Mesoscopic theory for fluctuating active nematics

2013 New J. Phys. 15 085032

(http://iopscience.iop.org/1367-2630/15/8/085032)

New Journal of Physics
The open access journal for physics

Mesoscopic theory for fluctuating active nematics

Eric Bertin[1,2], **Hugues Chaté**[2,3,4,9], **Francesco Ginelli**[5], **Shradha Mishra**[6], **Anton Peshkov**[2,3,4] **and Sriram Ramaswamy**[2,7,8]

[1] Université de Lyon, Laboratoire de Physique, ENS Lyon, CNRS, 46 Allée d'Italie, F-69007 Lyon, France
[2] Max Planck Institute for the Physics of Complex Systems, Nöthnitzer Straße 38, D-01187 Dresden, Germany
[3] Service de Physique de l'Etat Condensé, CEA-Saclay, URA 2464 CNRS, F-91191 Gif-sur-Yvette, France
[4] LPTMC, CNRS-UMR 7600, Université Pierre et Marie Curie, F-75252 Paris, France
[5] SUPA, Department of Physics and Institute for Complex Systems and Mathematical Biology, King's College, University of Aberdeen, Aberdeen AB24 3UE, UK
[6] Department of Physics and Meteorology, Indian Institute of Technology Kharagpur, Kharagpur 721 302, India
[7] TIFR Centre for Interdisciplinary Sciences, Tata Institute of Fundamental Research, 21 Brundavan Colony, Narsingi, Hyderabad 500 075, India
E-mail: hugues.chate@cea.fr

New Journal of Physics **15** (2013) 085032 (27pp)
Received 3 May 2013
Published 28 August 2013
Online at http://www.njp.org/
doi:10.1088/1367-2630/15/8/085032

Abstract. The term active nematics designates systems in which apolar elongated particles spend energy to move randomly along their axis and interact by inelastic collisions in the presence of noise. Starting from a simple Vicsek-style model for active nematics, we derive a mesoscopic theory, complete with effective multiplicative noise terms, using a combination of kinetic theory and Itô calculus approaches. The stochastic partial differential equations thus obtained

[8] On leave from Department of Physics, Indian Institute of Science, Bangalore 560 012 India.
[9] Author to whom any correspondence should be addressed.

are shown to recover the key terms argued in Ramaswamy *et al* (2003 *Europhys. Lett.* **62** 196) to be at the origin of anomalous number fluctuations and long-range correlations. Their deterministic part is studied analytically, and is shown to give rise to the long-wavelength instability at onset of nematic order (see Shi X and Ma Y 2010 arXiv:1011.5408). The corresponding nonlinear density-segregated band solution is given in a closed form.

Contents

1. Introduction

The study of collective properties of systems of interacting active particles [1–3] is currently attracting a great deal of interest. In active matter, particles extract energy from their surrounding and dissipate it to propel themselves in some coherent way in a viscous fluid and/or over a dissipative substrate. In this last case, or whenever hydrodynamic effects can be neglected, physicists speak of 'dry active matter' [3]. Systems as diverse as animal flocks [4–6], human crowds [7, 8], subcellular proteins [9], bacterial colonies [10] and driven granular matter [11–13] have been described in this framework.

In the context of dry active matter, there is now some consensus in the physics community that minimal models such as the celebrated Vicsek model [14, 15] play a crucial role, since they stand as simple representatives of universality classes which have started to emerge from a combination of numerical and theoretical results: for instance, many different microscopic (particle) models have been shown to exhibit the same collective properties as the Vicsek model, and the continuous equation proposed by Toner and Tu [16] is widely believed to account for its collective properties. Such hydrodynamic theories formulated at the mesoscopic level through stochastic partial differential equations (PDEs) are the natural framework to characterize and define universality classes.

In early approaches these mesoscopic theories have been built on the principle of including all that is not explicitly forbidden, retaining all leading terms (in a gradient expansion sense) allowed by symmetries and conservation laws [16, 17]. This grants access to the general structure of these equations and has been successful in describing relevant features of active matter systems such as their anomalously large number density fluctuations [12, 13, 16, 18]. Despite the attractions of a gradient expansion, it typically contains many transport coefficients of unknown dependence on microscopic control parameters and hydrodynamic fields such as local density. Moreover, the dependence of the noise terms on the dynamical fields in such equations remains arbitrary, and frequently neglected, whereas it could have profound consequences for important phenomena such as spontaneous segregation, clustering and interface dynamics.

Ideally, thus, one would be able to derive well-behaved mesoscopic theories using a systematic procedure starting from a given microscopic model. Kinetic-theory-like approaches [19–23] go one step toward this goal, by allowing one to compute hydrodynamic transport coefficients and nonlinear terms. One of the most successful versions is arguably the 'Boltzmann–Ginzburg–Landau' (BGL) framework recently put forward by some of us [24, 25], where, in the spirit of weakly nonlinear analysis, one performs well-controlled expansions in the vicinity of ordering transitions. Kinetic approaches alone thus yield good deterministic 'mean-field' equations but one still needs to 'reintroduce' fluctuations in order to get *bona fide* mesoscopic descriptions.

In this work, we show how this complete program can be achieved for the case of active nematics, i.e. systems where particles are energized individually but not really self-*propelled*, moving along the axis of the nematic degree of freedom they carry, with equal probability forward or back (think of shaken apolar rods aligning by inelastic collisions [12]). Starting from the Vicsek-style model for active nematics introduced in [26], we formulate a version of the BGL scheme mentioned above adapted to problems dominated by diffusion, derive the corresponding hydrodynamic equations and study their homogeneous and inhomogeneous solutions. In a last section, we show how these equations can be complemented by appropriate noise terms using a direct coarse-graining approach.

2. Kinetic approach

2.1. Microscopic dynamics

We consider the microscopic model for active nematics of [26] in two space dimensions. This Vicsek-style model can be thought of as a minimal model for a single layer of vibrated granular rods [12] although it does not deal explicitly with any volume exclusion forces. Here, rather, pointwise particles $j = 1, \ldots, N$ are characterized by their position $\mathbf{x}_j^t$ and an axial direction $\theta_j^t \in [-\pi/2, \pi/2]$. They interact synchronously with all neighboring particles situated within distance r_0 in a characteristic driven-overdamped dynamics implemented at discrete timesteps Δt:

$$\theta_j^{t+\Delta t} = \frac{1}{2}\mathrm{Arg}\left[\sum_{k\in V_j} \mathrm{e}^{\mathrm{i}2\theta_k^t}\right] + \psi_j^t, \tag{1}$$

$$\mathbf{x}_j^{t+\Delta t} = \mathbf{x}_j^t + d_0\,\kappa_j^t\,\hat{\mathbf{n}}_j^t, \tag{2}$$

where V_j is the neighborhood of particle j, $d_0 < r_0$ is the elementary displacement, $\hat{\mathbf{n}}_j^t \equiv (\cos\theta_j^t, \sin\theta_j^t)^{\mathrm{T}}$ is the nematic director and ψ and κ are two white noises: the random angle ψ_j^t, familiar of Vicsek-style models, is drawn from a symmetric distribution $\tilde{P}_\eta(\psi)$ of variance η^2, and the zero average bimodal noise $\kappa_j^t = \pm 1$ determines the actual orientation of motion. Both noises are delta correlated, namely $\langle \kappa_j^t \kappa_k^{t'} \rangle \sim \langle \psi_j^t \psi_k^{t'} \rangle \sim \delta_{t\,t'}\delta_{j\,k}$. Note that the factor 2 in the exponential terms of equation (1) implements an alignment interaction which fulfills the nematic symmetry $\theta \to \theta + \pi$. In other words, particles align their axial direction but do not carry any polar orientation.

In the following, we adopt the convention $[\hat{\mathbf{n}}\hat{\mathbf{n}}]_{\alpha\beta} \equiv \hat{n}_\alpha \hat{n}_\beta$ and label coordinates by greek indices, $\alpha, \beta, \ldots = 1, 2$, summing over repeated indices.

2.2. Timescales and lengthscales

We consider low density systems in which particles, at a given time, are either non-interacting or involved in a binary interaction. In this dilute limit we can neglect interactions between more than two particles. We also treat interactions as collision-like events, with the mean intercollision time

$$\tau_{\mathrm{free}} \approx \frac{\tau_{\mathrm{d}}}{d_0^2\,\rho_0}, \tag{3}$$

where ρ_0 is the global particle density and τ_{d} is the shortest microscopic timescale of the dynamics, associated to the inversion of the rods direction of motion $\tau_{\mathrm{d}} \sim \Delta t$. This intercollision time is much larger than the collision timescale

$$\tau_{\mathrm{coll}} \approx \tau_{\mathrm{d}} \left(\frac{r_0}{d_0}\right)^2. \tag{4}$$

For driven granular rods, τ_{d} may be thought of as the inverse of the shaking frequency, and for typical parameters it is much smaller than both the collision (τ_{coll}) and the mean intercollision (τ_{free}) timescales; at low enough densities $\tau_{\mathrm{d}} \ll \tau_{\mathrm{coll}} \ll \tau_{\mathrm{free}}$. Note that the timescales (4)–(3) are different from the ones characteristic of ballistic dynamics [20].

To develop a kinetic approach we consider a mesoscopic timescale τ_{B} such that $\tau_{\mathrm{coll}} \ll \tau_{\mathrm{B}} \ll \tau_{\mathrm{free}}$. As a consequence, we will treat the inversion of the direction of motion as a noisy term through Itô stochastic calculus [27]. We also consider a mesoscopic coarse-graining lengthscale ℓ_{B} which, while being much smaller than the system size L, is larger than the microscopic scales, such as the step-size d_0, the mean interparticle distance $\rho_0^{-1/2}$ and the interaction range r_0. To summarize, in a dilute system one has

$$\tau_{\mathrm{d}} \ll \tau_{\mathrm{d}} \left(\frac{r_0}{d_0}\right)^2 \ll \tau_{\mathrm{B}} \ll \frac{\tau_{\mathrm{d}}}{d_0^2\rho_0} \tag{5}$$

and

$$d_0 < r_0 \ll \frac{1}{\sqrt{\rho_0}} \ll \ell_{\mathrm{B}} \ll L, \tag{6}$$

where we have made explicit the condition that the typical coarse-graining lengthscale ℓ_{B} is such that many particles are contained in a box of linear size ℓ_{B}, that is $\rho_0 \ell_{\mathrm{B}}^2 \gg 1$.

2.3. Master equation

We now write down a Boltzmann-like master equation in terms of the single particle probability distribution $f(\mathbf{x},\theta,t)$, with $-\frac{\pi}{2}<\theta\leqslant\frac{\pi}{2}$, evolving over the timescale $\Delta t\approx\tau_{\rm B}$. The minimal spatial resolution is such that many particles are contained in a spatial volume d^2x centered around the position $\mathbf{x}$. Moreover, we consider a dilute system, so that interactions (collisions) between particles are sufficiently rare to justify (i) *binary interactions* (as explained above, particles then either self-diffuse or experience noisy binary, collision-like interactions), (ii) decorrelation of the orientation between successive binary collisions of the same pair of particles, that is $f_2(\mathbf{x},\theta_1,\theta_2,t)\approx f(\mathbf{x},\theta_1,t)f(\mathbf{x},\theta_2,t)$.

We first omit collisions and angular diffusion, only considering equation (2) to get

$$f(\mathbf{x},\theta,t+\Delta t)=\tfrac{1}{2}[f(\mathbf{x}+\hat{\mathbf{n}}(\theta)d_0,\theta,t)+f(\mathbf{x}-\hat{\mathbf{n}}(\theta)d_0,\theta,t)],\tag{7}$$

where we have considered that a particle moves along one of the two orientations of $\hat{\mathbf{n}}$ with equal probability. On the mesoscopic timescale $\tau_{\rm B}\gg\tau_{\rm d}\sim\Delta t$, Itô calculus [27] to second order gives

$$\partial_t f(\mathbf{x},\theta,t)=D_0\partial_\alpha\partial_\beta[\hat{n}_\alpha(\theta)\hat{n}_\beta(\theta)f(\mathbf{x},\theta,t)],\tag{8}$$

where

$$D_0=\frac{d_0^2}{2\tau_{\rm d}}\tag{9}$$

is the microscopic diffusion parameter.

To account for angular diffusion and binary collisions, the appropriate integrals need to be added to the right-hand side of equation (8):

$$\partial_t f(\mathbf{x},\theta,t)=D_0\,\partial_\alpha\partial_\beta[\hat{n}_\alpha(\theta)\hat{n}_\beta(\theta)f(\mathbf{x},\theta,t)]+I_{\rm diff}[f]+I_{\rm coll}[f,f].\tag{10}$$

The diffusion integral describes self-diffusion which takes place at a rate $\lambda\approx 1/\tau_{\rm d}$

$$I_{\rm diff}[f]=-\lambda f(\theta)+\lambda\int_{-\pi/2}^{\pi/2}\mathrm{d}\theta'\,f(\theta')\int_{-\infty}^{\infty}\mathrm{d}\zeta\,P(\zeta)\,\delta_\pi(\theta'-\theta+\zeta),\tag{11}$$

where we used the simplified notation $f(\theta)\equiv f(\mathbf{x},\theta,t)$, δ_π is a generalized Dirac delta imposing that the argument is equal to zero modulo π and $P(\zeta)$ is a symmetric noise distribution of variance σ^2, corresponding to the effective noise arising at the timescale $\tau_{\rm B}$ from the sum of the microscopic stochastic contributions to angular dynamics.

