{13} - dx^{24}, \quad \omega_0^K = dx^{14} + dx^{23} \in \Lambda^2(\mathbb{R}^4)^*$$

is  $SU(2)$ . The stabiliser in  $SU(2)$  of a nonzero vector is clearly trivial, and the boundary of a 4-manifold  $W$  with  $SU(2)$ -structure  $(\omega_0^I, \omega_0^J, \omega_0^K)$  has a natural coframe defined by contracting each of the three 2-forms with an outward pointing normal vector field.

If  $e^1, e^2, e^3$  is a coframe on  $\mathbb{R}^3$  then

$$e^{123} + e^1 \wedge \omega_0^I + e^2 \wedge \omega_0^J + e^3 \wedge \omega_0^K \cong \varphi_0$$

on  $\mathbb{R}^3 \oplus \mathbb{R}^4$ . Therefore the product of a parallelised 3-manifold and a 4-manifold with  $SU(2)$ -structure has a natural product  $G_2$ -structure. Similarly, if we let  $\omega_1^I, \omega_1^J, \omega_1^K$  denote an equivalent triple of 2-forms on a second copy of  $\mathbb{R}^4$ , and  $\text{vol}_0 = \frac{1}{2}(\omega_0^I)^2$  etc, then

$$(22) \quad \text{vol}_0 + \omega_0^I \wedge \omega_1^I + \omega_0^J \wedge \omega_1^J + \omega_0^K \wedge \omega_1^K + \text{vol}_1 \cong \psi_0$$

on  $\mathbb{R}^4 \oplus \mathbb{R}^4$ , so the product of two 4-manifolds  $W_0, W_1$  with  $SU(2)$ -structures has a natural product  $\text{Spin}(7)$ -structure. If  $W_0$  is closed while  $\partial W_1$  is nonempty, clearly the  $G_2$ -structure induced on  $\partial(W_0 \times W_1)$  by this  $\text{Spin}(7)$ -structure equals the product of  $\omega_0^\bullet$  with the coframe on  $\partial W_1$  induced by  $\omega_1^\bullet$ .

## 4.2 Product $G_2$ -structures and spinors

Above we described two types of product  $G_2$ -structures. In order to compute  $\nu$  of such products, we shall need to describe  $SU(3)$  and  $SU(2)$  in terms of spinors.

The half-spin representations  $\Delta^\pm$  of  $\text{Spin}(6) \cong SU(4)$  are the standard 4-dimensional representation of  $SU(4)$  and its dual. The inclusion  $SU(3) \hookrightarrow SO(6)$  lifts to the obvious inclusion  $SU(3) \hookrightarrow SU(4)$ , so the stabiliser of a nonzero element in  $\Delta^+$  is exactly  $SU(3)$ . Hence, analogously to [Section 2.2](#),  $SU(3)$ -structures on a 6-manifold  $N$  compatible with a fixed spin structure and metric can be defined by positive unit spinor fields (which always exist and any two are homotopic since the real rank of  $S^+N$  is 8).

If  $N$  is the boundary of a spin 7-manifold  $M$ , then the half-spinor bundles on  $N$  are both isomorphic, as real vector bundles, to the restriction of the spinor bundle from  $M$ . Analogously to [Remark 2.3](#), the restrictions of  $G_2$ -structures on  $M$  to  $SU(3)$ -structures on  $N$  can be described equivalently in terms of differential forms or spinors. As there is no obstruction to extending a nonvanishing section of a rank 8 bundle on  $M$  from the boundary to the interior, it follows that any  $SU(3)$ -structure on  $N$  is induced as the boundary of a  $G_2$ -structure on  $M$ .

**Lemma 4.1** *If  $N$  is a 6-manifold with an  $SU(3)$ -structure  $(\Omega, \omega)$ , then the product  $G_2$ -structure  $\varphi = d\theta \wedge \omega + \text{Re } \Omega$  on  $S^1 \times N$  has  $\nu(\varphi) = 0$ .*

**Proof** Any spin 6-manifold  $N$  bounds some spin 7-manifold  $M$ , as the bordism group  $\Omega_6^{\text{Spin}}$  is trivial [29]. Then any product  $G_2$ -structure  $\varphi$  on  $S^1 \times N$  bounds a product  $\text{Spin}(7)$ -structure on  $S^1 \times M$ . The  $S^1$  factor makes

$$\sigma(S^1 \times M) = \chi(S^1 \times M) = 0,$$

so  $\nu(\varphi) = 0$ . □

Now we consider dimensions 3 and 4. Before looking at the spinors we prove a topological lemma.

**Lemma 4.2** *For any compact spin 4-manifold  $W$  with boundary  $Y$ ,*

$$\chi(W) \equiv \chi_2(Y) \pmod{2},$$

where  $\chi_2(Y)$  is the mod 2 semicharacteristic  $\sum_{i=0}^1 \dim H^i(Y; \mathbb{Z}_2)$ .

**Proof** Repeating the argument in the proof of (19) with  $\mathbb{Z}_2$ -coefficients instead of  $\mathbb{Q}$ -coefficients shows that there is a mod 2 identity

$$\chi(W) \equiv \dim H_0^2(W; \mathbb{Z}_2) + \chi_2(Y) \pmod{2},$$

where  $H_0^2(W; \mathbb{Z}_2)$  is the image of  $H^2(W, Y; \mathbb{Z}_2) \rightarrow H^2(W; \mathbb{Z}_2)$ . The intersection form of  $W$  defines a nonsingular bilinear form over  $\mathbb{Z}_2$  on  $H_0^2(W; \mathbb{Z}_2)$ . This injects as an orthogonal summand into the mod 2 intersection form of the manifold  $X := W \cup_{\text{Id}_Y} -W$ . Since  $X$  is a closed spin 4-manifold, its intersection form is even, and hence the form on  $H_0^2(W; \mathbb{Z}_2)$  is too. By [30, Chapter III, Lemma 1.1] the rank of every nonsingular even bilinear form over  $\mathbb{Z}_2$  is even, which completes the proof.  $\square$

The spin representations of  $\text{Spin}(4) \cong \text{SU}(2) \times \text{SU}(2)$  are the standard 2-dimensional complex representations of the two factors. Therefore the stabiliser of a nonzero positive spinor is one of the  $\text{SU}(2)$  factors, and a unit spinor field on a spin 4-manifold defines an  $\text{SU}(2)$ -structure.

The spin representation of  $\text{Spin}(3) \cong \text{SU}(2)$  is again the standard representation of  $\text{SU}(2)$ . The stabiliser of a nonzero spinor is trivial, so a unit spinor field defines a parallelism, ie a trivialisation of the tangent bundle. For a spin 4-manifold with boundary  $Y$ , the restriction of either the positive or negative spinor bundle to  $Y$  is isomorphic to the spinor bundle of  $Y$ . The analogue in dimension 4 of Corollary 2.5 is that

$$(23) \quad e_{\pm}(X) = \frac{3}{4}\sigma(X) \pm \frac{1}{2}\chi(X)$$

for any closed spin 4-manifold  $X$  (it suffices to check for  $X = S^4$  and  $K3$ ). Recall Rokhlin's theorem that  $\sigma(X)$  is divisible by 16.

**Lemma 4.3** *Let  $X$  be a closed 4-manifold with an  $\text{SU}(2)$ -structure  $(\omega^I, \omega^J, \omega^K)$  and  $Y$  a closed 3-manifold with a coframe field  $(e^1, e^2, e^3)$ . Then*

$$v(\varphi) = 24\chi_2(Y) \frac{\sigma(X)}{16} \pmod{48}$$

for the product  $G_2$ -structure  $\varphi = e^1 \wedge e^2 \wedge e^3 + e^1 \wedge \omega^I + e^2 \wedge \omega^J + e^3 \wedge \omega^K$  on  $Y \times X$ .

**Proof** Pick a spin coboundary  $W$  of  $Y$ . Let  $n_+(W, \pi)$  be the intersection number with the zero section of a positive spinor field on  $W$  whose restriction to  $Y$  is the defining spinor field of the parallelism  $\pi$  equivalent to the coframe field. We can apply connected sums with  $T^4$  or  $S^2 \times S^2$  to make  $n_+(W, \pi) = 0$  (this is the same argument as in Lemma 3.4), so we can assume that  $\pi$  bounds an  $\text{SU}(2)$ -structure on  $W$ .

