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Loop space homology associated with the mod 2 Dickson invariants

Ran Levi and Nora Seeliger

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Abstract. The spaces BG_2 and $BDI(4)$ have the property that their mod 2 cohomology is given by the rank 3 and 4 Dickson invariants respectively. Associated with these spaces, one has the classifying spaces of the finite groups $BG_2(q)$, for an odd prime power q, and the exotic family of classifying spaces of 2-local finite groups $B\mathrm{Sol}(q)$. In this article we compute the loop space homology of $BG_2(q)^{\wedge}_2$ and $B\mathrm{Sol}(q)$ for all odd primes q, as Hopf algebras over the Steenrod algebra, and the associated Bockstein spectral sequences.

Keywords. Classifying spaces, p-local finite groups, Loop space homology.

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It is well known [5, 18, 12] that the mod 2 Dickson invariants of rank n,

$$H^*(B\mathbb{Z}/2)^n, \mathbb{F}_2)^{\mathrm{GL}_n(\mathbb{F}_2)} = P[d_1, d_2 \dots, d_n], \quad |d_i| = 2^n - 2^{n-i},\ i = 1, \dots, n,$$

where invariants are taken with respect to the natural action of the general linear group, are realisable as the mod 2 cohomology of a space if and only if $n \le 4$. For $n = 2, 3$ the corresponding spaces are the classifying spaces of the Lie groups $SO(3)$ and G_2 respectively. The rank 4 Dickson invariants are realisable by a space $BDI(4)$, which was constructed by Dwyer and Wilkerson [5] and is exotic in the sense that it is not homotopy equivalent to the classifying space of any compact Lie group.

In 1994 Benson [2] introduced a family of spaces $B\mathrm{Sol}(q)$, one for each odd prime power q, closely related to $BDI(4)$, which he claimed realise the exotic fusion patterns studied by Solomon [19] twenty years earlier. He obtained this family of spaces by considering the pullback of the diagram

$$BDI(4) \xrightarrow{\psi^q \top 1} BDI(4) \times BDI(4) \xleftarrow{\Delta} BDI(4),$$

where ψ^q is the degree q unstable Adams operation constructed by Notbohm [15]. In [11] the first named author and Oliver showed that the patterns studied by Solomon form saturated fusion systems. They also showed that these fusion systems admit associated centric linking systems and thus give rise to a family of

2-local finite groups (see [4]). The "classifying spaces" of these 2-local finite groups are also named $B\mathrm{Sol}(q)$ and are shown to coincide with Benson's family [11, Theorem 4.5]. The family $B\mathrm{Sol}(q)$ provides one of the most interesting collections of p-local finite groups, in that they are all exotic and to date the only exotic systems known at the prime 2. The module structure of $H^*(B\mathrm{Sol}(q), \mathbb{F}_2)$ was calculated by Benson in [2], and the algebra and $\mathcal{A}_2$-module structure were determined by Grbic [7], who also computed the Bockstein spectral sequence for these spaces.

In this article we consider the spaces $B\mathrm{G}_2(q)^\wedge_2$ and $B\mathrm{Sol}(q)$ for all odd prime powers q and present a calculation of their mod 2 loop space homology. There are strong results known on the homotopy type of $\Omega BG^\wedge_p$ when G is a finite group [10], but not much is known on loop spaces of exotic classifying spaces. Furthermore, as we shall see these two families exhibit very systematic behaviour (see Remark 1 at the end of the introduction) which might be worth exploring further. This motivates our calculations.

Throughout this paper $H_*(-)$ and $H^*(-)$ will mean mod 2 homology and cohomology respectively. Different coefficients will always be explicitly specified. Subscripts on homology or cohomology classes will always denote their degrees. The letters P, E, T and Γ will be used to denote polynomial, exterior, tensor and divided power algebras respectively. By convention, we will always use the notation $T[x]$ to denote the tensor algebra on a single odd-dimensional generator, although over $\mathbb{F}_2$ the tensor algebra on a single generator in any dimension is graded commutative and thus isomorphic to the polynomial algebra on the same generator. The spectral sequences of Serre, Bockstein and Eilenberg–Moore will be used in our calculations and will be abbreviated as SSS, BSS and EMSS respectively.

For any integer n let $\nu_2(n)$ denote the highest power of 2 dividing n. Our first result determines the mod 2 loop space homology of $B\mathrm{G}_2(q)^\wedge_2$, for any odd prime power q.

Theorem A. *Fix an odd prime power q. Then*

$$H_*(\Omega B\mathrm{G}_2(q)^\wedge_2) \cong P[a_2]/(a_2^2) \otimes P[a_4, b_{10}] \otimes E[x_3, x_5] \otimes P[z_6]/(z_6^2),$$

as modules over $H_*(\Omega \mathrm{G}_2) \cong P[a_2]/(a_2^2) \otimes P[a_4, b_{10}]$. *Furthermore:*

- *The relations which determine the algebra extension are given by* $x_3^2 = x_5^2 = z_6^2 = 0$, $[a_2, z_6] = a_4^2$, $[a_4, z_6] = b_{10} + a_2 a_4^2$, *and* $[b_{10}, z_6] = a_4^4$. *All other commutators of generators are trivial.*
- *The reduced coproduct is given by* $\overline{\Delta}(a_4) = a_2 \otimes a_2$ *and* $\overline{\Delta}(z_6) = x_3 \otimes x_3$, *while all other generators are primitive.*

- *The action of the dual Steenrod algebra is determined by*

	a_2	x_3	a_4	x_5	z_6	b_{10}
Sq^1_*	0	0	0	0	x_5	0
Sq^2_*	0	0	a_2	x_3	0	a_4^2

  and the Steenrod axioms.

- *The homology Bockstein spectral sequences are determined by*

$q \equiv 1(4)$	a_2	x_3	a_4	x_5	a_2x_3	z_6	b_{10}	x_5z_6
Sq^1_*	0	0	0	0	0	x_5	0	0
$\beta^{r_2}_*$	0	a_2	0	–	0	–	0	b_{10}
$\beta^{r_2+1}_*$	–	–	0	–	a_4	–	–	–

  where $r_2 = \nu_2(q^2-1)$. The rubrics marked "–" mean that the corresponding element vanishes in the respective page of the BSS.

Next we have the analogous result for $B\mathrm{Sol}(q)$.