Binary collisions are described by

$$\begin{aligned}I_{\rm coll}[f,f]=&-f(\theta)\int_{-\pi/2}^{\pi/2}\mathrm{d}\theta'\,f(\theta')K(\theta,\theta')\\&+\int_{-\pi/2}^{\pi/2}\mathrm{d}\theta_1\int_{-\pi/2}^{\pi/2}\mathrm{d}\theta_2\,f(\theta_1)K(\theta_1,\theta_2)f(\theta_2)\int_{-\infty}^{\infty}\mathrm{d}\zeta\,P(\zeta)\,\delta_\pi(\Psi(\theta_1,\theta_2)-\theta+\zeta),\end{aligned}\tag{12}$$

where, for the sake of simplicity, we have used the same noise distribution $P(\zeta)$ as in the self-diffusion integral, and the out-coming angle Ψ from deterministic binary collisions is, for $-\frac{\pi}{2}<\theta_1,\theta_2\leqslant\frac{\pi}{2}$,

$$\Psi(\theta_1,\theta_2)=\tfrac{1}{2}(\theta_1+\theta_2)+h(\theta_1-\theta_2)\quad\text{with } h(\theta)=\begin{cases}0 & \text{if } |\theta|\leqslant\dfrac{\pi}{2},\\ \dfrac{\pi}{2} & \text{if } \dfrac{\pi}{2}<|\theta|\leqslant\pi.\end{cases}\tag{13}$$

Note that the role of the function $h(\theta)$ is to ensure that $\Psi(\theta_1, \theta_2)$ is π-periodic with respect to θ_1 and θ_2 independently, as dictated by the nematic symmetry of the system. The collision kernel $K(\theta_1, \theta_2)$, i.e. the number of collisions per unit time and volume, is calculated as follows, modifying the standard collision kernel of kinetic theory to take into account the fact that active nematic particles can move either along $\hat{\mathbf{n}}$ or $-\hat{\mathbf{n}}$ [19, 20, 28]. Consider two particles with nematic axis $\hat{\mathbf{n}}(\theta)$ and $\hat{\mathbf{n}}(\theta')$ located in the volume d^2x centered around position $\mathbf{x}$. In the reference frame of the first particle the second one diffuses either along the $|\hat{\mathbf{n}}(\theta) - \hat{\mathbf{n}}(\theta')|$ or the $|\hat{\mathbf{n}}(\theta) + \hat{\mathbf{n}}(\theta')|$ nematic axis. In unit time, taking into account the characteristic timescales τ_{d} and step-size d_0 of its motion, it sweeps a surface (its cross section, which is conserved going back to the lab reference frame) equal to

$$K(\theta,\theta') = \frac{r_0 d_0}{\tau_{\mathrm{d}}}[|\hat{\mathbf{n}}(\theta) - \hat{\mathbf{n}}(\theta')| + |\hat{\mathbf{n}}(\theta) + \hat{\mathbf{n}}(\theta')|]$$
$$= 2\alpha_0 \left[\left|\sin\frac{\theta-\theta'}{2}\right| + \left|\cos\frac{\theta-\theta'}{2}\right|\right], \tag{14}$$

where we have introduced the microscopic collision parameter

$$\alpha_0 = \frac{r_0 d_0}{\tau_{\mathrm{d}}}. \tag{15}$$

Note that $K(\theta,\theta') \equiv \tilde{K}(\theta-\theta')$ is an even function of the difference $(\theta-\theta')$, and fulfills the nematic symmetry, being invariant under rotation of either angle by π.

Before proceeding to derive hydrodynamic equations, we simplify all notations by rescaling time $\tilde{t} = \lambda t = t/\tau_{\mathrm{d}}$ and space $\tilde{x} = \frac{\sqrt{2}}{d_0}x$. As in [24, 25] we also set the collision surface $S = 2r_0 d_0$ to 1 by a global rescaling of the one-particle probability density f, without loss of generality. This amounts to set $\lambda = 1$, $D_0 = 1$ and $2\alpha_0 = 1$, so that, dropping the tildes, our Boltzmann-like master equation now depends only on the global density ρ_0 and the noise intensity σ.

2.4. Hydrodynamic description

In two spatial dimensions, hydrodynamic fields can be obtained by expanding the single particle probability density f in Fourier series of its angular variable $\theta \in [-\pi/2, \pi/2]$:[10]

$$f(\mathbf{x},\theta,t) = \frac{1}{\pi}\sum_{k=-\infty}^{k=\infty} \hat{f}_k(\mathbf{x},t)\,\mathrm{e}^{-\mathrm{i}2k\theta} \tag{16}$$

and

$$\hat{f}_k(\mathbf{x},t) = \int_{-\pi/2}^{\pi/2} \mathrm{d}\theta f(\mathbf{x},\theta,t)\,\mathrm{e}^{\mathrm{i}2k\theta}. \tag{17}$$

The number density and the density-weighted nematic tensor field $\mathbf{w} \equiv \rho\mathbf{Q}$ are then given by

$$\rho(\mathbf{x},t) = \int_{-\pi/2}^{\pi/2} \mathrm{d}\theta f(\mathbf{x},\theta,t) = \hat{f}_0(\mathbf{x},t) \tag{18}$$

[10] These k-modes are equivalent to even harmonics if one would define particles orientation in $[-\pi, \pi]$ in spite of the symmetry under rotations by π (with odd ones being zero by symmetry).

and

$$w_{11}(\mathbf{x},t) = -w_{22}(\mathbf{x},t) = \frac{1}{2}\int_{-\pi/2}^{\pi/2} \mathrm{d}\theta f(\mathbf{x},\theta,t)\cos(2\theta) = \frac{1}{2}\mathrm{Re}\hat{f}_1(\mathbf{x},t), \tag{19}$$

$$w_{12}(\mathbf{x},t) = w_{21}(\mathbf{x},t) = \frac{1}{2}\int_{-\pi/2}^{\pi/2} \mathrm{d}\theta f(\mathbf{x},\theta,t)\sin(2\theta) = \frac{1}{2}\mathrm{Im}\hat{f}_1(\mathbf{x},t). \tag{20}$$

Note that when $\mathrm{Im}\hat{f}_1 = 0$ the nematic field is aligned either along the x ($\mathrm{Re}\hat{f}_1 > 0$) or the y ($\mathrm{Re}\hat{f}_1 < 0$) axis.

Injecting the Fourier expansion (16) in the master equation (10), one gets, after some lengthy calculations detailed in appendix A, the infinite hierarchy:

$$\partial_t \hat{f}_k(\mathbf{x},t) = \frac{1}{2}\Delta \hat{f}_k(\mathbf{x},t) + \frac{1}{4}(\nabla^{*2}\hat{f}_{k+1} + \nabla^2 \hat{f}_{k-1}) + [\hat{P}_k - 1]\hat{f}_k(\mathbf{x},t)$$
$$+\frac{1}{\pi}\sum_q \hat{f}_q(\mathbf{x},t)\hat{f}_{k-q}(\mathbf{x},t)\left[\hat{P}_k \hat{J}_{k,q} - \frac{4}{1-16q^2}\right], \tag{21}$$

where $\hat{P}_k$ is the Fourier transform of the noise distribution $P(\zeta)$ (namely, $\hat{P}_k = \int_{-\infty}^{\infty} \mathrm{d}\zeta\, P(\zeta)\,\mathrm{e}^{\mathrm{i}2k\zeta}$) and

$$\hat{J}_{k,q} = 4\,\frac{1 + 2\sqrt{2}(2q-k)(-1)^q \sin\left(\frac{k\pi}{2}\right)}{1-4(2q-k)^2} \tag{22}$$

and we have introduced the following ‘complex’ operators

$$\begin{aligned}
\nabla &\equiv \partial_x + \mathrm{i}\partial_y,\\
\nabla^* &\equiv \partial_x - \mathrm{i}\partial_y,\\
\Delta &\equiv \nabla\nabla^*,\\
\nabla^2 &\equiv \nabla\nabla,\\
\nabla^{*2} &\equiv \nabla^*\nabla^*.
\end{aligned}$$

The equation at order $k = 0$ is thus expressed in the simple form

$$\partial_t \rho = \tfrac{1}{2}\Delta\rho + \tfrac{1}{2}\mathrm{Re}(\nabla^{*2}\hat{f}_1) \tag{23}$$

and is nothing but the continuity equation for diffusive active matter with local anisotropy characterized by $\hat{f}_1$.

Equation (21) possesses a trivial, isotropic and homogeneous solution: $\rho(\mathbf{x},t) \equiv \hat{f}_0(\mathbf{x},t) = \rho_0$ and $\hat{f}_k(\mathbf{x},t) = 0$ for $|k| > 0$. We are interested in a nematically ordered homogeneous solution which could eventually arise following some instability of the isotropic solution above. In analogy to the scaling ansatz used for polar particles [20, 25], the interaction term in equation (21) suggests a simple scaling ansatz to close the infinite hierarchy of equations on $\hat{f}_k(\mathbf{x},t)$. Near an instability threshold with continuous onset, Fourier coefficients should scale as $\hat{f}_k(\mathbf{x},t) \sim \epsilon^{|k|}$ where ϵ is a small parameter characterizing the distance to threshold. Moreover, the curvature induced current (last term of (23)) also induces an order ϵ variation in the density field, $\rho(\mathbf{x},t) - \rho_0 \sim \epsilon$. Then, assuming spatial derivatives to be of order ϵ, the request that all terms in equation (23) are of the same order also fixes the diffusive structure of the scaling of time and spatial gradients: $\partial_t \sim \nabla^2 \sim \Delta \sim \epsilon^2$.

Using the above scaling ansatz, we proceed by discarding all terms appearing in (21) of order higher than ϵ^3. For $k = 1, 2$ we get

$$\partial_t \hat{f}_1 = \tfrac{1}{2}\Delta \hat{f}_1 + \tfrac{1}{4}\nabla^2\rho + a_1(\rho)\hat{f}_1 + b_1\hat{f}_1^*\hat{f}_2 \tag{24}$$

and

$$0 = \tfrac{1}{4}\nabla^2\hat{f}_1 - a_2(\rho)\hat{f}_2 + b_2\hat{f}_1^2, \tag{25}$$

where the coefficients are

$$a_1(\rho) = \frac{8}{3\pi}\left[(2\sqrt{2}-1)\hat{P}_1 - \frac{7}{5}\right]\rho - (1-\hat{P}_1), \tag{26}$$

$$b_1 = \frac{8}{315\pi}[13 - 9\hat{P}_1(1+6\sqrt{2})], \tag{27}$$

$$a_2(\rho) = (1-\hat{P}_2) + \frac{8}{3\pi}\left(\frac{\hat{P}_2}{5} + \frac{31}{21}\right)\rho \tag{28}$$

and

$$b_2 = \frac{4}{\pi}\left(\frac{1}{15} + \hat{P}_2\right). \tag{29}$$

Equation (25) shows that at this order $\hat{f}_2$ is enslaved to $\hat{f}_1$ (given that $a_2 > 0$) and, further,

$$a_2(\rho_0)\hat{f}_2 \approx \tfrac{1}{4}\nabla^2\hat{f}_1 + b_2\hat{f}_1^2, \tag{30}$$

where the coefficient a_2 is evaluated at the mean density ρ_0, since the $\delta\rho = \rho - \rho_0$ corrections are of higher order. By substituting equations (25) into (24) one finally gets, neglecting the term $\hat{f}_1^*\nabla^2\hat{f}_1 \sim \epsilon^4$,

$$\partial_t \hat{f}_1 = (\mu - \xi|\hat{f}_1|^2)\hat{f}_1 + \tfrac{1}{4}\nabla^2\rho + \tfrac{1}{2}\Delta\hat{f}_1, \tag{31}$$

where we have introduced the transport coefficients

$$\mu = \frac{8}{3\pi}\left[(2\sqrt{2}-1)\hat{P}_1 - \frac{7}{5}\right]\rho - (1-\hat{P}_1), \tag{32}$$

$$\xi = \frac{32\nu}{35\pi^2}\left[\frac{1}{15} + \hat{P}_2\right]\left[(1+6\sqrt{2})\hat{P}_1 - \frac{13}{9}\right] \tag{33}$$

$$\text{with}\quad \nu = \left[\frac{8}{3\pi}\left(\frac{31}{21} + \frac{\hat{P}_2}{5}\right)\rho_0 + (1-\hat{P}_2)\right]^{-1}. \tag{34}$$

Note that the coefficient ξ is only a function of the average density ρ_0, as space and time dependent corrections are of order ϵ^4. Note also that the coefficients μ and ξ are exactly the same as those found for the nematic field equation of nematically aligning polar particles [25][11].