If  $X$  has an  $SU(2)$ -structure then  $e_+(X) = 0$ , so (23) implies  $\chi(X) = -\frac{3}{2}\sigma(X)$ .  $W \times X$  is a  $Spin(7)$ -coboundary for  $\varphi$  so, applying Lemma 4.2 in the final step,

$$\begin{aligned} v(\varphi) &= \chi(W \times X) - 3\sigma(W \times X) = (-24\chi(W) - 48\sigma(W))\frac{\sigma(X)}{16} \\ &= 24\chi_2(Y)\frac{\sigma(X)}{16} \mod 48. \end{aligned} \quad \square$$

### 4.3 Twisted connected sums

Now we sketch the basics of the twisted connected sum construction, ignoring many details that are required to justify that the resulting  $G_2$ -structures are torsion-free (see [24; 10]). The construction starts from a pair of asymptotically cylindrical Calabi–Yau 3-folds  $V_{\pm}$ . We can think of these as a pair of (usually simply connected) 6-manifolds with boundary  $S^1 \times \Sigma_{\pm}$ , for  $\Sigma_{\pm}$  a K3 surface. They are equipped with  $SU(3)$ -structures  $(\omega_{\pm}, \Omega_{\pm})$  such that on a collar neighbourhood  $C_{\pm} \cong [0, 1) \times \partial V_{\pm}$  of the boundary

$$(24) \quad \begin{aligned} \omega_{\pm} &= dt \wedge du + \omega_{\pm}^I, \\ \Omega_{\pm} &= (du - idt) \wedge (\omega_{\pm}^J + i\omega_{\pm}^K), \end{aligned}$$

where  $u$  is the  $S^1$ -coordinate,  $t$  is the collar coordinate and  $(\omega_{\pm}^I, \omega_{\pm}^J, \omega_{\pm}^K)$  is an  $SU(2)$ -structure on  $\Sigma_{\pm}$ . The construction assumes that there is a diffeomorphism  $f: \Sigma_+ \rightarrow \Sigma_-$  such that

$$f^*\omega_+^I = \omega_+^J, \quad f^*\omega_+^J = \omega_+^I \quad \text{and} \quad f^*\omega_+^K = -\omega_+^K.$$

Now define  $G_2$ -structures on  $S^1 \times V_{\pm}$  by

$$\varphi_{\pm} = dv \wedge \omega_{\pm} + \operatorname{Re} \Omega_{\pm},$$

where  $v$  denotes the  $S^1$ -coordinate, and a diffeomorphism

$$\begin{aligned} F: \partial(S^1 \times V_+) &\cong S^1 \times S^1 \times \Sigma_+ \rightarrow S^1 \times S^1 \times \Sigma_- \cong \partial(S^1 \times V_-), \\ (v, u, x) &\mapsto (u, v, f(x)). \end{aligned}$$

In the collar neighbourhoods  $C_{\pm}$

$$\varphi_{\pm} = dv \wedge dt \wedge du + dv \wedge \omega_{\pm}^I + du \wedge \omega_{\pm}^J + dt \wedge \omega_{\pm}^K,$$

so  $\varphi_+$  and  $\varphi_-$  patch up to a well-defined  $G_2$ -structure  $\varphi$  on the closed manifold

$$(25) \quad M = (S^1 \times V_+) \cup_F (S^1 \times V_-).$$

One arranges that this  $G_2$ -structure can be perturbed to a torsion-free one. Because  $F$  swaps the circle factors at the boundary,  $M$  is simply-connected if  $V_+$  and  $V_-$  are.

## 4.4 A Spin(7)-bordism

We now proceed with the proof of [Theorem 1.7](#), that the twisted connected sum  $G_2$ -structures defined above always have  $\nu = 24$ . Consider the diffeomorphism

$$\tilde{F} = \text{Id} \times -\text{Id} \times f: S^1 \times S^1 \times \Sigma_+ \rightarrow S^1 \times S^1 \times \Sigma_-,$$

and the “untwisted connected sum”  $\tilde{M} = (S^1 \times V_+) \cup_{\tilde{F}} (S^1 \times V_-)$ . Then  $\tilde{M} = S^1 \times N$ , where  $N = V_+ \cup_{-\text{Id} \times f} V_-$ . Let  $r$  denote the right angle rotation  $(v, u) \mapsto (u, -v)$  of  $S^1 \times S^1$  and  $g := F \circ \tilde{F}^{-1}$ , and let  $T_r$  and  $T_g$  denote their mapping tori. Then  $g = r \times \text{Id}_\Sigma$ , so  $T_g \cong T_r \times \Sigma$ .

To compute  $\nu(\varphi)$  of the twisted connected sum  $G_2$ -structure  $\varphi$  on  $M$  and prove [Theorem 1.7](#) we will construct a Spin(7)-bordism  $W$  to product  $G_2$ -structures on  $\tilde{M} \sqcup T_g$ . Let

$$\begin{aligned} B_\pm &= \{(y - \frac{1}{2})^2 + t^2 < \frac{1}{16}\} \subset I \times S^1 \times C_\pm, \\ W_\pm &= I \times S^1 \times V_\pm \setminus B_\pm, \end{aligned}$$

where  $y$  denotes the  $I$ -coordinate, and  $t$  the collar coordinate on  $C_\pm \subset V_\pm$  as before.  $\partial W_\pm$  is a union of five pieces, meeting in edges at  $\{y\} \times S^1 \times S^1 \times \Sigma$  for  $y = 0, \frac{1}{4}, \frac{3}{4}$  and 1: a “top” and “bottom” piece each diffeomorphic to  $S^1 \times V_\pm$ ,  $[0, \frac{1}{4}] \times S^1 \times S^1 \times \Sigma_\pm$  and  $[\frac{3}{4}, 1] \times S^1 \times S^1 \times \Sigma_\pm$ , and  $E_\pm := \{(y - \frac{1}{2})^2 + t^2 = \frac{1}{4}\} \subset I \times S^1 \times C_\pm$ .

We form a “keyhole” bordism  $W$  by gluing some of these pieces: identify  $[0, \frac{1}{4}] \times S^1 \times S^1 \times \Sigma_\pm$  via  $\text{Id} \times \tilde{F}$ , and  $[\frac{3}{4}, 1] \times S^1 \times S^1 \times \Sigma_\pm$  via  $\text{Id} \times F$ . Then  $\partial W$  is a disjoint union  $M \sqcup \tilde{M} \sqcup T_g$ , where  $M$  is formed by gluing the top pieces of  $\partial W_+$  and  $\partial W_-$  and  $\tilde{M}$  by gluing the bottom pieces, while the keyhole boundary component  $E_+ \cup E_-$  can be identified with the mapping torus  $T_g$ .

It is easy to compute that  $H_1(T_r) \cong \mathbb{Z} \times \mathbb{Z}_2$ , so  $\chi_2(T_r) \equiv 1$ . Since  $\sigma(\Sigma) = -16$ , [Lemma 4.3](#) implies that any product  $G_2$ -structure on  $T_r \times \Sigma$  has  $\nu = 24$ , while a product  $G_2$ -structure on  $\tilde{M}$  has  $\nu = 0$ . To complete the calculation of  $\nu(\varphi)$  it remains to show that  $W$  does indeed admit a suitable Spin(7)-structure, and to compute the topological invariants of the Spin(7)-bordism  $W$ .

**Lemma 4.4**  $\chi(W) = 0$  and  $\sigma(W) = -16$ .

**Proof** For the Euler characteristic, we use the usual inclusion-exclusion formula. The spaces  $W_+$ ,  $W_-$  and  $W_+ \cap W_-$  all contain  $S^1$  factors, so

$$\chi(W) = \chi(W_+) + \chi(W_-) - \chi(W_+ \cap W_-) = 0.$$

For the signature, we must apply Wall's signature formula [32] because  $W$  is formed by gluing  $W_+$  and  $W_-$  along only parts of boundary components. The piece of the boundaries of  $W_+$  and  $W_-$  that we glue is  $X_0 = ([0, \frac{1}{4}] \sqcup [\frac{3}{4}, 1]) \times T^2 \times \Sigma$ . Let  $Z = \partial X_0 = \{0, \frac{1}{4}, \frac{3}{4}, 1\} \times T^2 \times \Sigma$  (the edges of  $\partial W_\pm$ ), and

$$X_\pm := \partial(W_\pm) \setminus X_0 = (\{0, 1\} \times S^1 \times V_\pm) \sqcup E_\pm,$$

where  $E_\pm$  are the keyhole pieces as defined above.