Theorem B. *Fix an odd prime power q. Then*

$$H_*(\Omega B\mathrm{Sol}(q)) \cong P[a_6]/(a_6^2) \otimes P[b_{10}, c_{12}, e_{26}] \otimes E[y_7, y_{11}, y_{13}] \otimes P[y_{14}]/(y_{14}^2),$$

as a module over $H_*(\Omega DI(4)) \cong P[a_6]/(a_6^2) \otimes P[b_{10}, c_{12}, e_{26}]$. Furthermore:

- *The relations which determine the algebra extension are given by $y_7^2 = y_{11}^2 = y_{13}^2 = y_{14}^2 = 0$, $[a_6, y_{14}] = b_{10}^2$, $[b_{10}, y_{14}] = c_{12}^2$, $[c_{12}, y_{14}] = e_{26} + a_6b_{10}^2$ and $[e_{26}, y_{14}] = b_{10}^4$. All other commutators of generators are trivial.*

- *The reduced coproduct is given by $\bar{\Delta}(c_{12}) = a_6 \otimes a_6$ and $\bar{\Delta}(y_{14}) = y_7 \otimes y_7$. All other generators are primitive.*

- *The action of the dual Steenrod algebra is determined by*

	a_6	y_7	b_{10}	y_{11}	c_{12}	y_{13}	y_{14}	e_{26}
Sq^1_*	0	0	0	0	0	0	y_{13}	0
Sq^2_*	0	0	0	0	b_{10}	y_{11}	0	c_{12}^2
Sq^4_*	0	0	a_6	y_7	0	0	0	0

  and the Steenrod axioms.

- *The homology Bockstein spectral sequence is determined by the table*

	a_6	y_7	b_{10}	y_{11}	c_{12}	y_{13}	y_{14}	a_6y_7	$y_{13}y_{14}$
Sq^1_*	0	0	0	0	0	0	y_{13}	0	0
$\beta^{r_4-1}_*$	0	0	0	b_{10}	0	–	–	0	e_{26}
$\beta^{r_4}_*$	0	a_6	–	–	0	–	–	0	–
$\beta^{r_4+1}_*$	–	–	–	–	0	–	–	c_{12}	–

where $r_4 = \nu_2(q^4-1)$. The rubrics marked "–" mean that the corresponding element vanishes in the respective page of the BSS.

Remark 1. For each $n \geq 3$ consider the Hopf algebra Ω_n over $\mathbb{F}_2$, given as follows:

$$\Omega_n \overset{\text{def}}{=} P[a]/a^2 \otimes P[b_2,\dots,b_{n-1},c] \otimes E[x_1,\dots,x_{n-1}] \otimes P[y]/y^2,$$

as a module over $P[a]/a^2 \otimes P[b_2,\dots,b_{n-1},c]$, where $|a| = 2^{n-1}-2$, $|b_k| = 2^n - 2^{n-k}-2$ for $2 \leq k \leq n-1$, $|c| = 2^{n+1}-6$, $|x_j| = 2^n-2^{n-j}-1$ for $1 \leq j \leq n-1$ and $|y| = 2^n-2$. The relations which determine the algebra structure are $x_j^2 = 0$ for all j, $y^2 = 0$, $[a,y] = b_2^2$, $[b_k,y] = b_{k+1}^2$, $2 \leq k \leq n-2$, $[b_{n-1},y] = c^2 + ab_2^2$ and $[c,y] = b_2^4$. All other commutators of generators are trivial. The reduced coproduct is given by $\overline{\Delta}(b_2) = a \otimes a$ and $\overline{\Delta}(y) = x_1 \otimes x_1$, while all other generators are primitive.

Furnish the Hopf algebra Ω_n with a coaction of the dual Steenrod algebra as follows: $Sq_*^{2^n-2}(b_2) = a$, $Sq_*^{2^{n-k}}(b_k) = b_{k-1}$, $3 \leq k \leq n-1$, $Sq_*^{2^{n-k}}(x_j) = x_{j-1}$, $2 \leq j \leq n-1$, $Sq^1_*(y) = x_{n-1}$, and $Sq^2_*(c) = b^2_{n-1}$.

Then, with the exception of the BSS structure, which we do not attempt to write in this generality, Theorems A and B are given as particular cases of Ω_n for $n = 3$ and 4 respectively.

The paper is organised as follows. In Section 1 we record some basic facts which form a basis for our calculation. The loop space homology of $BG_2(q)^\wedge_2$ and $B\mathrm{Sol}(q)$ are calculated in Sections 2 and 3 respectively.

Some of the calculations presented here may be possible to carry out more easily using the general methods developed recently by Daisuke Kishimoto and Akira Kono in [8]. In this paper the authors develop a clever method of calculating the cohomology of the twisted loop space of a space X with respect to a self map f. The pullback space in the Quillen–Friedlander fibre square (1.1) below is an example of such a twisted loop space. The authors are very grateful to Kono for the interest he showed in our results, and for pointing out an error in the calculation of the algebra structures in an earlier version of this paper.

1 Preliminaries

Recall the Quillen–Friedlander fibre square [6] for groups of Lie type. If G is a complex reductive Lie group and $G(q)$ is the corresponding algebraic group over the field of q elements, then after completion at a prime p not dividing q there is a homotopy fibre square

$$\begin{array}{ccc} BG(q)^\wedge_p & \longrightarrow & BG^\wedge_p \\ \downarrow{\scriptstyle \pi_q} & & \downarrow{\scriptstyle \Delta} \\ BG^\wedge_p & \xrightarrow{1\top\psi^q} & BG^\wedge_p \times BG^\wedge_p \end{array} \tag{1.1}$$

where ψ^q is the q-th unstable Adams operation and Δ is the diagonal map. In particular, since for any self map $f\colon X \to X$, $\mathrm{hofib}(X \xrightarrow{1\top f} X \times X) \simeq \Omega X$, one has a fibration sequence of loop spaces and loop maps:

$$\Omega BG(q)^\wedge_p \longrightarrow G^\wedge_p \xrightarrow{f^q} G^\wedge_p, \tag{1.2}$$

obtained by backing up the fibration in the left column of the pullback diagram above twice. All p-compact groups, in particular DI(4), admit unstable Adams operations of degree q, where q is a p-adic unit. The pullback space arising from the construction of a fibre square analogous to diagram (1.1) for the operation ψ^q on DI(4) is defined by Benson [2] to be $B\mathrm{Sol}(q)$.

We next record three well-known cohomology algebras, which will be used in our calculation. As a convention we will use Roman alphabet to denote classes in mod 2 homology and cohomology and Greek letters to denote classes in integral homology and cohomology. A good reference for the cohomology of Lie groups and their classifying spaces is [14]. Throughout the paper we will assume the reader is familiar with the basics of Hopf algebras. An excellent reference is [13].

The Spaces BSU(3), SU(3) and ΩSU(3). Recall

$$H^*(B\mathrm{SU}(3)) \cong P[u_4, u_6] \quad \text{and} \quad H^*(B\mathrm{SU}(3), \mathbb{Z}) \cong P[\gamma_4, \gamma_6], \tag{1.3}$$

both as algebras, with $Sq^2(u_4) = u_6$. Recall also that

$$H_*(\mathrm{SU}(3), \mathbb{Z}) \cong E[\chi_3, \chi_5], \tag{1.4}$$

as a Hopf algebra. An elementary calculation, using the EMSS, yields

$$H_*(\Omega\mathrm{SU}(3), \mathbb{Z}) \cong P[\alpha_2, \alpha_4] \tag{1.5}$$

as an algebra, with the Hopf algebra structure determined by $\bar{\Delta}(\alpha_4) = \alpha_2 \otimes \alpha_2$, where $\bar{\Delta}$ denotes the reduced diagonal. Since these algebras are torsion free,

$$H_*(\mathrm{SU}(3)) \cong H_*(\mathrm{SU}(3), \mathbb{Z}) \otimes \mathbb{F}_2 \cong E[x_3, x_5]$$

and

$$H_*(\Omega\mathrm{SU}(3)) \cong H_*(\Omega\mathrm{SU}(3), \mathbb{Z}) \otimes \mathbb{F}_2 \cong P[a_2, a_4],$$

as Hopf algebras, with $Sq^2_*(x_5) = x_3$ and $Sq^2_*(a_4) = a_2$.