[11] Note that in [25], the equations obtained are not entirely correct: (i) there is a sign error and a misplaced factor π in the expression of ξ; (ii) the term $\frac{\nu}{4}\nabla^2 f_2$ should read $\frac{\nu}{4}\Delta f_2$, where Δ is the Laplacian. In addition, let us emphasize that the Fourier coefficients $\hat{P}_k$ have a different definition in [25], due to the absence of global nematic symmetry: $\hat{P}_k$ here corresponds to $\hat{P}_{2k}$ in [25], leading to (only apparent) differences.

Equations (23) and (31) can be expressed in tensorial notation. To this aim, we introduce the linear differential operator $\mathbf{\Gamma}$, such that $\Gamma_{11} = -\Gamma_{22} \equiv \partial_1\partial_1 - \partial_2\partial_2$ and $\Gamma_{12} = \Gamma_{21} \equiv 2\partial_1\partial_2$, and the Frobenius inner product $\mathbf{A}{:}\mathbf{B} = A_{\alpha\beta}B_{\alpha\beta}$ (note that $\mathbf{w}{:}\mathbf{w} = ||\mathbf{w}||^2$ and $\mathbf{\Gamma}{:}\mathbf{w} = 2\partial_\alpha\partial_\beta w_{\alpha\beta}$). After some manipulation of the terms and the use of equations (19) and (20), we obtain the hydrodynamic equations for the density and nematic field

$$\partial_t \rho = \tfrac{1}{2}\Delta\rho + \tfrac{1}{2}(\mathbf{\Gamma}{:}\mathbf{w}), \tag{35}$$

$$\partial_t \mathbf{w} = \mu\,\mathbf{w} - 2\xi\,\mathbf{w}\,(\mathbf{w}{:}\mathbf{w}) + \tfrac{1}{2}\Delta\mathbf{w} + \tfrac{1}{8}\mathbf{\Gamma}\rho. \tag{36}$$

Although the tensorial notation might be more familiar to some readers, it is in fact easier here to continue manipulating the complex field $\hat{f}_1$ and the complex operators defined above. Moreover, in the following we drop the '^' superscript to ease notations. Note also that equations (35) and (36) were also derived from an apolar Vicsek-style model in [29].

The parameter-free character of the Laplacian term in (36) means, consistently with our expansion in ϵ, that the nematic phase of our system will be characterized by a single Frank constant [30]. The nonlinearities studied in [31] are therefore also absent to this order. The last term in equation (23) (or equation (35)), i.e. $\frac{1}{2}\mathrm{Re}\left(\nabla^2 f_1\right)$ (or $\frac{1}{2}(\mathbf{\Gamma} : \mathbf{w})$), is a curvature induced current which couples the density and the nematic field. While its existence was first deduced from general principles [17], here we have computed it directly from microscopic dynamics. Our calculations also give an exact expression for the corresponding transport coefficient, which is equal to the diffusive one (in equation (23) or (35)), here set to 1/2 by our rescaling. In appendix B, we show explicitly that this curvature-induced current originates from the coupling of orientation with motility.

We note finally that equations (35) and (36) are similar to those found by Baskaran and Marchetti [22] but simpler, largely due to our simpler starting point.

2.5. Homogeneous solutions

From now on, we use for $P(\zeta)$ a centered Gaussian distribution of variance σ^2, in which case $\hat{P}_k = \mathrm{e}^{-2k^2\sigma^2}$. The linear stability with respect to homogeneous perturbations of the disordered solution $\rho(\mathbf{x}, t) = \rho_0$, $\hat{f}_1(\mathbf{x}, t) = 0$ is given by the sign of $\mu(\rho_0)$ which yields the basic transition line

$$\sigma_\mathrm{t} = \sqrt{\frac{1}{2}\ln\left[5\,\frac{8(2\sqrt{2}-1)\rho_0 + 3\pi}{56\rho_0 + 15\pi}\right]}. \tag{37}$$

Note that in the dilute limit $\rho_0 \ll 1$, where the equations have been derived, one has $\sigma_\mathrm{t} \sim \sqrt{\rho_0}$.

For $\sigma < \sigma_\mathrm{t}$, $\mu > 0$, and the homogeneous nematically ordered solution

$$|f_1| = \sqrt{\frac{\mu}{\xi}} \tag{38}$$

exists and is stable w.r.t. homogeneous perturbations. The critical line is shown in figure 1(a) (black solid line). Note that for $\sigma < \sigma_\mathrm{t}$, all transport coefficients (32)–(34) are positive. This will be useful in the rest of the paper.

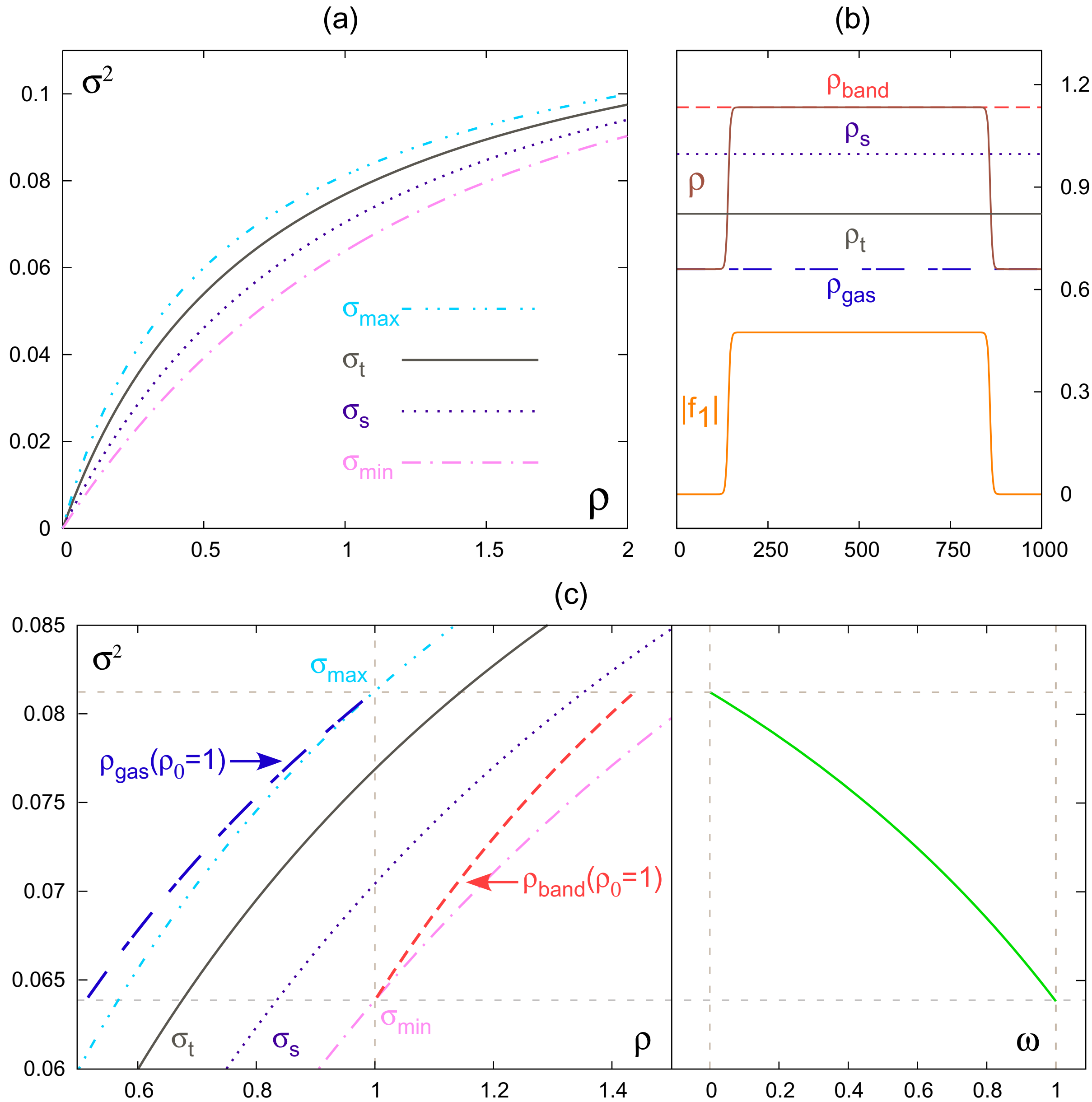

Figure 1. (a) Basic stability diagram. The line σ_t (solid, black) marks the linear instability of the disordered homogeneous solution. The ordered homogeneous solution is linearly unstable to large wavelengths between the σ_t and σ_s (dotted, purple) lines, and linearly stable below the σ_s line. The $\sigma_{\min}$ and $\sigma_{\max}$ lines mark the domain of existence of the band solution (62). (b) Density and order profile of the band solution for $\rho_0 = 1$, $\sigma = 0.265$, $L = 1000$; note that the lower and upper levels ($\rho_{\rm gas}$ and $\rho_{\rm band}$) are respectively lower than ρ_t and higher than ρ_s, i.e. such that the corresponding homogeneous solution are lineally stable. (c) Properties of the band solutions for $\rho_0 = 1$: left: values of $\rho_{\rm gas}$ (long dash, dark blue line) and $\rho_{\rm band}$ (dashed, red line) as σ varies between $\sigma_{\min}$ and $\sigma_{\max}$; right: corresponding variation of the surface fraction ω.

3. Linear stability analysis

We now study the linear stability of the above homogeneous solutions w.r.t. to arbitrary perturbations. Linearizing equations (23) and (31) around a homogeneous solution, $f_1 = f_{1,0} + \delta f_1$ and $\rho = \rho_0 + \delta\rho$, one has

$$\partial_t \delta\rho = \tfrac{1}{2}\Delta\delta\rho + \tfrac{1}{2}\mathrm{Re}(\nabla^{*2}\delta f_1), \tag{39}$$

$$\partial_t \delta f_1 = (\mu_0 - \xi|f_{1,0}|^2)\delta f_1 + \mu' f_{1,0}\,\delta\rho - 2\xi f_{1,0}\,\mathrm{Re}(f_{1,0}^*\,\delta f_1) + \tfrac{1}{4}\nabla^2\delta\rho + \tfrac{1}{2}\Delta\delta f_1, \tag{40}$$

where $\mu_0 \equiv \mu(\rho_0)$ and μ' is the derivative of μ w.r.t. ρ. We then introduce the real and imaginary parts of the order parameter perturbation, $\delta f_1 = \delta f_1^{(\mathrm{R})} + \mathrm{i}\delta f_1^{(\mathrm{I})}$, and express the spatial dependence of all perturbation fields in Fourier space, with a wavevector $\mathbf{q} = (q_x, q_y)$, by introducing the ansatz

$$\delta\rho(\mathbf{x}, t) = \delta\rho_{\mathbf{q}}\,\mathrm{e}^{st+\mathrm{i}\mathbf{qr}}, \tag{41}$$

$$\delta f_1^{(\mathrm{R})}(\mathbf{x}, t) = \delta f_{1,\mathbf{q}}^{(\mathrm{R})}\,\mathrm{e}^{st+\mathrm{i}\mathbf{qr}}, \qquad \delta f_1^{(\mathrm{I})}(\mathbf{x}, t) = \delta f_{1,\mathbf{q}}^{(\mathrm{I})}\,\mathrm{e}^{st+\mathrm{i}\mathbf{qr}}. \tag{42}$$

The stability of the stationary solution $f_{1,0}$ is then ruled by the real part of the growth rate s.

3.1. Stability of the disordered isotropic solution

We first study the stability of the disordered solution $f_{1,0} = 0$, in the case $\mu_0 < 0$. Substituting equations (41), (42) in equations (39), (40), one has

$$\begin{aligned} s\,\delta\rho_{\mathbf{q}} &= -\frac{q^2}{2}\,\delta\rho_{\mathbf{q}} - \frac{1}{2}(q_x^2 - q_y^2)\delta f_{1,\mathbf{q}}^{(\mathrm{R})} - q_x q_y \delta f_{1,\mathbf{q}}^{(\mathrm{I})}, \\ s\,\delta f_{1,\mathbf{q}}^{(\mathrm{R})} &= -\frac{1}{4}(q_x^2 - q_y^2)\delta\rho_{\mathbf{q}} + \left(\mu_0 - \frac{q^2}{2}\right)\delta f_{1,\mathbf{q}}^{(\mathrm{R})}, \\ s\,\delta f_{1,\mathbf{q}}^{(\mathrm{I})} &= -\frac{1}{2} q_x q_y\,\delta\rho_{\mathbf{q}} + \left(\mu_0 - \frac{q^2}{2}\right)\delta f_{1,\mathbf{q}}^{(\mathrm{I})}, \end{aligned} \tag{43}$$

where $q^2 = q_x^2 + q_y^2$. All directions of the wavevector $\mathbf{q}$ being equivalent, we choose for simplicity $q_x = q$ and $q_y = 0$. From equation (43), one then sees that the component $\delta f_{1,\mathbf{q}}^{(\mathrm{I})}$ becomes independent from $\delta\rho_{\mathbf{q}}$ and $\delta f_{1,\mathbf{q}}^{(\mathrm{R})}$, yielding the negative eigenvalue $s = \mu_0 - \frac{q^2}{2}$. The eigenvalues of the remaining 2×2 block of the stability matrix are solutions of the second order polynomial

$$s^2 + s[q^2 - \mu_0] + \frac{q^2}{2}\left[\frac{q^2}{4} - \mu_0\right] \equiv s^2 + \beta_1\,s + \beta_0 = 0. \tag{44}$$

In the disordered state $\mu_0 < 0$, so that β_1 and β_2 are positive and one always has $\mathrm{Re}\,(s) < 0$. Therefore, the homogeneous disordered solution is stable w.r.t. to all perturbations if $\mu_0 < 0$, i.e. $\sigma > \sigma_{\mathrm{t}}$.