Throughout this proof we will use real coefficients for all cohomology groups. We need to identify the images  $A$ ,  $B$  and  $C$  in  $H^3(Z)$  of  $H^3(X_0)$ ,  $H^3(X_+)$  and  $H^3(X_-)$ , respectively; each is a Lagrangian subspace with respect to the intersection form  $(\cdot, \cdot)$  on  $H^3(Z)$ . The vector space  $K = A \cap (B + C) / ((A \cap B) + (A \cap C))$  admits the following natural nondegenerate symmetric bilinear form  $q$ : if  $a, a' \in A \cap (B + C)$  (representing  $[a], [a'] \in K$ ) and  $a' = b' + c'$ ,  $b' \in B$ ,  $c' \in C$ , then we set

$$q([a], [a']) := -(a, b').$$

Since  $W_\pm$  both have signature 0, the signature formula [32, Theorem, page 271] implies that the signature of  $W$  equals the signature of  $(K, q)$ .

We can identify  $Z_y := \{y\} \times T^2 \times \Sigma$  with  $S^1 \times \partial V_+$ . On  $Z_y$ , let  $v$  denote the coordinate on the  $S^1$  factor from  $S^1 \times V_+$ , and  $u$  the coordinate on the  $S^1$  factor in  $\partial V_+$ . Let  $\theta_+ = [dv]$  and  $\theta_- = [du] \in H^1(Z_y)$ . If  $w \in H^4(\Sigma)$  is positive then  $\theta_+ \wedge \theta_- \wedge w \in H^6(Z_y)$  is positive with respect to the orientation on  $Z_y$  given by the identification with  $S^1 \times \partial V_+$ . The orientation on  $Z$  that we should use to define its intersection form in the application of the signature formula is that induced as the boundary of  $X_+$ , ie

$$Z = Z_1 \sqcup -Z_{\frac{3}{4}} \sqcup Z_{\frac{1}{4}} \sqcup -Z_0.$$

Since the K3 surface  $\Sigma$  has no cohomology in odd degrees, the vector space  $H^3(Z)$  decomposes as the sum of 8 copies of  $L := H^2(\Sigma)$ : we let  $L_{y\pm}$  denote the image of  $L \rightarrow H^3(Z_y)$ ,  $\ell \mapsto \theta_\pm \wedge \ell$ . (This means for example that if  $\alpha_\pm \in H^2(V_\pm)$  then the restriction of  $[dv] \wedge \alpha_\pm \in H^3(W_\pm)$  to  $Z_y$  lies in  $L_{y+}$  for  $y = 0, \frac{1}{4}$ , and in  $L_{y\pm}$  for  $y = \frac{3}{4}, 1$ .) For  $h \in H^3(Z)$ , let  $h_{y\pm} \in L$  denote the  $L_{y\pm}$  component under this isomorphism. Then the intersection form on  $H^3(Z)$  is given in terms of the inner product  $\langle \cdot, \cdot \rangle$  on  $L$  by

$$(26) \quad \begin{aligned} (h, h') = & \langle h_{1+}, h'_{1-} \rangle - \langle h_{1-}, h'_{1+} \rangle - \langle h_{\frac{3}{4}+}, h'_{\frac{3}{4}-} \rangle + \langle h_{\frac{3}{4}-}, h'_{\frac{3}{4}+} \rangle \\ & + \langle h_{\frac{1}{4}+}, h'_{\frac{1}{4}-} \rangle - \langle h_{\frac{1}{4}-}, h'_{\frac{1}{4}+} \rangle - \langle h_{0+}, h'_{0-} \rangle + \langle h_{0-}, h'_{0+} \rangle. \end{aligned}$$

Let  $N_{\pm}$  denote the image of  $H^2(V_{\pm})$  in  $H^2(\Sigma) \cong L$ , and  $T_{\pm} \subset L$  the orthogonal complement. By the long exact sequence of the pair  $(V_+, S^1 \times \Sigma_+)$  and Poincaré–Lefschetz duality, the image of  $H^3(V_+)$  in  $H^3(S^1 \times \Sigma)$  is the annihilator of the image of  $H^2(V_+)$  under the intersection pairing, which equals  $[du] \wedge T_+$ . We find that

$$(27) \quad \begin{aligned} A &= \{h \in H^3(Z) \mid h_{0\pm} = h_{\frac{1}{4}\pm}, h_{\frac{3}{4}\pm} = h_{1\pm}\}, \\ B &= \{h \in H^3(Z) \mid h_{0+}, h_{1+} \in N_+, h_{0-}, h_{1-} \in T_+, h_{\frac{1}{4}\pm} = h_{\frac{3}{4}\pm}\}, \\ C &= \{h \in H^3(Z) \mid h_{0+}, h_{1-} \in N_-, h_{0-}, h_{1+} \in T_-, h_{\frac{1}{4}\pm} = \pm h_{\frac{3}{4}\mp}\}. \end{aligned}$$

Given an element of  $K$  represented by  $a = b + c$ , we can certainly find some  $h \in A \cap B$  with  $h_{1\pm} = b_{1\pm}$ . Replacing  $a$  by  $a - h$ , we may assume without loss of generality that  $b_{1\pm} = 0$ . Similarly we can assume  $c_{1\pm} = 0$ , and then  $a_{1\pm} = 0$  too. Setting

$$n := a_{0+}, \quad t := a_{0-}, \quad n_+ := b_{0+}, \quad t_+ := b_{0-}, \quad n_- := c_{0+} \quad \text{and} \quad t_- := c_{0-},$$

the remaining components are determined by (27). Thus we find that any element of  $K$  can be represented by  $a = b + c$  such that

$$a = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ n & t \\ n & t \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 0 \\ \frac{1}{2}(n+t) & \frac{1}{2}(-n+t) \\ \frac{1}{2}(n+t) & \frac{1}{2}(-n+t) \\ n_+ & t_+ \end{pmatrix}, \quad c = \begin{pmatrix} 0 & 0 \\ \frac{1}{2}(-n-t) & \frac{1}{2}(n-t) \\ \frac{1}{2}(n-t) & \frac{1}{2}(n+t) \\ n_- & t_- \end{pmatrix}$$

(where the top left matrix entry corresponds to the  $1+$  component etc), and

$$n_{\pm} \in N_{\pm}, \quad t_{\pm} \in T_{\pm}, \quad n = n_+ + n_-, \quad t = t_+ + t_-.$$

Representing a pair of classes  $[a], [a'] \in K$  by elements of that form, applying (26) and rearranging gives

$$(28) \quad \begin{aligned} 2q([a], [a']) &= -2(a, b') = -\langle n, -n' + t' \rangle + \langle t, n' + t' \rangle + \langle n, 2t'_+ \rangle - \langle t, 2n'_+ \rangle \\ &= \langle n, n' \rangle + \langle t, t' \rangle + \langle n, t'_+ - t'_- \rangle + \langle t, -n'_+ + n'_- \rangle. \end{aligned}$$

Now consider

$$\begin{aligned} K_0 &= \{[a] \in K \mid n \in N_+ \cap N_-, t \in T_+ + T_-\}, \\ K_{\pm} &= \{[a] \in K \mid n = t \in N_{\pm} \cap (T_+ + T_-)\}. \end{aligned}$$

If we use (28) to evaluate the product of two elements of  $K_0$ , then the cross terms  $\langle n, t' \rangle$  etc vanish, and  $q([a], [a']) = \langle n, n' \rangle + \langle t, t' \rangle = \langle n + t, n' + t' \rangle$ . Hence  $K_0$  is isometric to  $L$ , so it has signature  $-16$ .

If  $[a] \in K_+$ , then the RHS of (28) reduces to  $2\langle t, n'_- \rangle$ , which vanishes if  $[a'] \in K_0 + K_+$ . Similarly  $K_-$  is orthogonal to  $K_0 + K_-$ . This implies in particular that  $K_+$  and  $K_-$  are transverse, and since  $K_+ \oplus K_-$  is a sum of isotropic spaces it has signature 0.