The Spaces $B\mathrm{G}_2$, G_2 and $\Omega\mathrm{G}_2$. There is an algebra isomorphism

$$H^*(B\mathrm{G}_2) \cong P[u_4, u_6, t_7], \tag{1.6}$$

with $Sq^2(u_4) = u_6$ and $Sq^1(u_6) = t_7$. These are the rank 3 mod 2 Dickson invariants. The group SU(3) is a subgroup of G_2 and the inclusion induces the obvious projection on mod 2 cohomology. Recall also that

$$H_*(\mathrm{G}_2) \cong E[x_3, x_5, x_6] \tag{1.7}$$

with $Sq^1_*(x_6) = x_5$ and $Sq^2_*(x_5) = x_3$. The Hopf algebra structure is determined by $\bar{\Delta}(x_6) = x_3 \otimes x_3$.

Using the BSS for $H_*(\mathrm{G}_2)$, we see that

$$H_i(\mathrm{G}_2, \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & i = 0, 3, 11, 14, \\ \mathbb{Z}/2, & i = 5, 8, \end{cases}$$

while all other homology groups vanish.

The integral loop space homology of G_2 as a Hopf algebra was computed by Bott [3]. Reduction mod 2 is immediate and gives

$$H_*(\Omega\mathrm{G}_2) \cong P[a_2, a_4]/(a_2^2) \otimes P[b_{10}], \tag{1.8}$$

as an algebra. (Notice also that in integral homology the square of the 2-dimensional generator is twice the 4-dimensional generator.) The Hopf algebra structure of $H_*(\Omega\mathrm{G}_2)$ is determined by $\bar{\Delta}(a_4) = a_2 \otimes a_2$, while b_{10} can be chosen to be primitive (if some choice of b_{10} is not primitive, then $b'_{10} = b_{10} + a_4^2 a_2$ is). It follows that in cohomology the dual of a_4 is the square of the dual of a_2, and so one has $Sq^2_*(a_4) = a_2$.

Dually, one has

$$H^*(\Omega\mathrm{G}_2) \cong P[\bar{a}_2]/(\bar{a}_2^4) \otimes \Gamma[a_8, \bar{b}_{10}],$$

where $\bar{a}_2$ and $\bar{b}_{10}$ are the duals of a_2 and b_{10} respectively, and a_8 is dual to a_4^2.

Deciding the action of the homology Steenrod squares on b_{10} requires a calculation. The authors are grateful to Akira Kono for sketching for them the following argument, which follows the lines of [8]. Let $\widetilde{\mathrm{G}_2}$ denote the 3-connected cover of G_2. Thus there is a principal fibration

$$K(\mathbb{Z},2) \longrightarrow \widetilde{\mathrm{G}_2} \longrightarrow \mathrm{G}_2.$$

Using the mod-2 cohomology SSS for this fibration, an elementary computation shows that

$$H^*(\widetilde{\mathrm{G}_2}) \cong P[u_8] \otimes E[y_9, z_{11}],$$

where u_8 restricts to $\iota_2^4 \in H^8(K(\mathbb{Z},2))$. The classes y_9 and z_{11} correspond to the infinite cycles in the spectral sequence given by $\iota_2^2 b_5$ and $\iota_2 a_3^3$, where a_3 and b_5 denote the generators of $H^*(\mathrm{G}_2)$. By analysing the SSS for the fibration

$$\widetilde{\mathrm{G}_2} \longrightarrow \mathrm{G}_2 \longrightarrow K(\mathbb{Z},3),$$

it is easy to see that $Sq^1(u_8) = y_9$ and $Sq^2(y_9) = z_{11}$. Finally, using the spectral sequence for

$$\Omega\mathrm{G}_2 \longrightarrow K(\mathbb{Z},2) \longrightarrow \widetilde{\mathrm{G}_2},$$

one observes that ι_2 restricts to $\bar{a}_2$, and so ι_2^4 restricts trivially and is therefore the image of u_8 under the inflation map. The rest of the spectral sequence is determined by letting y_9 and z_{11} be the image of $\bar{a}_8$ and $\bar{b}_{10}$ under the transgression. In particular, it follows that $Sq^2(\bar{a}_8) = \bar{b}_{10}$. Dually, in homology we have $Sq^2_*(b_{10}) = a_4^2$.

The Spaces BDI(4) and DI(4). Let BDI(4) denote the classifying space of the 2-compact group DI(4) (see [5]). Thus there is an algebra isomorphism

$$H^*(B\mathrm{DI}(4)) \cong P[v_8, v_{12}, v_{14}, s_{15}] \tag{1.9}$$

with $Sq^4(v_8) = v_{12}$, $Sq^2(v_{12}) = v_{14}$ and $Sq^1(v_{14}) = s_{15}$. One also has

$$H_*(\mathrm{DI}(4)) \cong E[y_7, y_{11}, y_{13}, y_{14}] \tag{1.10}$$

with $Sq^4(y_7) = y_{11}$, $Sq^2(y_{11}) = y_{13}$ and $Sq^1(y_{13}) = y_{14}$. The Hopf algebra structure is determined by $\bar{\Delta}(y_{14}) = y_7 \otimes y_7$.

2 Loop space homology of $B\mathrm{G}_2(q)^\wedge_2$.

In this section we calculate the mod 2 loop space homology of $B\mathrm{G}_2(q)^\wedge_2$. To avoid an awkward notation, we use $\widehat{(-)}$ to denote $(-)^\wedge_2$ where it makes sense to do so.

To calculate the loop space homology of $B\mathrm{G}_2(q)^\wedge_2$, consider first the Friedlander fibre square (1.1) for G_2. Taking iterated fibres on the left column of the square, we get a sequence of fibrations

$$\Omega\widehat{G_2} \longrightarrow \Omega BG_2(q)^\wedge_2 \longrightarrow \widehat{G_2} \xrightarrow{f^q} \widehat{G_2} \longrightarrow BG_2(q)^\wedge_2 \longrightarrow B\widehat{G_2},$$

where $\widehat{G_2}$ denotes $(G_2)^\wedge_2$. For the actual calculation, we use the SSS for the fibration

$$\Omega\widehat{G_2} \longrightarrow \Omega BG_2(q)^\wedge_2 \longrightarrow \widehat{G_2}. \tag{2.1}$$

Thus, we start by calculating the map induced by f^q on homology.