3.2. Stability of the ordered solution

To study the stability of the anisotropic ordered solution, it is convenient to choose a reference frame in which order is along one of the axes:

$$\mathrm{Re}(f_{1,0}) = \pm\sqrt{\frac{\mu_0}{\xi}}, \quad \mathrm{Im}(f_{1,0}) = 0. \tag{45}$$

This solution is aligned along x, if $f_{1,0}$ is positive, or along y if negative. For simplicity we will concentrate further on the case $f_{1,0} \geqslant 0$, i.e. on the nematic solution aligned along the x-axis. The real part $\delta f_1^{(\mathrm{R})}$ of the nematic field perturbation describes changes in the modulus $|f_{1,0}|$, and the imaginary part $\delta f_1^{(\mathrm{I})}$ describes perturbations perpendicular to the nematic orientation. The ansatz (41), (42) then yields the three coupled linear equations

$$\begin{aligned} s\,\delta\rho_{\mathbf{q}} &= -\frac{q^2}{2}\,\delta\rho_{\mathbf{q}} - \frac{1}{2}(q_x^2 - q_y^2)\delta f_{1,\mathbf{q}}^{(\mathrm{R})} - q_x q_y\,\delta f_{1,\mathbf{q}}^{(\mathrm{I})}, \\ s\,\delta f_{1,\mathbf{q}}^{(\mathrm{R})} &= \left[\mu' f_{1,0} - \frac{1}{4}(q_x^2 - q_y^2)\right]\delta\rho_{\mathbf{q}} - \left[2\mu_0 + \frac{q^2}{2}\right]\delta f_{1,\mathbf{q}}^{(\mathrm{R})}, \\ s\,\delta f_{1,\mathbf{q}}^{(\mathrm{I})} &= -\frac{1}{2} q_x q_y\,\delta\rho_{\mathbf{q}} - \frac{q^2}{2}\delta f_{1,\mathbf{q}}^{(\mathrm{I})}. \end{aligned} \tag{46}$$

We performed a full numerical stability analysis of these equations. The results are presented in figure 1. The transition to the homogeneous solution is given by the line σ_{t}. This solution is unstable to finite wavelength transversal perturbations of angle $|\theta| > \frac{\pi}{4}$ between the lines σ_{t} and σ_{s} (dotted purple line in figure 1), but is stable deeper in the ordered phase.

Two remarks are in order. Firstly, the angle of the most unstable mode is here always perfectly $\frac{\pi}{2}$. It is thus possible to obtain the 'restabilization' line σ_{s} analytically as shown below. Secondly, there is no spurious instability at low noise and/or high density (although we have found that such an instability appears if the truncation of the equations is made to the fourth order).

To obtain the analytic expression of the line σ_{s}, we write the wavevector in terms of its modulus q and its angle $\theta_{\mathbf{q}}$, so that $q_x^2 - q_y^2 = q^2\cos 2\theta_{\mathbf{q}}$ and $2q_x q_y = q^2 \sin 2\theta_{\mathbf{q}}$. We can then analyze equations (46) in the longitudinal and perpendicular wavedirections $\theta_{\mathbf{q}} = 0, \pm\frac{\pi}{2}$, where the imaginary perturbation $\delta f_{1,\mathbf{q}}^{(\mathrm{I})}$ decouples from the other two. The latter is stable toward long-wavelength perturbations, since the corresponding eigenvalue $s = -q^2/2$ is negative. The stability toward density and real perturbations depends on a 2×2 matrix which yields the quadratic eigenvalue equation

$$s^2 + [2\mu_0 + q^2]s + \left[\left(\frac{\pm\mu' f_{1,0}}{2} + \mu_0\right) q^2 + \frac{q^4}{8}\right] = 0 \tag{47}$$

whose solutions are

$$s = \frac{1}{2}\left[-2\mu_0 - q^2 \pm \sqrt{4\mu_0^2 \mp 2\mu' f_{1,0} q^2 + \frac{q^4}{2}}\right]. \tag{48}$$

The sign $\pm$ in front of the $\mu' f_{1,0}$ term in equation (47) corresponds to the case $\theta_{\mathbf{q}} = 0$ (positive sign) and $\theta_{\mathbf{q}} = \frac{\pi}{2}$ (negative sign) respectively. Note that μ' is strictly positive, as typical for all active matter system with metric interactions, where the interaction rate grows with local density. Also μ_0 is positive and of order ϵ^2 (see equation (45)). It it thus easy to see that in the case of large q, $\Re[s] \leqslant 0$. For small values of q, we perform an expansion to order q^2 of the largest growth rate s_+, obtained by taking the positive sign in front of the square root in equation (48), leading to

$$s_+ = \frac{q^2}{2}\left[\mp\frac{\mu'}{2\mu_0}f_{1,0} - 1\right]. \tag{49}$$

We can then conclude that for longitudinal perturbations ($\theta_{\mathbf{q}} = 0$, negative sign in front of μ'), the homogenous solution is stable confirming the results of numerical analysis. In the case of transversal perturbations ($\theta_{\mathbf{q}} = \pm\pi/2$), the stability condition is given by

$$\mu_0 > \frac{\mu'^2}{4\xi} \tag{50}$$

meaning that close to the instability threshold of the disordered solution, when μ_0 is positive but small, the state of homogeneous order is unstable with respect to long-wavelength perturbations. This instability was first identified in a kinetic-equation analysis by Shi and Ma [23]. Note that condition (50) is valid up to the third order in ϵ (or, equivalently, in the order parameter $||\mathbf{w}||$). It yields the stability line

$$\rho_s = \frac{4\mu_2 - \mu'^2\xi_2}{\mu'^2\xi_1 - 4\mu'}, \tag{51}$$

where $\mu_2 = \mu(\rho = 0)$, $\xi_1 = (1/\xi)'$ and $\xi_2 = (1/\xi(\rho_0 = 0))$. We do not provide here the explicit analytical expression for σ_s because this requires solving a sixth order polynomial.

We remark that the near-threshold instability discussed above is rather generic and appears in 'dry' active matter systems with metric interactions, as opposed to systems with metric-free ones, where the interaction rate is density-independent, and $\mu' = 0$ [24, 32, 33]. In this case (topological active nematics), stability would be enforced by the positive higher order corrections $\mu_0 q^2$ which dominates arbitrarily close to threshold.

4. Inhomogeneous solution

We now show how a spatially inhomogeneous stationary 'band' solution to our hydrodynamic equations can be found. First we remark that our equation for the nematic field, equation (31), is formally the same as that derived in [25] for polar particles with nematic alignment when the polar field is set to zero, as it is imposed here by the complete nematic symmetry of our system. We thus expect an ordered band solution made of two fronts connecting a linearly stable homogeneous disordered state ($\rho = \rho_{\text{gas}} < \rho_t$) and a linearly stable homogeneous ordered state ($\rho = \rho_{\text{band}} > \rho_s$) (see figure 1). Following [25], we rewrite

$$\mu(\rho) = \mu'(\rho - \rho_t) \tag{52}$$

with $\rho_t = (1 - \hat{P}_1)/\mu'$, suppose that the nematic field is aligned along one of the axes and varies only along y. In other words

$$\text{Re}\,(f_1) = f_1(y), \quad \text{Im}\,(f_1) = 0, \quad \rho = \rho(y). \tag{53}$$

Equation (35) then becomes

$$\partial_y^2 \rho = \partial_y^2 f_1 \tag{54}$$

which can be integrated to give

$$\rho = f_1 + Ay + \rho_{\mathrm{gas}}, \tag{55}$$

where A and ρ_{gas} are integration constants. Furthermore, to keep the fields finite for $|y| \to \infty$, one has $A = 0$. By substituting equations (54) and (55) into equation (31) one gets

$$\partial_{yy} f_1 = -4\mu'(\rho_{\mathrm{gas}} - \rho_{\mathrm{t}}) f_1 - 4\mu' f_1^2 + 4\xi f_1^3. \tag{56}$$

We multiply equation (56) by $\partial_y f_1$ and integrate it once to obtain

$$\frac{1}{2}(\partial_y f_1)^2 = -2\mu'(\rho_{\mathrm{gas}} - \rho_{\mathrm{t}}) f_1^2 - \frac{4}{3}\mu' f_1^3 + \xi f_1^4. \tag{57}$$

Separating the variables we obtain

$$\int \mathrm{d}y = \pm \int \frac{\mathrm{d}f_1}{\sqrt{-4\mu'(\rho_{\mathrm{gas}} - \rho_{\mathrm{t}}) f_1^2 - \frac{8}{3}\mu' f_1^3 + 2\xi f_1^4}}. \tag{58}$$

Integration of this equation under the condition $\lim_{y\to\pm\infty} f_1(y) = 0$ gives after simplifications

$$f_1(y) = \frac{3(\rho_{\mathrm{t}} - \rho_{\mathrm{gas}})}{1 + a\cosh\left(2y\sqrt{\mu'(\rho_{\mathrm{t}} - \rho_{\mathrm{gas}})}\right)}, \tag{59}$$

where $a = \sqrt{1 - \frac{9\xi}{2\mu'}(\rho_{\mathrm{t}} - \rho_{\mathrm{gas}})}$. We still need to obtain the value of ρ_{gas} which is fixed by the condition $\int_L \rho(y)\,\mathrm{d}y = \rho_0 L$, where L is the length of the box. In the integral on the lhs we can neglect the exponentially decaying tails and integrate instead on the infinite domain. Furthermore, in the limit $L \to \infty$ we can neglect the exponentially weak dependence of ρ_{gas} on L everywhere except the a term. We then obtain

$$\rho_{\mathrm{gas}} \approx \rho_{\mathrm{t}} - \frac{2\mu'}{9\xi}\left(1 - 4\,\mathrm{e}^{-KL}\right), \tag{60}$$

$$K = \frac{2\sqrt{2}\mu'}{9\sqrt{\xi}}\left(1 + \frac{9\xi}{2\mu'}(\rho_0 - \rho_{\mathrm{t}})\right). \tag{61}$$

Substituting it back into equation (59) we get, under the assumption $L \to \infty$:

$$f_1(y) = \frac{f_1^{\mathrm{band}}}{\left(1 + 2\,\mathrm{e}^{-\frac{KL}{2}}\cosh\left(y\frac{2\sqrt{2}\mu'}{3\sqrt{\xi}}\right)\right)} \quad \text{where} \quad f_1^{\mathrm{band}} = \frac{2\mu'}{3\xi} \tag{62}$$

and we finally obtain the ordered solution density

$$\rho_{\mathrm{band}} = f_1^{\mathrm{band}} + \rho_{\mathrm{gas}} = \rho_{\mathrm{t}} + \frac{4\mu'}{9\xi}\left(1 + 2\,\mathrm{e}^{-KL}\right) \tag{63}$$

with, as expected, $\rho_{\mathrm{band}} > \rho_{\mathrm{t}} > \rho_{\mathrm{gas}}$, which guarantees the stability of both the ordered and disordered parts of the solution. Note that since $f_1^{\mathrm{band}} > 0$ the nematic order is parallel to the x direction (i.e. along the band orientation). This is the opposite of what happens in the Vicsek model, where bands extend transversally with respect to their polarization [15].

We can introduce the band fraction Ω which indicates the fraction of the box occupied by the band. If we suppose that the front width is negligible (once again justified in the limit $L \to \infty$), this band fraction is determined by the equation

$$\Omega(\rho_{\mathrm{band}} - \rho_{\mathrm{gas}}) + \rho_{\mathrm{gas}} = \rho_0. \tag{64}$$

Substituting inside the values of ρ_{gas} and ρ_{band}, we obtain

$$\Omega = \frac{9\xi\,(\rho_0 - \rho_{\text{t}}) + 2\mu'}{6\mu'}. \tag{65}$$

The condition $0 < \Omega < 1$ gives us the lower σ_{min} and upper σ_{max} limits of the existence of bands. As found for polar particles aligning nematically, these limits of existence of the band solution extend *beyond* the region of linear instability of the homogeneous ordered solution (given by $\sigma \in [\sigma_{\text{s}}, \sigma_{\text{t}}]$, see figure 1). In figure 1, we provide a graphical illustration of the shape and properties of the band solution.

An important problem left for future work is the linear stability analysis of the band solution in two space dimensions. This is all the more important as the unpublished work of Shi and Ma [23] suggests the existence of some instability mechanism.