Finally, note that  $K_+ \oplus K_-$  is a complement to  $K_0$  in  $K$ : given  $(n, t) \in (N_+ + N_-) \times (T_+ + T_-)$  we can certainly subtract an element of  $N_+ \cap N_-$  from  $n$  to ensure that  $n \in T_+ + T_-$ , and then an element of  $T_+ + T_-$  from  $t$  to ensure  $n = t$ . Hence the orthogonal complement to  $K_0$  is precisely  $K_+ \oplus K_-$ , and

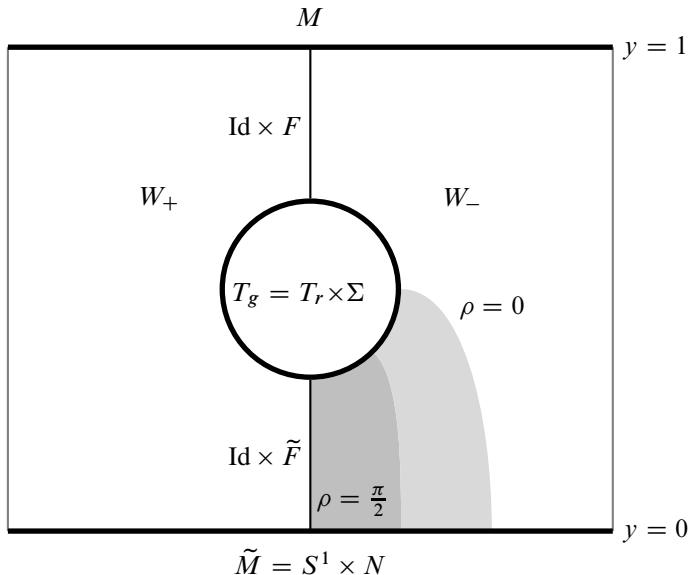
$$\sigma(W) = \sigma(K) = \sigma(K_0) + \sigma(K_+ \oplus K_-) = -16. \quad \square$$

To finish the proof of [Theorem 1.7](#), we need to exhibit a Spin(7)-structure on  $W$  with the right restrictions to the boundary components: the restriction to  $M$  should be the twisted connected sum  $G_2$ -structure  $\varphi$ , while the restrictions to  $\tilde{M} = S^1 \times N$  and  $T_g = T_r \times \Sigma$  should be product  $G_2$ -structures. We can define an SU(3)-structure on  $N$  as follows. Let  $V'_-$  be the complement of the collar neighbourhood  $C_- \subset V_-$ . On  $C_-$ , set

$$\begin{aligned} \omega' &= dt \wedge du + c_\rho \omega_-^I + s_\rho \omega_-^J, \\ \Omega' &= (du - i dt) \wedge (c_\rho \omega_-^J - s_\rho \omega_-^I + i \omega_-^K), \end{aligned}$$

where  $c_\rho = \cos \rho, s_\rho = \sin \rho$  for a smooth function  $\rho$  supported on  $C_-$ , such that  $\rho = \pi/2$  on  $\partial V_-$ . Take  $\tilde{\omega}$  to be  $\omega_+$  on  $V_+$ ,  $\omega'$  on  $C_-$ , and  $\omega_-$  on  $V'_-$ , and define  $\tilde{\Omega}$  analogously. Then  $(\tilde{\omega}, \tilde{\Omega})$  is a well-defined SU(3)-structure on  $N$ , and  $\tilde{\varphi} = d\theta \wedge \tilde{\omega} + \text{Re } \tilde{\Omega}$  is a product  $G_2$ -structure on  $\tilde{M}$ . Next we define the Spin(7)-structure  $\psi$  on  $W$ . Let  $y$  be the  $I$  coordinate on each half. First, define  $\rho$  on  $I \times C_-$  to be  $\pi/2$  on a neighbourhood of  $[0, \frac{1}{4}] \times \partial V_-$  and have compact support in  $[0, \frac{1}{2}] \times C_-$  (see [Figure 1](#)), and use this to define forms  $\tilde{\omega}$  and  $\tilde{\Omega}$  on  $I \times V_-$ . Since  $dy$  is a global covector field on  $W_0$ , defining a Spin(7)-structure is equivalent to defining a  $G_2$ -structure on each slice  $y = \text{const}$ . Take this to be  $\varphi_+ = d\theta \wedge \omega_+ + \text{Re } \Omega_+$  on  $\{y\} \times S^1 \times V_+$ , and  $d\theta \wedge \tilde{\omega} + \text{Re } \tilde{\Omega}$  on  $\{y\} \times S^1 \times V_-$ . Then the restriction of  $\psi$  to the boundary components  $M$  and  $\tilde{M}$  are  $\varphi$  and  $-\tilde{\varphi}$  respectively, as desired.

Finally we show that the restriction of  $\psi$  to the “keyhole” boundary component  $T_g = E_+ \cup E_-$  is a product  $G_2$ -structure too. We first outline the argument, starting from  $E_\pm \cong [0, \pm\pi] \times S^1 \times S^1 \times \Sigma_\pm$  (the first factor corresponding to one half of the circle  $\{(y - \frac{1}{2})^2 + t^2 = \frac{1}{16}\}$ ) being embedded as a product inside  $I \times C_\pm$ . The restriction of  $\psi$  to  $I \times C_\pm$  is a product of two SU(2)-structures, so the induced  $G_2$ -structure on  $E_\pm$  is a product of a coframe field on  $[0, \pm\pi] \times S^1 \times S^1$  and an SU(2)-structure on  $\Sigma$ . The coframes on the two copies of  $[0, \pm\pi] \times S^1 \times S^1$  patch up to a coframe on their

Figure 1: The “keyhole” bordism  $W$ 

union  $T_r$ , and the  $G_2$ -structure on  $T_g$  is the product of that with an  $SU(2)$ -structure on  $\Sigma$ .

In order to fill in the details of this sketch we need to write down the structures explicitly, which is rather cumbersome. To make the notation slightly more manageable we will use a complex form as a shorthand for an ordered pair of real forms, so that an  $SU(2)$ -structure can be defined by one complex and one real 2-form, or a coframe field on a 3-manifold by one complex and one real 1-form. Also, we identify both  $\Sigma_+$  and  $\Sigma_-$  with a standard K3 surface  $\Sigma$ , so that  $f$  corresponds to  $\text{Id}_\Sigma$ . Setting  $y = -\frac{1}{2}c_\alpha + \frac{1}{2}$ ,  $t = \frac{1}{2}s_\alpha$  for  $\alpha \in [0, \pi]$  lets us identify  $E_+ \subset I \times C_+$  with  $[0, \pi] \times S^1 \times S^1 \times \Sigma$ . On  $I \times C_+$ ,  $\psi$  is the product of the  $SU(2)$ -structure

$$(29) \quad ((dy - i dt) \wedge (dv + i du), dy \wedge dt - dv \wedge du)$$

on  $I \times [0, 1] \times S^1 \times S^1$  and  $(\omega_+^I + i \omega_+^J, \omega_+^K)$  on  $\Sigma$ . The induced  $G_2$ -structure on  $E_+$  is given by contraction with the normal vector field  $c_\alpha \frac{\partial}{\partial y} - s_\alpha \frac{\partial}{\partial t}$ . The result is the product of the same  $SU(2)$ -structure on  $\Sigma$  with the coframe field  $(e^{i\alpha}(dv + i du), \frac{1}{2}d\alpha)$  on  $[0, \pi] \times S^1 \times S^1$ .

Similarly, for  $\alpha \in [\pi, 2\pi]$  we set  $y = -\frac{1}{2}c_\alpha + \frac{1}{2}$ ,  $t = -\frac{1}{2}s_\alpha$  to identify  $[\pi, 2\pi] \times S^1 \times S^1 \times \Sigma_- \cong E_-$ . On  $I \times C_-$ , the restriction of  $\psi$  is given by the product of (29) on  $I \times [0, 1] \times S^1 \times S^1$  and  $(e^{-i\rho}(\omega_-^I + i \omega_-^J), \omega_-^K)$  on the tangent space to the  $\Sigma$  factor. Contracting with the normal vector field  $c_\alpha \frac{\partial}{\partial y} + s_\alpha \frac{\partial}{\partial t}$  gives the coframe

$(e^{-i\alpha}(dv + i du), -\frac{1}{2}d\alpha)$  on  $[\pi, 2\pi] \times S^1 \times S^1$ . Now, as product  $G_2$ -structures,

$$\begin{aligned} (e^{-i\alpha}(dv + i du), -\frac{1}{2}d\alpha) \cdot (e^{-i\rho}(\omega_-^I + i\omega_-^J), \omega_-^K) \\ = (e^{i(\rho-\alpha)}(dv + i du), -\frac{1}{2}d\alpha) \cdot (\omega_-^I + i\omega_-^J, \omega_-^K) \\ = (e^{i(\alpha-\rho)}(du + i dv), \frac{1}{2}d\alpha) \cdot (\omega_+^I + i\omega_+^J, \omega_+^K). \end{aligned}$$

$T_g$  is formed by gluing the boundaries of  $[0, \pi] \times S^1 \times S^1 \times \Sigma$  and  $[\pi, 2\pi] \times S^1 \times S^1 \times \Sigma$  using  $(\pi, v, u, x) \mapsto (\pi, u, v, x)$  and  $(0, v, u, x) \mapsto (2\pi, v, -u, x)$ . These maps preserve the  $SU(2)$ -structure on the  $\Sigma$  factor, and match up the coframes  $(e^{i\alpha}(dv + i du), \frac{1}{2}d\alpha)$  and  $(e^{i(\alpha-\rho)}(du + i dv), \frac{1}{2}d\alpha)$  to a well-defined coframe on  $T_r$  (since  $\rho = 0$  at  $\alpha = \pi$  and  $\rho = \pi/2$  at  $\alpha = 0, 2\pi$ ). Thus the  $G_2$ -structure on  $T_g = T_r \times \Sigma$  is a product, completing the proof of [Theorem 1.7](#).  $\square$

## 4.5 Orbifold resolutions

For some of Joyce's examples of compact  $G_2$ -manifolds constructed by resolving flat orbifolds, the torsion-free  $G_2$ -structures are homotopic to twisted connected sum  $G_2$ -structures, and thus have  $v = 24$ . It is proved in [\[25\]](#) that in some cases there is even a connecting path of torsion-free  $G_2$ -structures, but that is irrelevant for the calculation of  $v$ .