Expanding the Friedlander fibre square, one sees that f^q is the composite

$$\widehat{G_2} \xrightarrow{1\top\Omega\psi^q} \widehat{G_2}\times\widehat{G_2} \xrightarrow{-1\top 1} \widehat{G_2}\times\widehat{G_2} \xrightarrow{\mu} \widehat{G_2}.$$

There is an isomorphism of modules over $P[u_4, u_7, u_6^2]$,

$$H^*(BG_2) \cong P[u_4, u_7, u_6^2] \otimes E[u_6],$$

with $Sq^1(u_6) = u_7$ and $Sq^2(u_4) = u_6$. Considering this as a differential graded algebra with the differential given by Sq^1 and taking cohomology, we get the E_2 term of the BSS for $H^*(BG_2)$,

$$E_2 \cong P[u_4, u_6^2],$$

which is concentrated in even degrees, and so $E_2 = E_\infty$. Hence the integral cohomology of BG_2 is given by

$$H^*(BG_2, \mathbb{Z}) \cong P[u_4, u_7, v_{12}]/(2u_7).$$

Notice that u_4 and v_{12} are torsion free classes and $(\psi^q)^*(u_4) = q^2u_4$, while $(\psi^q)^*(v_{12}) = q^6v_{12}$. On the other hand, since the class u_7 is of order 2, every element in the ideal it generates is of order 2, and since ψ^q is a mod-2 equivalence for q odd, the ideal generated by u_7 is fixed under $(\psi^q)^*$.

Let χ_i denote a generator for $H_i(G_2, \mathbb{Z})$ for those values of i where the respective homology group is nontrivial. Using the known algebra structure of $H_*(G_2)$, it is easy to conclude that χ_3, χ_5 and χ_{11} are indecomposable and that $\chi_3\chi_5 = \chi_8$ and $\chi_3\chi_{11} = \chi_{14}$. Using the SSS for the path loop fibration over BG_2 and naturality, we conclude that $\Omega\psi^q_*(\chi_3) = q^2\chi_3$ and $\Omega\psi^q_*(\chi_{11}) = q^6\chi_{11}$, while $\Omega\psi^q_*(\chi_5) = \chi_5$ and $\Omega\psi^q_*(\chi_8) = \chi_8$. Reducing to mod-2 homology, both $(\psi^q)_*$ and $(\Omega\psi^q)_*$ clearly induce the identity map.

Using the Künneth formula, we see that for $n \leq 11$

$$H_n(G_2\times G_2, \mathbb{Z}) \cong \bigoplus_{i+j=n} H_i(G_2, \mathbb{Z}) \otimes H_j(G_2, \mathbb{Z}).$$

This and the information about $\Omega\psi^q_*$ allows us to easily calculate the map f^q_* on $H_*(\mathrm{G}_2,\mathbb{Z})$. One has

$$f^q_*(\chi_3) = (q^2-1)\chi_3, \quad f^q_*(\chi_5) = 0 \quad \text{and} \quad f^q_*(\chi_{11}) = (q^6-1)\chi_{11}.$$

Therefore, on mod-2 homology, f^q_* is trivial.

Now consider fibration (2.1), which is induced from the path-loop fibration over G_2 via the map f_q. Since f^q_* is trivial on mod-2 homology, the SSS for (2.1) collapses at E^2, and it follows that for all odd q there is an isomorphism of modules over $H_*(\Omega\mathrm{G}_2)$

$$H_*(\Omega B\mathrm{G}_2(q)^\wedge_2) \cong \{P[a_2]/(a_2^2) \otimes P[a_4,b_{10}]\} \otimes E[x_3,x_5] \otimes P[z_6]/(z_6^2). \quad (2.2)$$

The structure of $H_*(\Omega B\mathrm{G}_2(q)^\wedge_2)$ as a module over the dual Steenrod algebra follows from the information we have about the two factors. Namely, one has $Sq^2_*(a_4) = a_2$, $Sq^2_*(x_5) = x_3$, $Sq^2_*(b_{10}) = a_4^2$ and $Sq^1_*(z_6) = x_5$.

For dimension reasons, the classes a_2 and x_3 are primitive with respect to the diagonal in $H_*(\Omega B\mathrm{G}_2(q)^\wedge_2)$. The class x_5 can be chosen to be primitive since for any choice, $\overline{\Delta}(x_5) = A(a_2 \otimes x_3 + x_3 \otimes a_2)$, for some $A \in \mathbb{F}_2$, and so $x_5 + Aa_2x_3$ is primitive, has the same action of Sq^2_* as x_5 and represents the same class modulo $H_*(\Omega\mathrm{G}_2)$. One also has $\overline{\Delta}(a_4) = a_2 \otimes a_2$ since $H_*(\Omega\mathrm{G}_2)$ is a Hopf subalgebra. The class z_6 can be chosen to have reduced diagonal $x_3 \otimes x_3$ since for any choice of representative one has $\overline{\Delta}(z_6) = x_3 \otimes x_3 + B(a_2 \otimes a_4 + a_4 \otimes a_2)$, and so $z'_6 = z_6 + Ba_2a_4$ is congruent to z_6 modulo $H_*(\Omega\mathrm{G}_2)$, has the same action of Sq^1_* as z_6 and has the required diagonal. Finally, the class b_{10} is primitive since it is the image of the corresponding class in $H_*(\Omega\mathrm{G}_2)$.

Next, we compute the algebra extension. To do that, fix the representatives for x_3, x_5 and z_6 as above. Notice first that $x_3^2 = 0$ and $z_6^2 = 0$ since there are no primitives in the respective dimensions. Similarly, x_5 is primitive, and so $x_5^2 = Ab_{10}$ for some $A \in \mathbb{F}_2$. Applying Sq^2_* to both sides, we see that $A = 0$, so $x_5^2 = 0$.

The following table lists all basic commutators involving x_3, x_5 and z_6, which we proceed by examining.

5	7	8	9	10	11	13	15	16
$[a_2,x_3]$	$[a_2,x_5]$	$[a_2,z_6]$	$[a_4,x_5]$	$[a_4,z_6]$	$[x_5,z_6]$	$[x_3,b_{10}]$	$[x_5,b_{10}]$	$[z_6,b_{10}]$
	$[x_3,a_4]$	$[x_3,x_5]$	$[x_3,z_6]$					

We will show that

$$[a_2,z_6] = a_4^2, \quad [a_4,z_6] = b_{10} + a_4^2a_2 \quad \text{and} \quad [b_{10},z_6] = a_4^4,$$

while all the other commutators in the table vanish.

Observe first that every nonprimitive class among the algebra generators of $H_*(\Omega B\mathrm{G}_2(q)^\wedge_2)$ has a reduced diagonal consisting of a single element. The commutator of two primitives is always primitive, and if a is a primitive and $\overline{\Delta}(b) = c \otimes c$, then

$$\overline{\Delta}([a,b]) = c \otimes [a,c] + [a,c] \otimes c. \qquad (2.3)$$

Thus the commutators $[a_2, x_3]$, $[a_2, x_5]$, $[x_3, x_5]$, $[x_3, b_{10}]$ and $[x_5, b_{10}]$ are automatically primitive.