5. Langevin formulation

Being based on a master equation, the derivation we have discussed in the previous sections leads to a set of deterministic PDEs. This is a standard approach in equilibrium statistical physics, where the microscopic fluctuations are integrated out in the coarse-graining process implicit in the definition of a mesoscopic cell size ℓ_{B}. Fluctuations, when needed, can be eventually introduced as an additive, delta correlated stochastic term as in [16]. However, the presence of large density fluctuations [17] suggests that fluctuations may not be faithfully accounted for by some additive noise term. The precise nature of noise correlations at the mesoscopic level cannot be safely overlooked in non-equilibrium systems, as it is known that stochastic terms multiplicative in the relevant fields can radically alter the universality class of mesoscopic theories [34].

In this section, we perform in the same spirit as in [35] a direct coarse-graining of the microscopic dynamics in order to compute the (multiplicative) stochastic terms which emerge at the mesoscopic level. In the following, we restrict the computation to the stochastic terms emerging from the collisionless dynamics. In the dilute, low density regime, where collisions are sparse, this is not a major limitation, since one can reformulate the microscopic dynamics as composed of deterministic collisions separated by several self-diffusion events—see section 5.2 for more details.

For real-space coarse-graining, we make use of a smooth, isotropic, normalized (to one) filter $g_s(r)$ decaying exponentially or faster for $r > s$, e.g. a Gaussian of width s. The fluctuating coarse-grained density and nematic order field are then defined as

$$\rho(\mathbf{x}, t) \equiv \sum_{i=1}^{N} g_s(\mathbf{x}_i^t - \mathbf{x}) \tag{66}$$

and

$$\mathbf{w}(\mathbf{x}, t) \equiv \sum_{i=1}^{N} g_s(\mathbf{x}_i^t - \mathbf{x})\tilde{\mathbf{Q}}_i^t, \tag{67}$$

where we have introduced the microscopic traceless tensor

$$\tilde{\mathbf{Q}}_i^t = \hat{\mathbf{n}}_i^t \hat{\mathbf{n}}_i^t - \frac{\mathbb{I}}{2} = \frac{1}{2}\begin{pmatrix} \cos 2\theta_i^t & \sin 2\theta_i^t \\ \sin 2\theta_i^t & -\cos 2\theta_i^t \end{pmatrix} \equiv \mathbf{Q}(\theta_i^t). \tag{68}$$

5.1. Density field fluctuations

The correlations of density field fluctuations can be derived by generalizing an approach first outlined by Dean [36] for Brownian particles. As mentioned above, we use the collisionless dynamics. We are interested in the time evolution of the density field (66), which is given by

$$\rho(\mathbf{x}, t+\Delta t) = \sum_{i=1}^{N} g_s(\mathbf{x}_i^{t+\Delta t} - \mathbf{x}) = \sum_{i=1}^{N} g_s(\mathbf{x}_i^t + \Delta\mathbf{x}_i^t - \mathbf{x}), \tag{69}$$

where $\Delta\mathbf{x}_i^t = \mathbf{x}_i^{t+\Delta t} - \mathbf{x}_i^t = d_0\, \kappa_j^t\, \hat{\mathbf{n}}_j^t$ (see equation (2)).

Expanding up to second order in powers of $\Delta\mathbf{x}_i^t$ according to Itô calculus [27] one has

$$\partial_t \rho(\mathbf{x}, t) = T_0(\mathbf{x}, t) + T_1(\mathbf{x}, t), \tag{70}$$

where

$$T_0(\mathbf{x}, t) = \frac{d_0^2}{2\tau_\mathrm{d}} \sum_{i=1}^{N} [\hat{\mathbf{n}}_i^t]_\alpha [\hat{\mathbf{n}}_i^t]_\beta \partial_\alpha \partial_\beta g_s(\mathbf{x}_i^t - \mathbf{x}) \tag{71}$$

and

$$T_1(\mathbf{x}, t) = \frac{d_0}{\tau_\mathrm{d}} \sum_{i=1}^{N} \kappa_i^t\, \left(\hat{\mathbf{n}}_i^t \cdot \nabla\right) g_s(\mathbf{x}_i^t - \mathbf{x}). \tag{72}$$

The second order term T_0 yields the deterministic part of the density dynamics. By equations (66), (67) and the definition of the microscopic nematic tensor $\tilde{\mathbf{Q}}$ (equation (68)) one easily gets

$$T_0 = \frac{D_0}{2}(\mathbf{\Gamma} : \mathbf{w}) + \frac{D_0}{2}\Delta\rho, \tag{73}$$

that is, the right-hand side of the diffusion equation (35) in non-rescaled time and space units. The first-order term T_1 gives rise to the (zero average) stochastic term we are interested in. At this stage, T_1 is not a simple function of the mesoscopic fields; however, following [36] it is possible to show that its two point correlation can be recast as a function of ρ and $\mathbf{w}$. Averaging over the random numbers κ_i^t, we have, in the limit $s \to 0$,

$$\begin{aligned} \langle T_1(\mathbf{x}, t) T_1(\mathbf{y}, t')\rangle &= d_0^2 \frac{\delta(t-t')}{\tau_\mathrm{d}} \sum_{i=1}^{N} (\hat{\mathbf{n}}_i^t \cdot \nabla_x)(\hat{\mathbf{n}}_i^t \cdot \nabla_y) g_s(\mathbf{x}_i^t - \mathbf{x}) g_s(\mathbf{x}_i^t - \mathbf{y}) \\ &\simeq d_0^2 \frac{\delta(t-t')}{\tau_\mathrm{d}} \sum_{i=1}^{N} (\hat{\mathbf{n}}_i^t \cdot \nabla_x)(\hat{\mathbf{n}}_i^t \cdot \nabla_y)(g_s(\mathbf{x}-\mathbf{y}) g_s(\mathbf{x}_i^t - \mathbf{x})). \end{aligned} \tag{74}$$

Using equation (68), one then finds, approximating the filter g_s by a Dirac delta in the limit $s \to 0$,

$$\langle T_1(\mathbf{x}, t) T_1(\mathbf{y}, t')\rangle = d_0^2 \frac{\delta(t-t')}{\tau_\mathrm{d}} \partial_\alpha \partial_\beta \left[\delta(\mathbf{x}-\mathbf{y}) \left(w_{\alpha\beta}(\mathbf{x}, t) + \frac{1}{2}\rho(\mathbf{x}, t)\delta_{\alpha\beta} \right) \right]. \tag{75}$$

We can rewrite the noise term T_1 in the stochastically equivalent (i.e. with the same correlations on the mesoscopic scale) form

$$T_1(\mathbf{x}, t) = \nabla \cdot \mathbf{h}(\mathbf{x}, t), \tag{76}$$

where $\mathbf{h}$ is a Gaussian, zero-average vectorial noise, delta-correlated in time with correlations

$$\langle h_\alpha(\mathbf{x},t) h_\beta(\mathbf{y},t')\rangle \simeq \frac{d_0^2}{\tau_\mathrm{d}}\,\delta(t-t')\,\delta(\mathbf{x}-\mathbf{y})\left(w_{\alpha\beta}(\mathbf{x},t)+\frac{\delta_{\alpha\beta}}{2}\rho(\mathbf{x},t)\right). \tag{77}$$

Such a noise term can finally be expressed in the more convenient form

$$h_\alpha(\mathbf{x},t) = K_{\alpha\beta}(\mathbf{x},t)\tilde{h}_\beta(\mathbf{x},t), \tag{78}$$

where the Gaussian noise $\tilde{\mathbf{h}}$ has correlations independent from the hydrodynamic fields

$$\langle \tilde{h}_\alpha(\mathbf{x},t)\tilde{h}_\beta(\mathbf{x}',t')\rangle = 2D_0\,\delta_{\alpha\beta}\,\delta(t-t')\,\delta(\mathbf{x}-\mathbf{x}') \tag{79}$$

and the tensor $\mathbf{K}$ is implicitly defined from the relation $\mathbf{K}\cdot\mathbf{K} = (\rho/2)\mathbf{I}+\mathbf{w}$ (with $\mathbf{I}$ being the identity matrix). In the limit of small $\mathbf{w}$ considered here, we can expand $\mathbf{K}$ to first order in $\mathbf{w}$, yielding

$$\mathbf{K} = \frac{1}{\sqrt{2}}\rho^{1/2}\left(\mathbf{I}+\frac{\mathbf{w}}{\rho}\right). \tag{80}$$

The divergence term $\nabla\cdot$ appearing in T_1 reflects global density conservation, while the proportionality of noise variance to number density can be interpreted as a consequence of the central limit theorem. Adding up the two contributions, one finally gets

$$\partial_t\rho = \frac{D_0}{2}(\boldsymbol{\Gamma}:\mathbf{w}) + \frac{D_0}{2}\Delta\rho + \nabla\cdot\,(\mathbf{K}\cdot\tilde{\mathbf{h}}). \tag{81}$$

The coupling of density fluctuations to the density weighted nematic field $\mathbf{w}=\rho\mathbf{Q}$ can be better understood if we express the deterministic part appearing in the rhs of equation (81) as the sum of an active, non-equilibrium term

$$T_\mathrm{a} = \frac{D_0}{2}\partial_\alpha\left(\rho\partial_\beta\,[\mathbf{Q}]_{\alpha\beta}\right) \tag{82}$$

and a locally anisotropic diffusion term

$$T_\mathrm{m} = \frac{D_0}{2}\partial_\alpha\left([\mathbf{Q}]_{\alpha\beta}\,\partial_\beta\rho\right) + \frac{D_0}{2}\Delta\rho \tag{83}$$

whose mobility, proportional to $(2\mathbf{Q}+\mathbf{I})\rho$, stands in a fluctuation–dissipation relation [37][12] to the multiplicative noise term $\nabla\cdot\,(\mathbf{K}\cdot\tilde{\mathbf{h}})$, since the noise amplitude in equation (77) is proportional to $(2\mathbf{Q}+\mathbf{I})\rho$.

5.2. Nematic field fluctuations

We next discuss fluctuations of the nematic tensor. As seen from equation (67), $\mathbf{w}$ is a function of the $2N$ microscopic stochastic variables $\mathbf{x}_i^t$ and—through the microscopic nematic tensor (68)—θ_i^t, whose dynamics is given by equations (1) and (2). According to Itô calculus, one has

$$\partial_t\mathbf{w} = \boldsymbol{\Omega}_0+\boldsymbol{\Omega}_1+\boldsymbol{\Omega}_2, \tag{84}$$

where $\boldsymbol{\Omega}_0$ is the deterministic part of the coarse-grained collisionless dynamics (which we do not write here explicitly), arising from quadratic contributions in the Itô expansion, while $\boldsymbol{\Omega}_1$

[12] Kumaran discusses fluctuation–dissipation relations when the kinetic coefficient is field dependent.

and $\mathbf{\Omega}_2$ are two stochastic contributions, obtained by first order expansion in, respectively, the angular and spatial stochastic variables,

$$\mathbf{\Omega}_1 = \frac{2}{\tau_\mathrm{d}} \sum_{i=1}^{N} g_s(\mathbf{x}_i^t - \mathbf{x})\, \mathbf{A} \cdot \tilde{\mathbf{Q}}_i^t\, \psi_i^t, \tag{85}$$

$$\mathbf{\Omega}_2 = \frac{d_0}{\tau_\mathrm{d}} \sum_{i=1}^{N} \kappa_i^t \hat{\mathbf{n}}_i^t \cdot \nabla g_s(\mathbf{x}_i^t - \mathbf{x})\, \tilde{\mathbf{Q}}_i^t, \tag{86}$$

where ψ_i^t and κ_i^t are the microscopic noises and $\partial_\theta \tilde{\boldsymbol{Q}}_i^t = 2\, \mathbf{A} \cdot \tilde{\mathbf{Q}}_i^t$, with

$$\mathbf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{87}$$

Note that in $\mathbf{\Omega}_1$ we have retained only the linear contribution in the microscopic noise ψ_i^t. We first focus on the stochastic terms $\mathbf{\Omega}_1$. On coarse-graining scales, averaging over the microscopic noise ψ_i^t, correlations of $\mathbf{\Omega}_1$ are given by

$$\begin{aligned}\langle [\mathbf{\Omega}_1(\mathbf{x}, t)]_{\alpha\beta} \left[\mathbf{\Omega}_1(\mathbf{y}, t')\right]_{\gamma\delta} \rangle &= 4\eta^2 \frac{\delta(t-t')}{\tau_\mathrm{d}} \sum_{i=1}^{N} g_s(\mathbf{x}_i^t - \mathbf{x}) g_s(\mathbf{x}_i^t - \mathbf{y}) \left[\mathbf{A} \cdot \tilde{\mathbf{Q}}_i^t\right]_{\alpha\beta} \left[\mathbf{A} \cdot \tilde{\mathbf{Q}}_i^t\right]_{\gamma\delta} \\ &\approx 4\eta^2 \frac{\delta(t-t')}{\tau_\mathrm{d}} g_s(\mathbf{y}-\mathbf{x}) \sum_{i=1}^{N} g_s(\mathbf{x}_i^t - \mathbf{x}) \left[\mathbf{A} \cdot \tilde{\mathbf{Q}}_i^t\right]_{\alpha\beta} \left[\mathbf{A} \cdot \tilde{\mathbf{Q}}_i^t\right]_{\gamma\delta}. \end{aligned} \tag{88}$$