We have no general technique for computing  $v$  of orbifold resolution  $G_2$ -manifolds. We note, however, that a small number of examples have  $b_2(M) + b_3(M)$  even, eg Joyce [\[23, Section 12.8.4\]](#). Those  $G_2$ -manifolds have  $\chi_{\mathbb{Q}}(M)$ —and hence  $v$ —odd.

# 5 The $h$ -principle for coclosed $G_2$ -structures

We now prove [Theorem 1.8](#), that coclosed  $G_2$ -structures satisfy the  $h$ -principle. We first set up some notation, continuing from [Section 2.1](#).

## 5.1 Positive 4-forms

For a vector space  $V$  of dimension 7, let  $\Lambda_+^3 V^*$  and  $\Lambda_+^4 V^*$  denote the space of forms equivalent to  $\varphi_0$  (as defined in [\(13\)](#)) and  $*\varphi_0$  respectively. These are *open* subsets of the spaces of forms. Any  $\varphi \in \Lambda_+^3 V^*$  defines a  $G_2$ -structure, and thus an inner product and orientation, and a Hodge star operator. This gives a nonlinear map  $\Lambda_+^3 V^* \rightarrow \Lambda_+^4 V^*$ ,  $\varphi \mapsto *\varphi$ , which is two-to-one. The stabiliser of a  $\sigma \in \Lambda_+^4 V^*$  is isomorphic to  $G_2 \times \{\pm 1\}$ , so  $\sigma$  together with a choice of orientation on  $V$  determines a  $G_2$ -structure [\[4, Section 2.8.3\]](#).

We say that a  $G_2$ -structure on a 7-manifold  $M$ , defined by a positive 3-form  $\varphi \in \text{Sec } \Lambda_+^3(M)$ , is coclosed if the associated 4-form  $\sigma = * \varphi \in \text{Sec } \Lambda_+^4(M)$  is closed. The set of coclosed  $G_2$ -structures on an oriented manifold  $M$  is therefore the same as the space of closed positive 4-forms  $\text{Clo } \Lambda_+^4(M) \subset \text{Sec } \Lambda_+^4(M)$ . (Each section induces a spin structure, and the space  $\mathcal{G}_2^{cc}(M)$  appearing in the statement of [Theorem 1.8](#) is a subset of  $\text{Clo } \Lambda_+^4(M)$  compatible with a fixed spin structure on  $M$ .)

## 5.2 Microextension

It is generally easier to prove  $h$ -principles for partial differential relations on open manifolds than on closed manifolds. The Hirsch microextension trick is the strategy to prove  $h$ -principles on closed manifolds by reducing the problem to an  $h$ -principle on an open manifold of higher dimension.

In order to apply the microextension trick, we consider 4-forms on 8-manifolds such that the restriction to every hypersurface is a positive 4-form. The key point that makes the argument work is that not only is the set of such forms open, but moreover any positive 4-form from a hypersurface can be extended this way. This is the feature that enables us to prove the  $h$ -principle for coclosed  $G_2$ -structures on closed manifolds, but not for, say, symplectic structures or closed  $G_2$ -structures.

**Definition 5.1** For a vector space  $W$  of dimension 8, let

$$\mathcal{R}(W) = \{\chi \in \Lambda^4 W^* \mid \chi|_V \in \Lambda_+^4 V^* \text{ for every hyperplane } V \subset W\}.$$

If  $W = V \oplus \mathbb{R}$  and  $\varphi \in \Lambda_+^3 V^*$  then the invariance of  $\psi = dt \wedge \varphi + * \varphi$  under  $\text{Spin}(7)$  (see [\(14\)](#)), which acts transitively on the hyperplanes, shows that  $\psi \in \mathcal{R}(W)$ .

**Lemma 5.2**  $\mathcal{R}(W)$  is open in  $\Lambda^4 W^*$ .

**Proof** Let  $G \cong \mathbb{RP}^7$  denote the Grassmannian of hyperplanes in  $W$ , and  $\pi: \mathcal{V} \rightarrow G$  the tautological bundle. If  $f: \pi^{-1}(U) \rightarrow U \times \mathbb{R}^7$  is a local trivialisation, then  $\Lambda^4 W^* \times U \rightarrow \Lambda^4(\mathbb{R}^7)^*$ ,  $(\chi, V) \mapsto f_{V*}(\chi|_V)$  is continuous, so the preimage of  $\Lambda_+^4(\mathbb{R}^7)^*$  is open. Hence if  $\chi \in \mathcal{R}(W)$  then for each  $V \in G$  there are open neighbourhoods  $B_V \subset \Lambda^4 W^*$  of  $\chi$  and  $C_V \subset G$  of  $V$  such that  $\chi'_{|V'} \in \Lambda_+^4 V'^*$  for each  $\chi' \in B_V$  and  $V' \in C_V$ . Since  $G$  is compact it can be covered by  $C_{V_1}, \dots, C_{V_k}$  for finitely many  $V_1, \dots, V_k \in G$ . Then  $B_{V_1} \cap \dots \cap B_{V_k}$  is an open neighbourhood of  $\chi$  in  $\Lambda^4 W^*$  and contained in  $\mathcal{R}(W)$ .  $\square$

For an 8–manifold  $N$ , let  $\mathcal{R}(N) \subset \Lambda^4(N)$  be the subbundle with fibres  $\mathcal{R}(T_x N) \subset \Lambda^4 T_x^* N$ . Let  $\text{Clo } \mathcal{R}(N) \subset \text{Sec } \mathcal{R}(N)$  denote the subspace of closed 4–forms, and  $\text{Clo}_a \mathcal{R}(N)$  the subspace of forms representing a fixed cohomology class  $a \in H_{\text{dR}}^4(N)$ . Because the subbundle  $\mathcal{R}(N) \subset \Lambda^4(N)$  is open and invariant under the natural action of  $\text{Diff}(N)$ , [15, Theorem 10.2.1] immediately implies that  $\text{Clo}_a \mathcal{R}(N) \hookrightarrow \text{Sec } \mathcal{R}(N)$  is a homotopy equivalence if  $N$  is an open manifold.

### 5.3 The proof of Theorem 1.8

We prove the following stronger version of Theorem 1.8.

**Theorem 5.3** *Let  $M$  be a closed 7–manifold. Let  $I^k \rightarrow \text{Sec } \Lambda_+^4(M)$ ,  $s \mapsto \sigma_s$  and  $I^k \rightarrow H_{\text{dR}}^4(M)$ ,  $s \mapsto a_s$  be families such that  $\sigma_s \in \text{Clo}_{a_s} \Lambda_+^4(M)$  for all  $s \in \partial I^k$ . Then the family  $\sigma_s$  is homotopic in  $\text{Sec } \Lambda_+^4(M)$ , relative to  $\partial I^k$ , to a family  $\sigma'_s$  such that  $\sigma'_s \in \text{Clo}_{a_s} \Lambda_+^4(M)$  for all  $s \in I^k$ .*

In particular:

- $\text{Clo } \Lambda_+^4(M) \hookrightarrow \text{Sec } \Lambda_+^4(M)$  is a homotopy equivalence.
- $\text{Clo}_a \Lambda_+^4(M) \hookrightarrow \text{Sec } \Lambda_+^4(M)$  is a homotopy equivalence for each fixed  $a \in H_{\text{dR}}^4(M)$ .