Since $[a_2, x_3] = Ax_5$, applying Sq^2_* to both sides, it follows that $A = 0$, so a_2 and x_3 commute. Next, one has $\overline{\Delta}(a_4) = a_2 \otimes a_2$, $\overline{\Delta}(z_6) = x_3 \otimes x_3$, $[a_2, x_3] = 0$. Applying (2.3) to $[x_3, a_4]$ and $[a_2, z_6]$, one sees that these commutators are also primitive. The only other primitive in dimension 7 is $[a_2, x_5]$, and so $[x_3, a_4] = A[a_2, x_5]$, for some $A \in \mathbb{F}_2$. Similarly,

$$[a_2, z_6] = B[x_3, x_5] + Ca_4^2$$

as $[x_3, x_5]$ and a_4^2 are the only other primitives in dimension 8. But Sq^1_* applied to both sides yields 0 on the right hand side and $[a_2, x_5]$ on the left hand side. Hence a_2 commutes with x_5, and consequently a_4 commutes with x_3.

By a similar method, we analyse $[a_4, x_5]$, $[x_3, z_6]$, $[x_3, x_5]$ and $[x_5, z_6]$. First, by direct calculation and the results already listed above,

$$\overline{\Delta}([a_4, z_6]) = a_2 \otimes [a_2, z_6] + [a_2, z_6] \otimes a_2.$$

Since we do not yet know whether a_2 commutes with z_6, we write the commutator $[a_4, z_6]$ in general form as $[a_4, z_6] = Ab_{10} + Ba_4^2 a_2$, for some $A, B \in \mathbb{F}_2$. Applying Sq^1_* to both sides, we have $[a_4, x_5] = 0$. Now, since x_3 and x_5 commute, $[x_5, z_6]$ is primitive. Since x_5 commutes with both a_2 and a_4, there are no other nonzero primitives in dimension 11, and so $[x_5, z_6] = 0$. Applying Sq^2_* and then Sq^1_*, we conclude that both $[x_3, z_6]$ and $[x_3, x_5]$ vanish.

Next, notice that $[z_6, b_{10}]$ is a primitive class. Hence $[z_6, b_{10}] = Aa_4^4$ for some $A \in \mathbb{F}_2$. Applying Sq^1_* and then Sq^2_* to both sides, and using the fact that x_5 and a_4 commute, we conclude that b_{10} commutes with x_5 and x_3.

It remains to analyse the commutators $[a_2, z_6]$, $[a_4, z_6]$ and $[b_{10}, z_6]$. To do that, recall from [7] that

$$H^*(B\mathrm{G}_2(q)) = P[u_4, u_6, t_7, y_3, y_5]/(y_5^2 + y_3 t_7 + y_3^2 u_4,\ y_3^4 + y_5 t_7 + y_3^2 u_6).$$

Denote classes in $H_*(B\mathrm{G}_2(q))$ by adding a bar to the corresponding cohomology class, and consider the cobar spectral sequence for $H_*(\Omega B\mathrm{G}_2(q)^\wedge_2)$. Thus

$$E^2 \cong \mathrm{Cotor}^{H_*(B\mathrm{G}_2(q)^\wedge_2)}(\mathbb{F}_2, \mathbb{F}_2) \cong H_*(T(\Sigma^{-1}(\overline{H}_*(B\mathrm{G}_2(q)^\wedge_2)), d_E)),$$

where d_E is the differential on $T(\Sigma^{-1}(\overline{H}_*(BG_2(q)^\wedge_2)))$ induced by the reduced diagonal [1]. If $x, y \in \overline{H}_*(BG_2(q)^\wedge_2)$ are any classes, we denote the corresponding elements of the tensor algebra by $[x]$, $[y]$ etc. and their product in the tensor algebra structure by standard bar notation $[x|y]$.

Consider the homology classes $\overline{y_5^2}$, $\overline{y_3t_7}$ and $\overline{y_3^2u_4}$. The corresponding reduced diagonals are $\overline{y_5}\otimes\overline{y_5}$, $\overline{y_3}\otimes\overline{t_7}+\overline{t_7}\otimes\overline{y_3}$ and $\overline{y_3}\otimes\overline{y_3u_4}+\overline{y_3u_4}\otimes\overline{y_3}+\overline{y_3^2}\otimes\overline{u_4}+\overline{u_4}\otimes\overline{y_3^2}$ respectively. Hence

$$d_E([\overline{y_5^2}]) = [\overline{y_5}]^2, \quad d_E([\overline{y_3t_7}]) = [[\overline{y_3}],[\overline{t_7}]]$$
$$d_E([\overline{y_3^2u_4}]) = [[\overline{y_3}],[\overline{y_3u_4}]] + [[\overline{y_3^2}],[\overline{u_4}]].$$

On the other hand, since $y_5^2 + y_3t_7 + y_3^2u_4 = 0$ in $H^*(BG_2(q))$, we have

$$[\overline{y_5}]^2 + [[\overline{y_3}],[\overline{t_7}]] = [[\overline{y_3}],[\overline{y_3u_4}]] + [[\overline{y_3^2}],[\overline{u_4}]].$$

Furthermore, $[\overline{y_5}]$ and $[\overline{y_3}]$ and $[\overline{t_7}]$ are all cycles, which are permanent for dimensional reasons and hence represent a_4, a_2 and z_6 respectively in loop space homology. The equations above show that the expression $[\overline{y_5}]^2 + [[\overline{y_3}],[\overline{t_7}]]$ is a boundary, and so we obtain the relation $[a_2, z_6] = a_4^2$. Next, notice that $Sq_*^2([a_4, z_6]) = [a_2, z_6] = a_4^2$, while $\overline{\Delta}([a_4, z_6]) = a_2\otimes a_4^2 + a_4^2\otimes a_2$. Hence we conclude that

$$[a_4, z_6] = b_{10} + a_4^2a_2 = b_{10} + [a_2, z_6]a_2.$$

Finally, since $b_{10} = [a_4, z_6] + a_4^2a_2$, we directly calculate

$$\begin{aligned}[b_{10}, z_6] &= [[a_4, z_6] + a_4^2a_2, z_6] = [[a_4, z_6], z_6] + [a_4^2a_2, z_6]\\ &= [a_4, z_6^2] + [[a_2, z_6]a_2, z_6]\\ &= 0 + [a_2, z_6]^2 = a_4^4.\end{aligned}$$

This completes the computation of the Hopf algebra structure. To summarise, we have shown that

$$H_*(\Omega BG_2(q)^\wedge_2) \cong P[a_2]/(a_2^2)\otimes P[a_4, b_{10}]\otimes E[x_3, x_5]\otimes P[z_6]/(z_6^2),$$

as modules over $H_*(\Omega BG_2)$. The relations which determine the algebra extension are given by $x_3^2 = x_5^2 = z_6^2 = 0$, $[a_2, z_6] = a_4^2$, $[a_4, z_6] = b_{10} + a_2a_4^2$ and $[b_{10}, z_6] = a_4^4$. All other commutators of generators are trivial. The coproduct is given by $\overline{\Delta}(a_4) = a_2\otimes a_2$, $\overline{\Delta}(z_6) = x_3\otimes x_3$, and all other generators are primitive.