To evaluate this correlator, we determine the average value $\langle \sum_i g_s\, (\mathbf{A} \cdot \tilde{\mathbf{Q}}_i^t)(\mathbf{A} \cdot \tilde{\mathbf{Q}}_i^t) \rangle$, in the framework of the deterministic dynamics studied in section 2, namely

$$\left\langle \sum_{i=1}^{N} g_s(\mathbf{x}_i^t - \mathbf{x}) \left[\mathbf{A} \cdot \tilde{\mathbf{Q}}_i^t\right]_{\alpha\beta} \left[\mathbf{A} \cdot \tilde{\mathbf{Q}}_i^t\right]_{\gamma\delta} \right\rangle = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{d}\theta\, f(\mathbf{x}, \theta, t)\, [\mathbf{A} \cdot \mathbf{Q}(\theta)]_{\alpha\beta}\, [\mathbf{A} \cdot \mathbf{Q}(\theta)]_{\gamma\delta}\,. \tag{89}$$

After some rather lengthy calculations (see appendix C), one finds

$$\begin{aligned}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{d}\theta\, f(\mathbf{x}, \theta, t)\, [\mathbf{A} \cdot \mathbf{Q}(\theta)]_{\alpha\beta}\, [\mathbf{A} \cdot \mathbf{Q}(\theta)]_{\gamma\delta} &= \rho\, J_{\alpha\beta\gamma\delta} + \frac{2b_2}{a_2} \left[(w_{\mu\nu} w_{\mu\nu}) J_{\alpha\beta\gamma\delta} - 2 w_{\alpha\beta} w_{\gamma\delta}\right] \\ &+ \frac{1}{4a_2} \left[\Gamma_{\mu\nu} w_{\mu\nu} J_{\alpha\beta\gamma\delta} - \Gamma_{\alpha\beta} w_{\gamma\delta} - \Gamma_{\gamma\delta} w_{\alpha\beta}\right], \end{aligned} \tag{90}$$

where we have introduced the tensor

$$J_{\alpha\beta\gamma\delta} = \tfrac{1}{2} \left(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma} - \delta_{\alpha\beta}\delta_{\gamma\delta}\right) \tag{91}$$

which plays the role of a unit tensor for the double contraction of symmetric traceless tensors, e.g. $w_{\alpha\beta} = J_{\alpha\beta\mu\nu}\, w_{\mu\nu}$. In order to characterize the noise $\mathbf{\Omega}_1$, we introduce the following change of variables:

$$[\mathbf{\Omega}_1(\mathbf{x}, t)]_{\alpha\beta} = H_{\alpha\beta\mu\nu}(\mathbf{x}, t)\, \tilde{\Omega}_{\mu\nu}(\mathbf{x}, t), \tag{92}$$

where $\tilde{\mathbf{\Omega}}$ is a tensorial symmetric traceless white noise, such that

$$\langle \tilde{\Omega}_{\alpha\beta}(\mathbf{x},t)\tilde{\Omega}_{\gamma\delta}(\mathbf{y},t')\rangle = 2D\delta(\mathbf{x}-\mathbf{y})\,\delta(t-t')\,J_{\alpha\beta\gamma\delta} \tag{93}$$

with $D = 2\eta^2/\tau_{\rm d}$. The correlation of $\mathbf{\Omega}_1$ then reads

$$\langle [\mathbf{\Omega}_1(\mathbf{x},t)]_{\alpha\beta}\left[\mathbf{\Omega}_1(\mathbf{y},t')\right]_{\gamma\delta}\rangle = 2D\delta(\mathbf{x}-\mathbf{y})\,\delta(t-t')\,H_{\alpha\beta\mu\nu}(\mathbf{x},t)H_{\gamma\delta\mu\nu}(\mathbf{x},t). \tag{94}$$

By identification with equation (88), and using equation (90), one eventually finds for $\mathbf{H}$

$$\begin{aligned} H_{\alpha\beta\gamma\delta} = \rho^{1/2}\,J_{\alpha\beta\gamma\delta} + \frac{b_2}{a_2\,\rho^{1/2}}\left[w_{\mu\nu}w_{\mu\nu}J_{\alpha\beta\gamma\delta} - 2w_{\alpha\beta}w_{\gamma\delta}\right] \\ + \frac{1}{8a_2\,\rho^{1/2}}\left[\Gamma_{\mu\nu}w_{\mu\nu}J_{\alpha\beta\gamma\delta} - \Gamma_{\alpha\beta}w_{\gamma\delta} - \Gamma_{\gamma\delta}w_{\alpha\beta}\right]. \end{aligned} \tag{95}$$

Note that, in agreement with the central limit theorem, $\mathbf{\Omega}_1$ is (at least to first order in $\mathbf{w}$) proportional to the square root of local density.

The second stochastic term $\mathbf{\Omega}_2$, finally, can be treated similarly, but it would give rise to a conserved noise (due to the presence of ∇ terms) akin to the one discussed for the density equations, thus related to density fluctuations affecting the $\mathbf{w} = \rho\mathbf{Q}$ field. We discard such conserved term as irrelevant (in the renormalization group sense) with respect to the non-conserved multiplicative noise $\mathbf{\Omega}_1$.

In order to write down the complete Langevin equation, one also needs to evaluate the contribution of the deterministic part $\mathbf{\Omega}_0$. However, expressing this contribution in terms of the fluctuating fields ρ and $\mathbf{w}$ turns out to be a very complicated task. One should also take into account collisions between particles, and not only the collisionless dynamics described by $\mathbf{\Omega}_0$. However, as mentioned earlier in this section, in the low density limit where our kinetic approach is justified, the microscopic dynamics (1)–(2) can be reformulated as deterministic binary collisions separated by several self-diffusion events, at the cost of a rescaling of the angular noise amplitude (note that this reformulation is not exact, in the sense that self-diffusion events no-longer have a Poissonian statistics). As a first approximation, this rescaling of the angular noise amplitude results in a global rescaling of the noise term $\tilde{\Omega}$ by a phenomenological factor χ. Therefore, we believe our method to provide essentially the correct relevant stochastic correlations for the nematic Langevin dynamics, up to an order one unknown multiplicative constant.

In addition, microscopic collisions could provide a further fluctuation source due to disorder below the coarse-graining scale, like the randomness of collision times. While we conjecture them to be irrelevant, we leave a final settlement of this difficult problem for future work, and use for the deterministic part of the dynamics the hydrodynamic equation (36), derived from the Boltzmann approach.

We thus finally obtain the stochastic equation for the nematic field (in rescaled units)

$$\partial_t \mathbf{w} = \mu\,\mathbf{w} - 2\xi\,\mathbf{w}\,(\mathbf{w}:\mathbf{w}) + \tfrac{1}{2}\Delta\mathbf{w} + \tfrac{1}{8}\mathbf{\Gamma}\rho + \chi\mathbf{H}:\tilde{\mathbf{\Omega}}. \tag{96}$$

A few remarks are in order: firstly, our expressions of the noise amplitudes $\mathbf{K}$ and $\mathbf{H}$ (equations (80) and (95)) suggest that the stochastic terms might be better expressed in terms of the field $\mathbf{Q}$, rather than $\mathbf{w} = \rho\mathbf{Q}$; secondly, equations similar to equations (81) and (96) were also derived from an apolar Vicsek-style model in [29].

In spite of the limitations listed above, the present approach already provides us with useful information on the statistics of the noise terms, which is seen to differ significantly from the white noise postulated on a phenomenological basis in previous works. On top of the overal $\rho^{1/2}$ dependency, our calculation reveals a non-trivial dependence of the correlation of the noise on the nematic order parameter (see equations (80), (81), (94) and (95)).

6. Conclusions

To summarize, using as a starting point the simple active nematics model of [26], we have demonstrated how one can derive in a systematic manner a continuous mesoscopic description. We formulated a version of the BGL approach put forward in [24, 25] for this case where (anisotropic) diffusion dominates, deriving a simple hydrodynamic equation for the nematic ordering field –equation (36). We have then used a direct coarse-graining approach to endow the hydrodynamic equations with proper noise terms.

The next stage, left for future work, consists in studying the stochastic PDEs obtained. At the linear level, it is clear that in the long-wavelength limit, standard results on giant density fluctuations [17] are recovered. However, the large amplitude of density fluctuations calls for a nonlinear analysis (which turns out to be very difficult), where the density dependence of the noise derived in section 5 may play an important role. Ideally, one should try to tackle this issue by applying methods from field theory and renormalization group analysis. In addition, we note that the multiplicative nature of the noise may also affect finite-wavelength properties, like coarsening behavior. The analysis of the stochastic PDEs can be done numerically, but some care must be taken when dealing with the multiplicative, conserved noise terms in (81).

Pending such attempts, some remarks and comments are already in order: like all previous cases studied before, the hydrodynamic equations found exhibit a domain of linear instability of the homogeneous ordered solution bordering the basic transition line σ_t. This solution does become linearly stable deeper in the ordered phase (for σ below σ_s). Moreover, we have found that the long-wavelength instability of the homogeneous ordered solution leads to a nonlinear, inhomogeneous band solution—see equation (62)—and that this band solution exists beyond the $[\sigma_s, \sigma_t]$ interval. These coexistence regions suggest, at the fluctuating level, *discontinuous* transitions.

This seems to be at odds with the reported behavior of the original microscopic model: (i) the order/disorder transition has been reported to be of the Kosterlitz–Thouless type [26]; (ii) there is no trace, at the microscopic level, of the existence of a non-segregated, homogeneous phase; (iii) coming back to giant number fluctuations, we note that the standard calculation is made in the homogeneous ordered phase whereas the numerical evidence for them reported in [26] appears now to have been obtained in the inhomogeneous phase. All this calls for revisiting the simple particle-based model and, eventually, understanding its behavior in the context of the stochastic continuum theory constructed here.

Acknowledgments

Part of this work was performed at the Max Planck Institute for the Physics of Complex Systems in Dresden, Germany, within the Advanced Study Group 2011/2012 'Statistical Physics of Collective Motion'. FG acknowledges support by EPSRC First Grant EP/K018450/1.

Appendix A. Fourier expansion of the master equation

We provide in this appendix details of the Fourier expansion of the master equation (10), leading to equation (21). Multiplying equation (10) by $e^{i2\theta}$ and integrating over θ, one gets

$$\partial_t \hat{f}_k = \partial_\alpha \partial_\beta \int_{-\pi/2}^{\pi/2} d\theta\, e^{i2k\theta} \hat{n}_\alpha(\theta) \hat{n}_\beta(\theta) f(\mathbf{x}, \theta, t) + \int_{-\pi/2}^{\pi/2} d\theta\, e^{i2k\theta} I_{\mathrm{diff}}[f] + \int_{-\pi/2}^{\pi/2} d\theta\, e^{i2k\theta} I_{\mathrm{coll}}[f, f]. \tag{A.1}$$

In the following, we successively compute each term of the rhs of equation (A.1).

A.1. Diffusion-like term

Let us define $Q_{\alpha\beta}(\theta)$ as

$$Q_{\alpha\beta}(\theta) = \hat{n}_\alpha(\theta)\hat{n}_\beta(\theta) - \frac{\delta_{\alpha\beta}}{2}. \tag{A.2}$$

We then have

$$\begin{aligned} Q_{11}(\theta, t) = -Q_{22}(\theta, t) &= \frac{1}{2}\cos 2\theta = \frac{e^{i2\theta} + e^{-i2\theta}}{4}, \\ Q_{12}(\theta, t) = Q_{21}(\theta, t) &= \frac{1}{2}\sin 2\theta = \frac{e^{i2\theta} - e^{-i2\theta}}{4i}. \end{aligned} \tag{A.3}$$

As a result,

$$\begin{aligned} \partial_\alpha \partial_\beta \int_{-\pi/2}^{\pi/2} d\theta\, e^{i2k\theta} \hat{n}_\alpha(\theta)\hat{n}_\beta(\theta) f(\theta) &= \partial_\alpha \partial_\beta \int_{-\pi/2}^{\pi/2} d\theta\, e^{i2k\theta} \left(Q_{\alpha\beta}(\theta) + \frac{\delta_{\alpha\beta}}{2} \right) f(\theta) \\ &= \frac{1}{2}\Delta \hat{f}_k + \frac{1}{4}\left(\nabla^{*2} \hat{f}_{k+1} + \nabla^2 \hat{f}_{k-1} \right). \end{aligned} \tag{A.4}$$

A.2. Self-diffusion term

We have rather straightforwardly

$$\begin{aligned} \int_{-\pi/2}^{\pi/2} d\theta\, e^{i2k\theta} I_{\mathrm{diff}}[f] &= -\hat{f}_k + \int_{-\pi/2}^{\pi/2} d\theta'\, e^{i2k\theta'} f(\theta') \int_{-\infty}^{\infty} d\zeta\, e^{i2k\zeta} P(\zeta) \\ &= \left[\hat{P}_k - 1 \right] \hat{f}_k, \end{aligned} \tag{A.5}$$

where

$$\hat{P}_k = \int_{-\infty}^{\infty} d\zeta\, e^{i2k\zeta} P(\zeta) \tag{A.6}$$

is the Fourier transform of $P(\zeta)$.