**Proof** Identify  $\sigma_s$  with its pull-back to  $M \times \mathbb{R}$ , and let

$$\chi_s = \sigma_s + dt \wedge * \sigma_s - td(*\sigma_s) \in \text{Sec } \Lambda^4(M \times \mathbb{R}).$$

Then there is  $\epsilon > 0$  such that  $\chi_s$  takes values in  $\mathcal{R}$  over  $N := M \times (-\epsilon, \epsilon)$  for all  $s \in I^k$ , and  $\chi_s \in \text{Clo}_{a_s} \mathcal{R}(N)$  for  $s \in \partial I^k$ . If  $a_s \equiv a$  is constant in  $s$  then it follows immediately from [15, Theorem 10.2.1] that the family  $\chi_s$  is homotopic in  $\text{Sec } \mathcal{R}(N)$ , relative to  $\partial I^k$ , to a family  $\chi'_s \in \text{Clo}_a \mathcal{R}(N)$ . If we set  $\sigma'_s = \chi'_{s|M}$  then  $\sigma'_s \in \text{Clo}_a \Lambda_+^4(M)$  for all  $s \in I^k$ , and the restriction to  $M$  of the homotopy from  $\chi$  to  $\chi'$  gives a homotopy from  $\sigma$  to  $\sigma'$  in  $\text{Sec } \Lambda_+^4(M)$ .

The proof of [15, Theorem 10.2.1] builds on [15, Proposition 4.7.4], which is stated for the case when  $a_s$  is constant. However, the proof still works if  $a_s$  is allowed to depend on  $s$  (see [15, Exercise in Section 10.2]).  $\square$

## 6 The action of spin diffeomorphisms on $\pi_0 \mathcal{G}_2(M)$

Let  $(M, \varphi)$  be a closed connected spin 7–manifold with  $G_2$ –structure. In this section we investigate the action of the group of spin diffeomorphisms of  $M$  on the set of homotopy classes of  $G_2$ –structures on  $M$ :

$$\pi_0 \mathcal{G}_2(M) \times \text{Diff}(M) \rightarrow \pi_0 \mathcal{G}_2(M), \quad ([\varphi], f) \mapsto [f^* \varphi].$$

The quotient is the set  $\pi_0 \bar{\mathcal{G}}_2(M)$  of deformation classes of  $G_2$ -structures. To determine the action for a specific spin diffeomorphism  $f: M \cong M$  amounts to computing the difference class  $D(\varphi, f^*\varphi) \in \mathbb{Z}$ . The existence of the  $v$ -invariant ensures that  $D(\varphi, f^*\varphi) = 24k$  for some integer  $k$ . In this section we relate the possible values of  $k$  to the topology of  $M$  and in particular  $p_M \in H^4(M)$ . At the end we provide the general definition of the  $\xi$ -invariant.

## 6.1 The spin characteristic class $p_M$

Recall that the classifying space  $B \text{Spin}$  is 3-connected and  $\pi_4(B \text{Spin}) \cong \mathbb{Z}$ . It follows that  $H^4(B \text{Spin}) \cong \mathbb{Z}$  is infinite cyclic. A generator is denoted  $\pm \frac{p_1}{2}$  and the notation is justified since for the canonical map  $\pi: B \text{Spin} \rightarrow B \text{SO}$  we have  $\pi^* p_1 = 2 \frac{p_1}{2}$ , where  $p_1$  is the first Pontrjagin class. Given a spin manifold  $X$ , we write

$$p_X := \frac{p_1}{2}(X) \in H^4(X).$$

The following lemma is well known to experts.

**Lemma 6.1** [11, Lemma 2.2(i)] *For a closed spin 7-manifold  $M$ ,  $p_M \in 2H^4(M)$ .*

For later use, we recall from [Section 1](#) that  $d_\pi$  denotes the greatest divisor of  $p_M$  modulo torsion, while  $d_o := \text{Max}\{s \mid s, m \in \mathbb{Z}, m^2 s \text{ divides } mp_M\}$ ; we set  $d_\pi = d_o = 0$  if  $p_M$  is torsion. Both are even by [Lemma 6.1](#).

**Example 6.2** If  $H^4(M) \cong \mathbb{Z} \oplus \mathbb{Z}_4$  and  $p_M \mapsto (8, 2)$  then  $d_\pi = 8$  while  $d_o = 4$ .

## 6.2 Translations of $G_2$ -structures and mapping tori

Given  $(M, \varphi)$  and a spin diffeomorphism  $f: M \cong M$ , we wish to calculate the difference element  $D(\varphi, f^*\varphi) \in \mathbb{Z}$ . Note that (given  $\varphi$ ) the homotopy class  $[f^*\varphi] \in \pi_0 \mathcal{G}_2(M)$  depends only on the pseudoisotopy class of  $f$ . For suppose that  $F$  is pseudoisotopy between diffeomorphisms  $f_0$  and  $f_1$ , ie a diffeomorphism  $F: M \times I \cong M \times I$  such that  $F|_{M \times \{i\}} = f_i$  for  $i = 0, 1$ . Then contracting the pull-back  $F^*\psi$  of the product  $\text{Spin}(7)$ -structure  $\psi = dt \wedge \varphi + * \varphi$  with  $\frac{\partial}{\partial t}$  and restricting to the slices  $M \times \{t\}$  defines a homotopy between  $f_0^*\varphi$  and  $f_1^*\varphi$ . On the other hand, [Proposition 6.3](#) shows that  $D(\varphi, f^*\varphi)$  does not depend upon the  $G_2$ -structure  $\varphi$ . Hence we obtain a well-defined function

$$D_M: \tilde{\pi}_0 \text{Diff}(M) \rightarrow \mathbb{Z}, \quad [f] \mapsto D_M(f) := D(\varphi, f^*\varphi),$$

where  $\tilde{\pi}_0 \text{Diff}(M)$  denotes the group of pseudoisotopy classes of spin diffeomorphisms of  $M$ .

The integer  $D_M(f)$  measures the translation action of  $f$  on the set of homotopy classes of  $G_2$ -structures. Next we show how to calculate  $D_M(f)$  using the mapping torus of  $f$ :

$$T_f := (M \times [0, 1]) / (x, 0) \sim (f(x), 1).$$

Since  $f$  is a spin diffeomorphism the closed 8-manifold  $T_f$  admits a spin structure. We choose a spin structure and let  $T_f$  to denote the corresponding 8-dimensional spin manifold: no confusion shall arise since we are interested only in the characteristic number

$$p^2(f) := \langle p_{T_f}^2, [T_f] \rangle \in \mathbb{Z},$$

which depends only on the oriented diffeomorphism type of  $T_f$  since  $2p_{T_f} = p_1(T_f)$  and  $H^8(T_f) \cong \mathbb{Z}$  (in fact  $p_{T_f}$  is independent of the choice of spin structure by Čadek, Crabb and Vanžura [8, page 170]). Therefore  $p^2(f)$  is an invariant of the pseudoisotopy class of  $f$  and we define the function

$$p^2: \tilde{\pi}_0 \text{Diff}(M) \rightarrow \mathbb{Z}, \quad [f] \mapsto p^2(f).$$

The following proposition proves [Proposition 1.10](#) and shows how the mapping torus  $T_f$  can be used to compute the difference class  $D(\varphi, f^*\varphi)$ .

**Proposition 6.3** *The function  $D_M: \tilde{\pi}_0 \text{Diff}(M) \rightarrow \mathbb{Z}$  is a homomorphism given by*

$$D(\varphi, f^*\varphi) = \frac{3 \cdot p^2(f)}{28} = 24\hat{A}(T_f).$$

**Proof** From the definition of  $D(\varphi, \varphi')$  in [Section 3](#) it is clear that  $D(f^*\varphi, f^*\varphi') = D(\varphi, \varphi')$  for any spin diffeomorphism  $f$  and any pair of  $G_2$ -structures  $\varphi$  and  $\varphi'$  on  $M$ . Now for two spin diffeomorphisms  $f_0, f_1: M \cong M$ , the affine property (5) of  $D$  gives

$$D(\varphi, (f_1 \circ f_0)^*\varphi) = D(\varphi, f_0^*\varphi) + D(f_0^*\varphi, f_0^*(f_1^*\varphi)) = D(\varphi, f_0^*\varphi) + D(\varphi, f_1^*\varphi).$$

This shows that  $D_M$  is a homomorphism.