It remains to compute the Bockstein spectral sequence for $H_*(\Omega B\mathrm{G}_2(q)^\wedge_2)$. This is done by calculating the integral SSS for the fibration in the top row of the diagram

$$\begin{array}{ccccc} \Omega\widehat{\mathrm{G}_2} & \longrightarrow & \Omega B\mathrm{G}_2(q)^\wedge_2 & \longrightarrow & \widehat{\mathrm{G}_2} \\ \| & & \downarrow & & \downarrow f^q \\ \Omega\widehat{\mathrm{G}_2} & \longrightarrow & * & \longrightarrow & \widehat{\mathrm{G}_2}\,. \end{array}$$

We use naturality and the action of f^q_* computed above. First, analyse the SSS for the bottom row, using the same notation we have been using before. One has $d_3(\chi_3) = a_2$, and since in integral homology $a_2^2 = 2a_4$, $d_3(a_2\chi_3) = 2a_4$. Hence $d_3(a_4^k\chi_3) = a_4^k a_2$ and $d_3(a_4^k a_2\chi_3) = 2a_4^{k+1}$. Since $\chi_8 = \chi_3\chi_5$, it follows that $d_3(\chi_8) = a_2\chi_5$, but $d_3(a_2\chi_8) = 0$. Similarly, $d_3(\chi_{14}) = a_2\chi_{11}$. In addition, one must have $d_3(\chi_{11}) = a_2\chi_8$ since otherwise $a_2\chi_8$ will be an infinite cycle. This determines d_3. The next nontrivial differential is d_5, which takes χ_5 isomorphically to a_4, which in E^5 is a class of order 2. Finally, d_{11} takes χ_{11} to b_{10}, and $E^{12} = E^\infty$. Now, using naturality of the spectral sequence and our knowledge of f^q_*, it follows that in the SSS for the top row in the diagram, $d_3(\chi_3) = (q^2-1)a_2$, $d_5(\chi_5) = 0$ and $d_{11}(\chi_{11}) = (q^6-1)b_{10}$. This information suffices for the computation of the BSS. The integral calculation yields in particular the observation that a_2 is a class of order (q^2-1), while b_{10} has order (q^6-1).

Now, consider $H_*(\Omega B\mathrm{G}_2(q)^\wedge_2)$ as a module over $H_*(\Omega\mathrm{G}_2)$ as in (2.2):

$$H_*(\Omega B\mathrm{G}_2(q)^\wedge_2) \cong \left\{P[a_2]/(a_2^2)\otimes P[a_4,b_{10}]\right\}\otimes E[x_3,x_5]\otimes P[z_6]/(z_6^2).$$

Taking the Sq^1_* homology, one has

$$E^2 \cong P[a_2]/(a_2^2)\otimes P[a_4,b_{10}]\otimes E[x_3,h_{11}],$$

where the class h_{11} is represented by the Sq^1_* cycle x_5z_6. Notice that for any odd q, $r_6 = \nu_2(q^6-1) = \nu_2(q^2-1) = r_2$. Hence the next nontrivial Bockstein operator is $\beta^{r_2}_*(x_3) = a_2$, and $\beta^{r_2}_*(h_{11}) = b_{10}$. It now follows that $\beta^{r_2+1}_*(a_2x_3) = a_4$ and $E^{r_2+2} = E^\infty$.

The results are summarised in the following table.

$q\equiv 1(4)$	a_2	x_3	a_4	x_5	a_2x_3	z_6	b_{10}	x_5z_6
Sq^1_*	0	0	0	0	0	x_5	0	0
$\beta^{r_2}_*$	0	a_2	0	–	0	–	0	b_{10}
$\beta^{r_2+1}_*$	–	–	0	–	a_4	–	–	–

This completes the proof of Theorem A.

3 Loop space homology of $B\mathrm{Sol}(q)$

For any odd prime power q the 2-local finite group $\mathrm{Sol}(q)$ is defined in [11]. The starting point is a family of saturated fusion systems $\mathcal{F}_{\mathrm{Sol}(q)}$ over the Sylow 2-subgroup of $\mathrm{Spin}_7(q)$. These fusion systems were originally defined by Ron Solomon [19] as a part of his contribution to the classification of finite simple groups. He did not use the language of fusion systems but essentially presented the entire family and studied its general behaviour. In [2] Benson gave a construction of a family of spaces $B\mathrm{Sol}(q)$, which he claimed realise the fusion patterns defined by Solomon. These spaces are given as the pullback spaces in the diagram

$$\begin{array}{ccc} B\mathrm{Sol}(q) & \longrightarrow & B\mathrm{DI}(4) \\ \big\downarrow & & \big\downarrow{\scriptstyle \Delta} \\ B\mathrm{DI}(4) & \xrightarrow{\psi^q \top 1} & B\mathrm{DI}(4)^{\times 2} \end{array} \tag{3.1}$$

where Δ is the diagonal and ψ^q is the degree q unstable Adams operation on $B\mathrm{DI}(4)$ constructed by Notbohm [15]. The paper [11] unifies the two constructions. On one hand it is shown that the fusion patterns defined by Solomon are indeed saturated fusion systems, each of which admits an associated centric linking system $\mathcal{L}_{\mathrm{Sol}(q)}$, and on the other hand that the classifying spaces of the corresponding 2-local finite groups $B\mathrm{Sol}(q) \stackrel{\mathrm{def}}{=} |\mathcal{L}_{\mathrm{Sol}(q)}|_2^\wedge$ coincide with the spaces constructed by Benson, whose approach allows a calculation of the mod 2 cohomology of $B\mathrm{Sol}(q)$, as demonstrated in [7]. We shall also utilize Benson's pullback diagram in the current work.

In what follows we will denote $\Omega B\mathrm{DI}(4)$ by $\mathrm{DI}(4)$ (not to be confused with the notation $\mathrm{DI}(n)$ which is sometimes used to denote the rank n algebra of Dickson invariants).

As before, one has a fibration of loop spaces and loop maps

$$\Omega\mathrm{DI}(4) \longrightarrow \Omega B\mathrm{Sol}(q) \longrightarrow \mathrm{DI}(4),$$

resulting from looping the left column in Benson's pullback diagram (3.1). Thus our first task is to compute the loop space homology of $\mathrm{DI}(4)$.

Proposition 3.1. *There is an isomorphism of Hopf algebras*

$$H_*(\Omega\mathrm{DI}(4)) \cong P[a_6]/(a_6^2) \otimes P[b_{10}, c_{12}, e_{26}],$$

where a_6, b_{10} and e_{26} are primitive, and $\overline{\Delta}(c_{12}) = a_6 \otimes a_6$. The action of the dual Steenrod algebra is determined by $Sq^4_*(b_{10}) = a_6$, $Sq^2_*(c_{12}) = b_{10}$ and $Sq^2_*(e_{26}) = c_{12}^2$.