A.3. Binary collisions term

Let us split the Fourier transformed collision integral into an outgoing (negative) collision term $I_k^{(-)}$ and an ingoing (positive) collision term $I_k^{(+)}$. A direct integration of the outgoing collision term yields, using $K(\theta,\theta')=\tilde{K}(\theta-\theta')$,

$$I_k^{(-)} \equiv -\int_{-\pi/2}^{\pi/2} \mathrm{d}\theta\, \mathrm{e}^{\mathrm{i}2k\theta} f(\theta) \int_{-\pi/2}^{\pi/2} \mathrm{d}\theta' f(\theta')\tilde{K}(\theta-\theta') = -\frac{1}{\pi}\sum_q \hat{K}_q \hat{f}_q \hat{f}_{k-q}, \tag{A.7}$$

where $\hat{K}_q$ is the Fourier coefficient of $\tilde{K}(\theta-\theta')$ given by, using equation (14),

$$\hat{K}_q = \int_{-\pi/2}^{\pi/2} \mathrm{d}\theta\, \mathrm{e}^{\mathrm{i}2q\theta}\left[\left|\sin\frac{\theta-\theta'}{2}\right| + \left|\cos\frac{\theta-\theta'}{2}\right|\right] = \frac{4}{1-16q^2}. \tag{A.8}$$

Then, the calculation of the ingoing collision term requires a few steps. After integration of the (generalized) Dirac delta δ_π, we have

$$I_k^{(+)} = \hat{P}_k \int_{-\pi/2}^{\pi/2} \mathrm{d}\theta_1 \int_{-\pi/2}^{\pi/2} \mathrm{d}\theta_2\, \mathrm{e}^{\mathrm{i}2k\Psi(\theta_1,\theta_2)} f(\theta_1)\tilde{K}(\theta_1-\theta_2) f(\theta_2). \tag{A.9}$$

By the change of variables $\phi=\theta_1-\theta_2$, one gets

$$I_k^{(+)} = \hat{P}_k \int_{-\pi/2}^{\pi/2} \mathrm{d}\theta_2 \int_{-\pi/2-\theta_2}^{\pi/2-\theta_2} \mathrm{d}\phi\, \mathrm{e}^{\mathrm{i}2k\Psi(\theta_2+\phi,\theta_2)} f(\theta_2+\phi)\tilde{K}(\phi) f(\theta_2). \tag{A.10}$$

Using the π-periodicity of the integrand with respect to ϕ, we can change the integration interval on ϕ, yielding

$$I_k^{(+)} = \hat{P}_k \int_{-\pi/2}^{\pi/2} \mathrm{d}\theta_2 \int_{-\pi/2}^{\pi/2} \mathrm{d}\phi\, \mathrm{e}^{\mathrm{i}2k\Psi(\theta_2+\phi,\theta_2)} f(\theta_2+\phi)\tilde{K}(\phi) f(\theta_2). \tag{A.11}$$

On this interval of ϕ, one has from equation (13)

$$\Psi(\theta_2+\phi,\theta_2) = \theta_2 + \frac{\phi}{2}. \tag{A.12}$$

Expanding f in Fourier series (see equations (16) and (17)), we get

$$I_k^{(+)} = \frac{\hat{P}_k}{\pi^2}\sum_{q,q'} \hat{f}_q \hat{f}_{q'} \int_{-\pi/2}^{\pi/2} \mathrm{d}\theta_2\, \mathrm{e}^{\mathrm{i}2(k-q-q')\theta_2} \int_{-\pi/2}^{\pi/2} \mathrm{d}\phi\, \mathrm{e}^{\mathrm{i}(k-2q)\phi}\tilde{K}(\phi). \tag{A.13}$$

The integral over θ_2 is equal to $\pi\delta_{k,q+q'}$. Defining

$$\hat{J}_{k,q} = \int_{-\pi/2}^{\pi/2} \mathrm{d}\phi\, \mathrm{e}^{\mathrm{i}(k-2q)\phi}\tilde{K}(\phi), \tag{A.14}$$

we finally obtain

$$I_k^{(+)} = \frac{\hat{P}_k}{\pi}\sum_q \hat{J}_{k,q}\hat{f}_q \hat{f}_{k-q}. \tag{A.15}$$

The coefficient $\hat{J}_{k,q}$ can be computed explicitly, leading to

$$\hat{J}_{k,q} = 4\,\frac{1+2\sqrt{2}(2q-k)(-1)^q \sin\left(\frac{k\pi}{2}\right)}{1-4(2q-k)^2}. \tag{A.16}$$

Note finally that $\hat{J}_{0,q}=\hat{K}_q$.

Appendix B. Curvature-induced current and equilibrium limit

In this appendix, we show explicitly that the curvature-induced current, that is the term $\frac{1}{2}\mathrm{Re}(\nabla^{*2}\hat{f}_1)$ appearing in the continuity equation (23), originates from the coupling of orientation with motility. To this aim, we consider a slightly generalized microscopic process w.r.t. equations (1) and (2), where particles are also allowed to move perpendicular w.r.t. to the nematic tensor. Replace equation (2) by

$$\mathbf{x}_i^{t+\Delta t} = \mathbf{x}_i^t + d_0\,\mathbf{R}\left(\theta_i^t\right), \tag{B.1}$$

where $\mathbf{R}(\theta)$ is a stochastic operator defining the coupling between orientation and particle motion

$$\mathbf{R}(\theta) = \begin{cases} \hat{\mathbf{n}}(\theta) & \text{w.p. } p/2, \\ -\hat{\mathbf{n}}(\theta) & \text{w.p. } p/2, \\ \hat{\mathbf{n}}^{\perp}(\theta) & \text{w.p. } (1-p)/2, \\ -\hat{\mathbf{n}}^{\perp}(\theta) & \text{w.p. } (1-p)/2, \end{cases} \tag{B.2}$$

where $0 \leqslant p \leqslant 1$, w.p. stands for 'with probability' and $\hat{\mathbf{n}}^{\perp}(\theta) = \hat{\mathbf{n}}(\theta + \pi/2)$ is the perpendicular director. The standard active nematic case is recovered for $p = 1$, while $p = 1/2$ corresponds to an isotropic random walk, a case for which motion is decorrelated from order. The corresponding collisionless master equation reads

$$f(\mathbf{x},\theta,t+\Delta t) = \frac{p}{2}\left[f(\mathbf{x}-\hat{\mathbf{n}}(\theta)d_0,\theta,t) + f(\mathbf{x}+\hat{\mathbf{n}}(\theta)d_0,\theta,t)\right]$$
$$+\frac{(1-p)}{2}\left[f(\mathbf{x}-\hat{\mathbf{n}}^{\perp}(\theta)d_0,\theta,t) + f(\mathbf{x}+\hat{\mathbf{n}}^{\perp}(\theta)d_0,\theta,t)\right]. \tag{B.3}$$

By making use of Itô calculus, one gets at the mesoscopic timescale τ_{B}

$$\partial_t f(\mathbf{x},\theta,t) = (2p-1)\partial_\alpha\partial_\beta\left[\hat{n}_\alpha(\theta)\hat{n}_\beta(\theta) - \frac{\delta_{\alpha\beta}}{2}\right] f(\mathbf{x},\theta,t) + \frac{1}{2}\Delta f(\mathbf{x},\theta,t), \tag{B.4}$$

where we have used the identity $\hat{n}_\alpha^{\perp}(\theta)\hat{n}_\beta^{\perp}(\theta) = \delta_{\alpha\beta} - \hat{n}_\alpha(\theta)\hat{n}_\beta(\theta)$. By considering the zeroth-order Fourier term of f (for which collision and angular diffusion terms vanish), one obtains the continuity equation (see also [29] for a similar result)

$$\partial_t \rho = \frac{1}{2}\Delta\rho + \frac{2p-1}{2}\mathrm{Re}\left(\nabla^{*2} f_1\right) \tag{B.5}$$

which shows that the non-equilibrium current vanishes for $p = \frac{1}{2}$.

Appendix C. Correlation of the nematic field fluctuations

In this appendix, we wish to compute the rhs of equation (89), that is, the non-trivial part of the noise correlation in the nematic order parameter equation. In other words, we need to compute the fourth-rank tensor

$$R_{\alpha\beta\gamma\delta} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{d}\theta\; f(\mathbf{x},\theta,t)\,[\mathbf{A}\cdot\mathbf{Q}(\theta)]_{\alpha\beta}\,[\mathbf{A}\cdot\mathbf{Q}(\theta)]_{\gamma\delta}\,. \tag{C.1}$$

This is *a priori* a complicated task, and it will be useful to use a mapping between tensors and 'multicomplex' numbers (see, e.g. http://en.wikipedia.org/wiki/Multicomplex_number).

Although such a mapping is perhaps not a standard method, it will prove a very convenient technique in the following. We present this mapping in appendix C.1, and we sketch the main steps of the calculation in appendix C.2.

C.1. Mapping between tensors and 'multicomplex' numbers

Let us first emphasize that the mapping presented here is restricted to two-dimensional spaces. Before turning to tensors, we recall the rather natural correspondence between two-dimensional vectors and complex numbers: a vector $\mathbf{V} = (V_1, V_2)$ can be mapped onto a complex number $\mathcal{V} = V_1 + \mathrm{i}V_2$, where $\mathrm{i}^2 = -1$. Denoting the mapping by a double arrow $\leftrightarrow$, we can write

$$\mathbf{V} = (V_1, V_2) \longleftrightarrow \mathcal{V} = V_1 + \mathrm{i}V_2 = V_\alpha \mathrm{i}^{\alpha-1}. \tag{C.2}$$

Here, and in what follows, a sum over repeated indices is understood. The scalar product between two vectors $\mathbf{V}$ and $\mathbf{V}'$ can be expressed in complex notations as

$$\mathbf{V}\cdot\mathbf{V}' = \frac{1}{2}\left(\mathcal{V}\mathcal{V}'^* + \mathcal{V}^*\mathcal{V}'\right). \tag{C.3}$$

Such a mapping can be generalized to describe tensors by introducing several imaginary numbers i, j, ..., which commute one with the other. For instance, a second rank tensor $\mathbf{Q}$ of components $Q_{\alpha\beta}$ can be mapped onto a 'bicomplex' number

$$Q_{\alpha\beta} \longleftrightarrow \mathcal{Q} = Q_{11} + \mathrm{i}Q_{21} + \mathrm{j}Q_{12} + \mathrm{ij}Q_{22} = Q_{\alpha\beta}\, \mathrm{i}^{\alpha-1}\mathrm{j}^{\beta-1} \tag{C.4}$$

where $\mathrm{i}^2 = \mathrm{j}^2 = -1$ and i,j commute. The contraction operation between two tensors can also be expressed in complex notations, through a generalization of equation (C.3). A symmetric traceless tensor $\mathbf{Q}$ reads in complex notations

$$\mathcal{Q} = Q_{11} + (\mathrm{i}+\mathrm{j})Q_{12} - \mathrm{ij}Q_{11} = (1-\mathrm{ij})(Q_{11} + \mathrm{i}Q_{12}) \tag{C.5}$$

and is thus fully characterized by the complex number $Q_{11} + \mathrm{i}Q_{12}$. For instance, the local nematic tensor $\mathbf{Q}(\theta)$ defined in equation (68) maps onto $\frac{1}{2}(1-\mathrm{ij})\,\mathrm{e}^{2\mathrm{i}\theta}$, and the nematic field $\mathbf{w}$ maps onto $\frac{1}{2}(1-\mathrm{ij})\,f_1$. These simple relations will be useful in the following.

In addition, one of the interests of the complex notation is that tensorial products simply map onto products of complex numbers. Starting from two vectors $\mathbf{V}$ and $\mathbf{V}'$, their tensorial product maps as follows:

$$P_{\alpha\beta} = V_\alpha V'_\beta \longleftrightarrow \mathcal{P} = (V_1 + \mathrm{i}V_2)(V'_1 + \mathrm{j}V'_2). \tag{C.6}$$

The introduction of fourth rank tensors follows the same line. Introducing four independent imaginary numbers i, j, k, l ($\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = \mathrm{l}^2 = -1$), a tensor of components $R_{\alpha\beta\gamma\delta}$ can be mapped onto a 'quadricomplex' number:

$$R_{\alpha\beta\gamma\delta} \longleftrightarrow \mathcal{R} = R_{\alpha\beta\gamma\delta}\, \mathrm{i}^{\alpha-1}\mathrm{j}^{\beta-1}\mathrm{k}^{\gamma-1}\mathrm{l}^{\delta-1}. \tag{C.7}$$

If $\mathbf{R}$ is the tensorial product of two second rank tensors $\mathbf{Q}$ and $\mathbf{Q}'$, namely $R_{\alpha\beta\gamma\delta} = Q_{\alpha\beta}Q'_{\gamma\delta}$, the associated 'quadricomplex' number is the product of two bicomplex numbers

$$\mathcal{R} = \left(Q_{\alpha\beta}\, \mathrm{i}^{\alpha-1}\mathrm{j}^{\beta-1}\right)\left(Q'_{\gamma\delta}\, \mathrm{k}^{\gamma-1}\mathrm{l}^{\delta-1}\right). \tag{C.8}$$

In particular, if $\mathbf{Q}$ and $\mathbf{Q}'$ are symmetric traceless tensors, the above expression simplifies to

$$\mathcal{R} = (1-\mathrm{ij})(1-\mathrm{kl})(Q_{11} + \mathrm{i}Q_{12})(Q'_{11} + \mathrm{k}Q'_{12}). \tag{C.9}$$

Such relations will turn useful in the following calculations.