Turning to the mapping torus, we can use [Lemma 1.5](#) to compute  $D(\varphi, f^*\varphi)$  by treating the product  $M \times [0, 1]$  together with the embeddings  $(\text{Id}, 0)$  and  $(f, 1): M \hookrightarrow M \times [0, 1]$  as a  $\text{Spin}(7)$ -bordism  $W_f$  from  $(M, f^*\varphi)$  to  $(M, \varphi)$ . Clearly the manifold  $\overline{W}_f$  obtained by closing up the bordism as in [\(20\)](#) is nothing other than the mapping torus  $T_f$ , so [\(6\)](#) gives

$$D(f^*\varphi, \varphi) = -e_+(\overline{W}_f) = -e_+(T_f).$$

By [Proposition 2.4](#),  $e_+(T_f) = \frac{1}{16}(4p_{T_f}^2 - 4p_2 + 8e)$  and using the signature theorem to eliminate  $p_2$  from this equation we have

$$D(\varphi, f^*\varphi) = e_+(T_f) = \frac{3p_{T_f}^2}{28} - \frac{45\sigma(T_f)}{28} + \frac{\chi(T_f)}{2}.$$

Since  $T_f$  is a mapping torus both  $\sigma(T_f)$  and  $\chi(T_f)$  vanish which proves the first equality of the proposition. Now the second equality follows from [Corollary 2.5](#).  $\square$

Since [Proposition 6.3](#) determines  $D_M$  in terms of  $p^2$ , the proofs of Theorems 1.11 and 1.12 are completed by quoting the following result. Here  $b_M$  denotes the torsion linking form on  $\text{Tor } H^4(M)$ .

**Theorem 6.4** [11, Definition 4.4 and Corollary 4.17(iv)] *For any spin 7-manifold  $M$ , there is an  $r \in \{0, 1, 2\}$  depending only on  $(H^4(M), b_M, p_M)$  such that*

$$(30) \quad p^2(\text{Diff}(M)) \subseteq \text{lcm}(224, 2^r d_o(M))\mathbb{Z},$$

with equality if  $M$  is 2-connected.

The next subsection summarises some ingredients of the proof of this theorem. However, before we do so let us prove an elementary special case of (30) in order to make the appearance of  $d_o(M)$  less mysterious.

**Lemma 6.5** *Let  $M$  be a closed spin 7-manifold and  $f$  a spin diffeomorphism of  $M$ . Then*

$$(31) \quad p^2(f) \in \text{lcm}(224, d_o(M))\mathbb{Z}.$$

**Proof** First recall (8): For a closed 8-dimensional spin manifold  $X$ , combining the definitions (16) of the  $L$ -genus and the  $\hat{A}$ -genus gives

$$p_X^2 - \sigma(X) = 8 \cdot 28 \hat{A}(X).$$

Since the mapping torus  $T_f$  is a closed 8-dimensional spin manifold with  $\sigma(T_f) = 0$  we deduce that

$$(32) \quad p_{T_f}^2 \in 8 \cdot 28 \cdot \mathbb{Z}.$$

From the definition of  $d_o$  there is a positive integer  $m$  such that  $m^2 d_o$  divides  $m p_M$ . Applying [Lemma 6.6](#) below with  $x = m p_{T_f}$  and  $s = m^2 d_o(M)$  gives that  $m^2 d_o(M)$  divides  $m^2 p_{T_f}^2$  and hence

$$p_{T_f}^2 \in d_o(M) \cdot \mathbb{Z}. \quad \square$$

**Lemma 6.6** Let  $T_f$  be the mapping torus of  $f: M \cong M$  and  $i: M \rightarrow T_f$  the inclusion. If  $x \in H^4(T_f)$  and  $s \in \mathbb{Z}$  divides  $i^*x$  then  $s$  divides  $x^2 \in H^8(T_f) \cong \mathbb{Z}$ .

**Proof** Consider the following fragment of the long exact cohomology sequence for the mapping torus  $T_f$  with  $\mathbb{Z}_s$  coefficients:

$$H^3(M; \mathbb{Z}_s) \xrightarrow{\text{Id} - f^*} H^3(M; \mathbb{Z}_s) \xrightarrow{\partial} H^4(T_f; \mathbb{Z}_s) \xrightarrow{i^*} H^4(M; \mathbb{Z}_s) \xrightarrow{\text{Id} - f^*} H^4(M; \mathbb{Z}_s).$$

For a space  $X$ , let  $\rho_s: H^*(X) \rightarrow H^*(X; \mathbb{Z}_s)$  denote reduction mod  $s$ . By assumption  $i^*\rho_s(x) = 0$  and so  $\rho_s(x)$  lies in the image of  $\partial$ . But the cup-product

$$H^4(T_f; \mathbb{Z}_s) \times H^4(T_f; \mathbb{Z}_s) \rightarrow \mathbb{Z}_s$$

vanishes on  $\text{Im}(\partial)$ . Hence  $\rho_s(x)^2 = \rho_s(x^2) = 0 \in H^8(T_f; \mathbb{Z}_s)$  and so  $s$  divides  $x^2$ .  $\square$

### 6.3 Diffeomorphisms of spin 7–manifolds

We shall now summarise the main ideas of the proof of [Theorem 6.4](#) from [11]. For this recall that an almost-diffeomorphism is a homeomorphism that is smooth away from a finite set of points, and we denote the group of almost-diffeomorphisms of a spin manifold  $M$  that preserve the spin structure by  $\text{ADiff}(M)$ . Below we recall the technical notion of a Gauss refinement from [11, Section 2.5] and how it detects aspects of the action of diffeomorphisms and almost-diffeomorphisms. This leads to a generalisation of the Eells–Kuiper invariant [11, Section 2.6], and in the next subsection we use these ideas in the general definition of the  $\xi$ –invariant of a  $G_2$ –structure.

Let  $M$  be a closed spin 7–manifold as usual. We can associate to it the invariants  $p_M$ ,  $b_M$  and  $q_M^\circ$ , where  $b_M$  is the torsion linking form on  $\text{Tor } H^4(M)$ , and  $q_M^\circ$  is a “family of quadratic refinements” of  $b_M$  [11, Section 2.4]. Group isomorphisms  $F$  act naturally on these objects by pull-backs, eg  $F^* p_M$  is simply  $F^{-1}(p_M)$ . For any spin diffeomorphism  $f$  of  $M$ , the induced action  $f^*$  on  $H^4(M)$  preserves these invariants, ie

$$(f^*)^* p_M = p_M, \quad (f^*)^* b_M = b_M, \quad (f^*)^* q_M^\circ = q_M^\circ;$$

in fact, this remains true even if  $f$  is merely an almost-diffeomorphism or even a homeomorphism [11, Theorem 1.2]. We define a function

$$P: \text{Aut}(H^4(M), b_M, p_M) \rightarrow \mathbb{Z}/2d_\pi(M)\mathbb{Z}$$

as follows [11, (39)]. Let

$$(33) \quad S_{d_\pi} := \{k \in H^4(M) \mid p_M - d_\pi k \text{ is torsion}\}.$$

For  $F \in \text{Aut}(H^4(M), b_M, p_M)$ , pick  $k \in S_{d_\pi}$ , let  $t := F(k) - k$ , and

$$P(F) := d_\pi^2 b_M(t, t) - 2d_\pi b_M(p_M - d_\pi k, t) \pmod{2d_\pi(M)\mathbb{Z}}.$$

Then [11, Proposition 4.16] states that

$$(34) \quad p^2(f) = P(f^*) \pmod{2d_\pi(M)}.$$

Meanwhile [11, (42)] states that

$$(35) \quad \text{Im } P = 2^r d_o(M)\mathbb{Z} / 2d_\pi(M)\mathbb{Z}$$

for some  $r(H^4(M), b_M, p_M) \in \{0, 1, 2\}$ , and  $r = 1$  unless  $H^4(M)$  has 2-torsion. Combined with (32) this implies (30), and hence Theorem 1.11.

Further, if  $M$  is 2-connected then [11, Proposition 3.10] states that there exist  $f \in \text{ADiff}(M)$  with  $f^* = \text{Id}$  on  $H^4(M)$  and  $p^2(f) = 2\tilde{d}_\pi n$  for any  $n \in \mathbb{Z}$ ; as in Section 1,  $\tilde{d}_\pi := \text{lcm}(4, d_\pi)$ . It is well known that  $f$  is pseudoisotopic to a diffeomorphism if  $p^2(f)$  is divisible by 224 [11, Lemma 3.7(iii)], so one can find  $f \in \text{Diff}(M)$  such that  $p^2(f) = \text{lcm}(224, 2\tilde{d}_\pi)$ . Hence equality holds in (30), completing the proof of Theorem 6.4 (and hence Theorem 1.12).