Proof. Consider the homology EMSS for the path-loop fibration over DI(4). The E^2 term is given by

$$\mathrm{Cotor}^{H_*(\mathrm{DI}(4))}(\mathbb{F}_2, \mathbb{F}_2) \cong \mathrm{Ext}_{H^*(\mathrm{DI}(4))}(\mathbb{F}_2, \mathbb{F}_2).$$

The isomorphism holds since $H_*(\mathrm{DI}(4))$ is of finite type. To calculate the right hand side, consider the differential graded Hopf algebra

$$P_* \stackrel{\mathrm{def}}{=} (P[x_7]/(x_7^4) \otimes E[y_{11}, z_{13}]) \otimes (P[\hat{a}_6]/(\hat{a}^4) \otimes \Gamma[\hat{b}_{10}, \hat{t}_{24}, \hat{e}_{26}]),$$

where the left factor (which is isomorphic to $H^*(\mathrm{DI}(4))$) is primitively generated and the right factor is the dual of the Hopf algebra $P[a_6]/(a^2) \otimes P[b_{10}, c_{12}, e_{26}]$, where all generators but c_{12} are primitive and $\overline{\Delta}(c_{12}) = a_6 \otimes a_6$. We denote by $\gamma_k(\hat{b})$ the generator of $\Gamma[\hat{b}_{10}]$ in dimension $10k$. Thus, in particular, $\gamma_1(\hat{b}) = b_{10}$ and $\gamma_0(\hat{b}) = 1$. We use similar notation for the generators corresponding to $\hat{c}_{12}$ and $\hat{e}_{26}$. Thus, as an $\mathbb{F}_2$-algebra, $\Gamma[\hat{b}_{10}, \hat{t}_{24}, \hat{e}_{26}]$ can be written as

$$\bigotimes_{n \geq 0} E[\gamma_{2^n}(\hat{b}), \gamma_{2^n}(\hat{t}), \gamma_{2^n}(\hat{e})],$$

where we omit subscripts for short. The differential on P_* is given on generators by

- $d(x) = d(y) = d(z) = 0$,
- $d(\hat{a}) = x$,
- $d(\gamma_{2^n}(\hat{b})) = y\gamma_{2^n-1}(\hat{b})$,
- $d(\hat{a}^2) = z$,
- $d(\gamma_{2^n}(\hat{t})) = z\hat{a}^2\gamma_{2^n-1}(\hat{c})$ and
- $d(\gamma_{2^n}(\hat{e})) = x^3\hat{a}\gamma_{2^n-1}(\hat{e})$.

Extend the definition to the entire algebra by requiring that the differential satisfies the Leibniz rule. Notice that the differential is, in particular, a map of graded algebras over $H^*(\mathrm{DI}(4)) = P[x_7]/(x_7^4) \otimes E[y_{11}, z_{13}]$. In particular, P_* is a free differential graded $H^*(\mathrm{DI}(4))$-module. Furthermore, as a chain complex it is split as the tensor product of the following acyclic subcomplexes:

$$\{P[x_7]/(x_7^4) \otimes P[\hat{a}_6]/(\hat{a}_6^4) \otimes E[z_{13}] \otimes \Gamma[\hat{t}_{24}, \hat{e}_{26}]\} \otimes \{E[y_{11}] \otimes \Gamma[\hat{b}_{10}]\}.$$

Hence P_* is a free $H^*(\mathrm{DI}(4))$-resolution of $\mathbb{F}_2$.

Since P_* is a free differential graded $H^*(\mathrm{DI}(4))$-module, it is immediate that

$$\begin{aligned} E^2 &= \mathrm{Ext}_{H^*(\mathrm{DI}(4))}(\mathbb{F}_2, \mathbb{F}_2) = H^*(\mathrm{Hom}_{H^*(\mathrm{DI}(4))}(P_*, \mathbb{F}_2)) \\ &\cong \mathrm{Hom}_{\mathbb{F}_2}(P[\hat{a}_6]/(\hat{a}_6^4) \otimes \Gamma[\hat{b}_{10}, \hat{t}_{24}, \hat{e}_{26}], \mathbb{F}_2) \\ &\cong P[a_6]/(a_6^2) \otimes P[b_{10}, c_{12}, e_{26}]. \end{aligned} \tag{3.2}$$

Since this module is concentrated in even degrees, there are no possible nontrivial differentials, so $E^2 = E^\infty$. By inspection of the SSS for the path-loop fibration over DI(4), one easily obtains $Sq^4_*(b_{10}) = a_6$ and $Sq^2_*(c_{12}) = b_{10}$.

To calculate further Steenrod operations, we use a similar trick to the one used to $\Omega\mathrm{G}_2$. Let X denote the 7-connected cover of DI(4). Thus there is a fibration

$$X \longrightarrow \mathrm{DI}(4) \xrightarrow{x_7} K(\mathbb{Z}, 7).$$

To calculate $H^*(X)$, we use Smith's Big Collapse Theorem [17]. Notice that x_7^* is onto and its kernel consists of the ideal generated by all the polynomial generators of $H^*(K(\mathbb{Z}, 7))$, different from ι_7, $Sq^4\iota_7$ and $Sq^6\iota_7$, along with ι_7^4, $(Sq^6\iota_7)^2$ and $(Sq^4\iota_7)^2$. This collection of generators forms a regular sequence in the cohomology algebra $H^*(K(\mathbb{Z}, 7))$ and so the conditions of Smith's theorem are satisfied, and $H^*(X)$ can be written additively as the exterior algebra on infinitely many generators, corresponding in a one-to-one fashion to generators listed above, but with a dimension shift one down. Let ϵ_{27}, τ_{25}, and σ_{21} be the elements in $H^*(X)$ corresponding to ι_7^4, $(Sq^6\iota_7)^2$ and $(Sq^4\iota_7)^2$ respectively. Notice that $Sq^2(\tau_{25}) = \epsilon_{27}$ and $Sq^4(\sigma_{21}) = \tau_{25}$. Write

$$H^*(X) \cong E[\sigma_{21}, \tau_{25}, \epsilon_{27}] \otimes E[k^2, k^3, \ldots, k^I, \ldots], \tag{3.3}$$

where k^I in dimension $|Sq^I\iota_7| - 1$ stands for the exterior generator corresponding to $Sq^I\iota_7$, for each I such that $Sq^I\iota_7 \in \mathrm{Ker}(x_7^*)$.