C.2. Calculation of the correlation

We now turn to the calculation of the fourth rank tensor defined in equation (C.1), using the mapping to 'multicomplex' numbers introduced above. We first need to evaluate $\mathbf{A}\cdot\mathbf{Q}(\theta)$ in these notations. It is rather straightforward to show that

$$(\mathbf{A}\cdot\mathbf{Q}(\theta))_{\alpha\beta} \longleftrightarrow \mathrm{i}\mathcal{Q}(\theta)=\tfrac{1}{2}(\mathrm{i}+\mathrm{j})\,\mathrm{e}^{2\mathrm{i}\theta}. \tag{C.10}$$

This can be shown from the mapping $\mathbf{A}\leftrightarrow \mathrm{i}-\mathrm{j}$, using a generalization of equation (C.3). The resulting expression is however simple to interpret, since the multiplication by i corresponds to a rotation by an angle of $\frac{\pi}{2}$ in the complex plane. Computing the tensorial product $(\mathbf{A}\cdot\mathbf{Q})(\mathbf{A}\cdot\mathbf{Q})$, one finds

$$(\mathbf{A}\cdot\mathbf{Q}(\theta))_{\alpha\beta}(\mathbf{A}\cdot\mathbf{Q}(\theta))_{\gamma\delta} \longleftrightarrow \tfrac{1}{4}(\mathrm{i}+\mathrm{j})(\mathrm{k}+\mathrm{l})\,\mathrm{e}^{2\mathrm{i}\theta}\,\mathrm{e}^{2k\theta}. \tag{C.11}$$

Expanding the two exponentials and using standard trigonometric relations leads to

$$\mathrm{e}^{2\mathrm{i}\theta}\,\mathrm{e}^{2k\theta}=\tfrac{1}{2}(1+\mathrm{ik})+\tfrac{1}{2}(1-\mathrm{ik})\,\mathrm{e}^{4\mathrm{i}\theta}. \tag{C.12}$$

As a result,

$$(\mathbf{A}\cdot\mathbf{Q}(\theta))_{\alpha\beta}(\mathbf{A}\cdot\mathbf{Q}(\theta))_{\gamma\delta} \longleftrightarrow \tfrac{1}{8}(1+\mathrm{ik})(\mathrm{i}+\mathrm{j})(\mathrm{k}+\mathrm{l})+\tfrac{1}{8}(1-\mathrm{ik})(\mathrm{i}+\mathrm{j})(\mathrm{k}+\mathrm{l})\,\mathrm{e}^{4\mathrm{i}\theta}. \tag{C.13}$$

Computing the average over the phase-space distribution $f(\mathbf{x},\theta,t)$ leads to

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\mathrm{d}\theta\, f(\mathbf{x},\theta,t)\,[\mathbf{A}\cdot\mathbf{Q}(\theta)]_{\alpha\beta}\,[\mathbf{A}\cdot\mathbf{Q}(\theta)]_{\gamma\delta}$$
$$\longleftrightarrow \frac{1}{8}(1+\mathrm{ik})(\mathrm{i}+\mathrm{j})(\mathrm{k}+\mathrm{l})\rho(\mathbf{x},t)+\frac{1}{8}(1-\mathrm{ik})(\mathrm{i}+\mathrm{j})(\mathrm{k}+\mathrm{l})\,f_1(\mathbf{x},t). \tag{C.14}$$

It is then necessary to use the closure relation (30), yielding

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\mathrm{d}\theta\, f(\mathbf{x},\theta,t)\,[\mathbf{A}\cdot\mathbf{Q}(\theta)]_{\alpha\beta}\,[\mathbf{A}\cdot\mathbf{Q}(\theta)]_{\gamma\delta} \longleftrightarrow \frac{1}{8}(1+\mathrm{ik})(\mathrm{i}+\mathrm{j})(\mathrm{k}+\mathrm{l})\rho$$
$$+\frac{b_2}{8a_2}(1-\mathrm{ik})(\mathrm{i}+\mathrm{j})(\mathrm{k}+\mathrm{l})\,f_1^2+\frac{1}{32a_2}(1-\mathrm{ik})(\mathrm{i}+\mathrm{j})(\mathrm{k}+\mathrm{l})\nabla^2 f_1. \tag{C.15}$$

We now need to perform the inverse mapping onto tensors. Let us start with the first term in the rhs of equation (C.15). Looking for a tensor of the form $a\delta_{\alpha\gamma}\delta_{\beta\delta}+b\delta_{\alpha\delta}\delta_{\beta\gamma}+c\delta_{\alpha\beta}\delta_{\gamma\delta}$, one finds by identification of the associated complex expressions that

$$(1+\mathrm{ik})(\mathrm{i}+\mathrm{j})(\mathrm{k}+\mathrm{l}) \longleftrightarrow \delta_{\alpha\gamma}\delta_{\beta\delta}+\delta_{\alpha\delta}\delta_{\beta\gamma}-\delta_{\alpha\beta}\delta_{\gamma\delta}\equiv 2J_{\alpha\beta\gamma\delta}. \tag{C.16}$$

The second and third terms in the rhs of equation (C.15) can be computed along the same line. Both terms have a similar structure, which can be formally written as $(1-\mathrm{ik})(\mathrm{i}+\mathrm{j})(\mathrm{k}+\mathrm{l})(C_{11}+\mathrm{i}C_{12})(D_{11}+\mathrm{i}D_{12})$, where $C_{\alpha\beta}$ and $D_{\alpha\beta}$ are symmetric traceless tensors that do not necessarily commute ($C_{\alpha\beta}$ may include derivation operators). One then finds, again by identification of the complex forms

$$(1-\mathrm{ik})(\mathrm{i}+\mathrm{j})(\mathrm{k}+\mathrm{l})(C_{11}+\mathrm{i}C_{12})(D_{11}+\mathrm{i}D_{12}) \longleftrightarrow C_{\mu\nu}D_{\mu\nu}J_{\alpha\beta\gamma\delta}-C_{\alpha\beta}D_{\gamma\delta}-C_{\gamma\delta}D_{\alpha\beta}. \tag{C.17}$$

As a result, we find

$$(1-\mathrm{ik})(\mathrm{i}+\mathrm{j})(\mathrm{k}+\mathrm{l})f_1^2 \longleftrightarrow 4\left[(w_{\mu\nu}w_{\mu\nu})J_{\alpha\beta\gamma\delta}-2w_{\alpha\beta}w_{\gamma\delta}\right], \tag{C.18}$$

$$(1-\mathrm{ik})(\mathrm{i}+\mathrm{j})(\mathrm{k}+\mathrm{l})\nabla^2 f_1 \longleftrightarrow 2\left[\Gamma_{\mu\nu}w_{\mu\nu}J_{\alpha\beta\gamma\delta}-\Gamma_{\alpha\beta}w_{\gamma\delta}-\Gamma_{\gamma\delta}w_{\alpha\beta}\right]. \tag{C.19}$$

Gathering all terms, one then recovers equation (90) from equation (C.15).

References

[1] Ramaswamy S 2010 *Annu. Rev. Condens. Matter Phys.* **1** 323
[2] Romanczuk P, Bär M, Ebeling W, Lindner B and Schimansky-Geier L 2012 *Eur. Phys. J. Spec. Top.* **202** 1162
[3] Marchetti M C *et al* 2013 *Rev. Mod. Phys.* **85** 1143
[4] Parrish J K and Hamner W M (ed) 1997 *Animal Groups in Three Dimensions* (Cambridge: Cambridge University Press)
[5] Couzin I D, Krause J, Franks N and Levin S 2005 *Nature* **433** 513
Buhl J *et al* 2006 *Science* **312** 1402
[6] Ballerini M *et al* 2008 *Proc. Natl Acad. Sci. USA* **105** 1232
[7] Helbing D, Farkas I and Vicsek T 2000 *Nature* **407** 487
Helbing D, Farkas I J and Vicsek T 2000 *Phys. Rev. Lett.* **84** 1240
[8] Moussaïd M, Helbing D and Théraulaz G 2011 *Proc. Natl Acad. Sci. USA* **108** 6884
[9] Schaller V, Weber C A, Semmerich C, Frey E and Bausch A 2010 *Nature* **467** 73
Schaller V, Weber C, Frey E and Bausch A R 2011 *Soft Matter* **7** 3213
Schaller V, Weber C A, Hammerich B, Frey E and Bausch A R 2011 *Proc. Natl Acad. Sci. USA* **108** 19183
Sumino Y, Nagai K H, Shitaka Y, Tanaka D, Yoshikawa K, Chaté H and Oiwa K 2012 *Nature* **483** 448
[10] Peruani F, Starruß J, Jakovljevic V, Sogaard-Andersen L, Deutsch A and Bär M 2012 *Phys. Rev. Lett.* **108** 098102
[11] Kudrolli A, Lumay G, Volfson D and Tsimring L S 2008 *Phys. Rev. Lett.* **100** 058001
[12] Narayan V, Ramaswamy S and Menon N 2007 *Science* **317** 105
[13] Deseigne J, Dauchot O and Chaté H 2010 *Phys. Rev. Lett.* **105** 098001
Deseigne J, Léonard S, Dauchot O and Chaté H 2012 *Soft. Matter* **8** 5629
[14] Vicsek T *et al* 1995 *Phys. Rev. Lett.* **75** 1226
[15] Grégoire G and Chaté H 2004 *Phys. Rev. Lett.* **92** 025702
Chaté H *et al* 2008 *Phys. Rev.* E **77** 046113
[16] Toner J and Tu Y 1995 *Phys. Rev. Lett.* **75** 4326
Toner J and Tu Y 1998 *Phys. Rev.* E **58** 4828
Toner J 2012 *Phys. Rev.* E **86** 031918
[17] Ramaswamy S, Simha R A and Toner J 2003 *Europhys. Lett.* **62** 196
Toner J, Tu Y and Ramaswamy S 2005 *Ann. Phys. (NY)* **318** 170
[18] Zhang H P, Beer A, Florin E-L and Swinney H L 2010 *Proc. Natl Acad. Sci. USA* **107** 13626
[19] Bertin E, Droz M and Grégoire G 2006 *Phys. Rev.* E **74** 022101
[20] Bertin E, Droz M and Grégoire G 2009 *J. Phys. A: Math. Theor.* **42** 445001
[21] Ihle T 2011 *Phys. Rev.* E **83** 030901
[22] Baskaran A and Marchetti M C 2008 *Phys. Rev. Lett.* **101** 268101
Baskaran A and Marchetti M C 2008 *Phys. Rev.* E **77** 011920
Baskaran A and Marchetti M C 2012 *Eur. Phys. J.* E **35** 95
[23] Shi X and Ma Y 2010 arXiv:1011.5408
[24] Peshkov A, Ngo S, Bertin E, Chaté H and Ginelli F 2012 *Phys. Rev. Lett.* **109** 098101
[25] Peshkov A, Aranson I S, Bertin E, Chaté H and Ginelli F 2012 *Phys. Rev. Lett.* **109** 268701

[26] Chaté H, Ginelli F and Montagne R 2006 *Phys. Rev. Lett.* **96** 180602
[27] Oksendal B 1992 *Stochastic Differential Equations* (Berlin: Springer)
[28] Huang K 1987 *Statistical Mechanics* 2nd edn (New York: Wiley)
[29] Mishra S 2009 *PhD Thesis* Indian Institute of Science (www.openthesis.org/document/view/6011220)
[30] Oseen C W 1933 *Trans. Faraday Soc.* **29** 883
Frank F C 1958 *Discuss. Faraday Soc.* **25** 19
[31] Mishra S, Simha R A and Ramaswamy S 2010 *J. Stat. Mech.* P02003
[32] Gopinath A, Hagan M F, Marchetti M C and Baskaran A 2012 *Phys. Rev.* E **85** 061903
[33] Chou Y-L, Wolfe R and Ihle T 2012 *Phys. Rev.* E **86** 021120
[34] Muñoz M A 1998 *Phys. Rev.* E **57** 1377
[35] Tailleur J and Cates M E 2008 *Phys. Rev. Lett.* **100** 218103
[36] Dean D S 1996 *J. Phys. A: Math. Gen.* **29** L613
[37] Kumaran V 1996 *J. Chem. Phys.* **104** 3120