A key step in the above argument is that  $p^2(f) \pmod{2d_\pi}$  can be determined purely algebraically, from the action  $f^*$  on  $H^4(M)$ . A related fact is that  $p^2(f) \pmod{2\tilde{d}_\pi}$  can be determined by the action of  $f^*$  on *Gauss refinements* associated to spin coboundaries of  $M$ . A Gauss refinement of the quadratic linking family  $q_M^\circ$  of  $M$  is a function

$$g: S_{d_\pi} \rightarrow \mathbb{Q}/\frac{1}{4}d_\pi\mathbb{Z}$$

whose mod  $\mathbb{Z}$  reduction is determined by  $q_M^\circ$ , and which satisfies

$$g(k+t) - g(k) = \frac{d_\pi^2 b_M(t, t) - 2d_\pi b_M(p_M - d_\pi k, t)}{8} \pmod{\frac{1}{4}d_\pi\mathbb{Z}}.$$

For our present purposes the significance of these conditions is that the difference between two Gauss refinements is just a constant in  $\mathbb{Z}/\frac{1}{4}d_\pi\mathbb{Z}$ , and if  $f \in \text{ADiff}(M)$  then

$$(36) \quad g - (f^*)^g = \frac{p^2(f)}{8} \pmod{\frac{1}{4}\tilde{d}_\pi},$$

where  $F^g := g \circ F$  for any isomorphism  $F$  of  $H^4(M)$ .

Let  $W$  be a 3-connected coboundary of  $M$ , and  $j: H^4(W) \rightarrow H^4(M)$  the restriction map. We can associate a Gauss refinement  $g_W$  to  $W$  by setting

$$g_W(jn) := \frac{(p_W - d_\pi n)^2 - \sigma(W)}{8} \pmod{\frac{1}{4}d_\pi\mathbb{Z}}$$

for any  $n \in H^4(W)$  such that  $jn \in S_{d_\pi}$  (then the image of  $p_W - d_\pi n$  in  $H^4(W; \mathbb{Q})$  is supported away from the boundary, so its cup-square in  $H^8(W, M; \mathbb{Q}) \cong \mathbb{Q}$  is well-defined) [11, (18)]. The key property of  $g_W$  is that if  $f: M \rightarrow M'$  is a diffeomorphism and  $W'$  is another 3-connected coboundary of  $M'$  then

$$(37) \quad g_{W'} - (f^*)^\# g_W = \frac{p_X^2 - \sigma(X)}{8} = 28\hat{A}(X) \mod \frac{1}{4}d_\pi \mathbb{Z},$$

where  $X$  is the closed spin manifold  $(-M) \cup_f M'$  [11, (24)]. In particular, the mod 28 reduction of  $g_W$  is independent of the choice of spin coboundary  $W$ . This defines a generalisation of the Eells–Kuiper invariant,

$$\mu_M: S_{d_\pi} \rightarrow \mathbb{Q}/\gcd(28, \frac{1}{4}d_\pi \mathbb{Z})\mathbb{Z},$$

which distinguishes between  $\gcd(28, \text{Num}(2^r d_o/8))$  different smooth structures on the topological manifold underlying  $M$  [11, Corollary 4.14]. Together with the homeomorphism invariants  $(H^4(M), q_M^\circ, p_M)$ , it classifies 2-connected 7-manifolds up to diffeomorphism.

**Theorem 6.7** [11, Theorem 1.3] *For a pair of closed 2-connected 7-manifolds  $M_0$  and  $M_1$  and an isomorphism  $F: H^4(M_1) \rightarrow H^4(M_0)$ , there is a diffeomorphism  $f: M_0 \cong M_1$  such that  $F = f^*$  if and only if  $(q_{M_1}^\circ, \mu_{M_1}, p_{M_1}) = F^\#(q_{M_0}^\circ, \mu_{M_0}, p_{M_0})$ .*

## 6.4 The $\xi$ -invariant

We now give the definition of the  $\xi$ -invariant of a  $G_2$ -structure  $\varphi$ , which is a function  $\xi(\varphi): S_{d_\pi} \rightarrow \mathbb{Q}/3\tilde{d}_\pi \mathbb{Z}$  (with  $S_{d_\pi}$  as in (33)). We also explain how the pair  $(\nu, \xi)$  distinguishes between  $24\text{Num}(2^r d_o(M)/224)$  deformation equivalence classes of  $G_2$ -structures on a spin 7-manifold  $M$ . This entails Theorem 1.11, and when  $M$  is 2-connected combining with Theorem 1.12 implies Theorem 1.17, that  $(\nu, \xi)$  is a complete invariant of  $\pi_0 \bar{\mathcal{G}}_2(M)$ .

**Definition 6.8** Let  $\varphi$  be a  $G_2$ -structure on a closed 7-manifold with  $\text{Spin}(7)$ -co-boundary  $W$ . The  $\xi$ -invariant of  $\varphi$  is the function

$$\xi(\varphi) := 7(\chi(W) - 3\sigma(W)) + 12g_W: S_{d_\pi} \rightarrow \mathbb{Q}/3\tilde{d}_\pi \mathbb{Z}.$$

Combining (2) and (37) shows that  $\xi(\varphi)$  is diffeomorphism-invariant (and in particular independent of the choice of  $W$ ): if  $f: M' \rightarrow M$  is a diffeomorphism then

$$(f^*)^\#(\xi(f^*\varphi)) = \xi(\varphi).$$

The relations

$$\begin{aligned} 2D(\varphi, \varphi') &= v(\varphi') - v(\varphi) \pmod{48}, \\ 14D(\varphi, \varphi') &= \xi(\varphi') - \xi(\varphi) \pmod{3\tilde{d}_\pi} \end{aligned}$$

for  $G_2$ -structures  $\varphi$  and  $\varphi'$  on the same manifold  $M$  mean that  $(v, \xi)$  determine  $D \pmod{\text{lcm}(24, \text{Num}(3\tilde{d}_\pi/14))}$ . Moreover, these relations help us see that precisely  $\text{lcm}(24, \text{Num}(3\tilde{d}_\pi/14)) = 24\text{Num}(d_\pi/112)$  pairs  $(v, \xi)$  are realised, namely the ones satisfying

$$(38a) \quad v = \chi_{\mathbb{Q}}(M) \pmod{2},$$

$$(38b) \quad \frac{\xi - 7v}{12} = \mu_M \pmod{\gcd(28, \frac{1}{4}d_\pi\mathbb{Z})}.$$

However, this does not mean that there are  $24\text{Num}(d_\pi/112)$  different deformation equivalence classes, as one has to take into account that  $f \in \text{Diff}(M)$  acts nontrivially on Gauss refinements and hence on  $\xi$ : (36) implies

$$\xi(f^*\varphi) - \xi(\varphi) = \frac{3}{2}p^2(f) \pmod{3\tilde{d}_\pi}.$$

Using Theorem 6.4, the mod  $2^{r-1}3d_o(M)$  reductions of  $\xi$  of deformation equivalent  $G_2$ -structures on  $M$  must still be equal. Hence we can use  $(v, \xi)$  to distinguish between at least

$$\text{lcm}\left(24, \text{Num}\left(\frac{2^{r-1}3d_o(M)}{14}\right)\right) = 24\text{Num}\left(\frac{2^r d_o(M)}{224}\right)$$

deformation equivalence classes. For 2-connected  $M$  this is precisely the number of deformation equivalence classes on  $M$  according to Theorem 1.12, so  $(v, \xi)$  distinguishes between all the classes, completing the proof of Theorem 1.17.

Given a  $G_2$ -structure, we can use (38b) to recover the Eells–Kuiper invariant of the underlying smooth manifold from  $(v, \xi)$ . Hence Theorem 6.7 implies that we can classify closed 2-connected manifolds with homotopy classes of  $G_2$ -structures using the quintuple  $(H^4(M), q_M^\circ, p_M, v, \xi)$ .

**Theorem 6.9** *Let  $M_i$  be closed 2-connected 7-manifolds, and  $\varphi_i$   $G_2$ -structures on  $M_i$ . Given an isomorphism  $F: H^4(M_1) \rightarrow H^4(M_0)$ , there is a diffeomorphism  $f: M_0 \cong M_1$  such that  $F = f^*$  and  $f^*\varphi_1$  is homotopic to  $f^*\varphi_0$  if and only if  $v(\varphi_0) = v(\varphi_1)$  and  $F^\#(p_{M_0}, q_{M_0}^\circ, \xi(\varphi_0)) = (p_{M_1}, q_{M_1}^\circ, \xi(\varphi_1))$ .*

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