Now, consider the cohomology SSS for the principal fibration

$$\Omega\mathrm{DI}(4) \xrightarrow{j} K(\mathbb{Z}, 6) \xrightarrow{\pi} X.$$

Notice first that by naturality of the spectral sequence, the second factor in (3.3) injects into $H^*(K(\mathbb{Z}, 6))$ via π^*, while the classes $\sigma_{21}, \tau_{25}, \epsilon_{27}$ are all in $\mathrm{Ker}(\pi^*)$. Furthermore, one has $j^*(\iota_6) = a_6$, and so $j^*(Sq^4\iota_6) = \hat{b}_{10}$. The bottom dimensional class in $H^*(\Omega\mathrm{DI}(4))$ which is not hit by j^* is $\gamma_2(\hat{b}_{10})$, which is therefore transgressive. Hence $d(\gamma_2(\hat{b}_{10})) = \sigma_{21}$, and it follows that $d(Sq^4\gamma_2(\hat{b}_{10})) = Sq^4(\sigma_{21}) = \tau_{25}$, while $d(Sq^6\gamma_2(\hat{b}_{10})) = d(Sq^{2,4}\gamma_2(\hat{b}_{10})) = Sq^2\tau_{25} = \epsilon_{27}$. Hence $Sq^4\gamma_2(\hat{b}_{10}) = \hat{t}_{24}$ and $Sq^2\hat{t}_{24} = \hat{e}_{26}$. Dually, in homology one has $Sq^2_*(e_{26}) = c_{12}^2$ and $Sq^4_*(c_{12}^2) = b_{10}^2$.

Next, we work out the Pontryagin algebra structure. Since b_{10}, c_{12} and e_{26} are elements of infinite height in E^∞, they represent elements of infinite height in homology. Hence it remains only to check whether $a_6^2 = c_{12}$. But in cohomology one has $\hat{c}_{12} = Sq^2(\hat{b}_{10}) = Sq^2Sq^4(\hat{a}_6) = \hat{a}_6^2$. Hence in homology we obtain $\overline{\Delta}(c_{12}) = a_6 \otimes a_6$. But since a_6 is primitive, so is a_6^2, and since $H_{12}(\Omega\mathrm{DI}(4))$ is 1-dimensional, it follows that $a_6^2 = 0$. This completes the calculation of the algebra structure.

The classes a_6 and b_{10} are primitive for dimension reasons, and we have already computed the reduced diagonal of c_{12}. Thus it remains to compute the reduced diagonal of e_{26}. Notice that $H_{26}(\Omega\mathrm{DI}(4))$ is 2-dimensional, generated additively by e_{26} and $a_6b_{10}^2$, and that e_{26} can be modified by an additive summand of $a_6b_{10}^2$ without changing the algebra structure. For any choice of e_{26} one has

$$\overline{\Delta}(e_{26}) = A(a_6b_{10} \otimes b_{10} + b_{10} \otimes a_6b_{10}) + B(a_6 \otimes b_{10}^2 + b_{10}^2 \otimes a_6),$$

for some $A, B \in \mathbb{F}_2$. But B is the coefficient of $\overline{\Delta}(a_6b_{10}^2)$, and so by modifying the choice of e_{26} if necessary, we may assume that $B = 0$. Furthermore, if $A \neq 0$, then $H_{26}(\Omega\mathrm{DI}(4))$ contains no primitive class, and so dually every cohomology class in $H^{26}(\Omega\mathrm{DI}(4))$ is decomposable, which is clearly impossible. Hence $A = 0$ and there is a choice for the class e_{26} which is primitive.

This completes the calculation of $H_*(\Omega\mathrm{DI}(4))$ as a Hopf algebra and hence the proof of the proposition. □

Dually, the cohomology Hopf algebra is given by

$$H^*(\Omega\mathrm{DI}(4)) \cong P[\hat{a}_6]/(\hat{a}_6^4) \otimes \Gamma[\hat{b}_{10}, \hat{t}_{24}, \hat{e}_{26}].$$

We are now ready to start the calculation of $H_*(\Omega B\mathrm{Sol}(q))$. Consider the fibration

$$\Omega\mathrm{DI}(4) \longrightarrow \Omega B\mathrm{Sol}(q) \longrightarrow \mathrm{DI}(4).$$

The homology SSS associated with this fibration is a spectral sequence of Hopf algebras over $H_*(\Omega\mathrm{DI}(4))$, whose E^2-page has the form

$$\begin{aligned} E^2_{*,*} &= H_*(\Omega\mathrm{DI}(4)) \otimes H_*(\mathrm{DI}(4)) \\ &\cong \left\{P[a_6]/(a_6^2) \otimes P[b_{10}, c_{12}, e_{26}]\right\} \otimes \left\{E[y_7, y_{11}, y_{13}] \otimes P[y_{14}]/(y_{14}^2)\right\}. \end{aligned}$$

Since $B\mathrm{Sol}(q)$ is 6-connected, $\pi_6(\Omega B\mathrm{Sol}(q)) \cong H_7(B\mathrm{Sol}(q), \mathbb{Z})$. Therefore one concludes that $H_7(B\mathrm{Sol}(q), \mathbb{Z})$ is a finite 2-group, and so $d_7(y_7) = 0$. Thus d_7 vanishes on all the y_i (y_{14} by considering the dual cohomology spectral sequence and the other generators by dimension reasons). The next possible nonvanishing differential is d_{11}. Considering the dual cohomology SSS, we have $d_{11}(\hat{b}_{10}) = d_{11}(Sq^4\hat{a}_6) = Sq^4 d_7(\hat{a}_6) = 0$. Hence in homology $d_{11}(y_{11}) = 0$. Similarly, $d_{13}(y_{13}) = 0$. Hence the spectral sequence collapses at E^2 and

$$\begin{aligned} H_*(\Omega B\mathrm{Sol}(q)) \cong P[a_6]/(a_6^2) \otimes P[b_{10}, c_{12}, e_{26}] \\ \otimes E[y_7, y_{11}, y_{13}] \otimes P[y_{14}]/(y_{14}^2), \end{aligned} \tag{3.4}$$

as a module over $H_*(\Omega\mathrm{DI}(4))$.

The loop space homology $H_*(\Omega\mathrm{DI}(4))$ is contained in $H_*(\Omega B\mathrm{Sol}(q))$ as a Hopf subalgebra. The classes y_7 and y_{11} are primitive for dimension reason. For y_{13} one has $\overline{\Delta}(y_{13}) = A(a_6 \otimes y_7 + y_7 \otimes a_6) = A\overline{\Delta}(a_6 y_7)$ for some $A \in \mathbb{F}_2$. Hence y_{13} can be chosen to be primitive. Finally $\overline{\Delta}(y_{14}) = y_7 \otimes y_7$. This completes the description of the coalgebra structure on $H_*(\Omega B\mathrm{Sol}(q))$.

Since y_7, y_{11} and y_{13} are primitive, so are their squares. As there are no nontrivial primitives in the respective dimensions, except for e_{26}, we conclude at once that $y_7^2 = y_{11}^2 = 0$, while $y_{13}^2 = Ae_{26}$, for some $A \in \mathbb{F}_2$. But $Sq^2_*(e_{26}) = c_{12}^2$, while $Sq^2_*(y_{13}^2) = 0$, so $A = 0$ and therefore $y_{13}^2 = 0$. Finally, y_{14}^2 is primitive, and since there are no nontrivial primitives in dimension 28, we get $y_{14}^2 = 0$.

Next, we calculate all the commutators involving the classes y_i. The results are summarised in the following table, while the calculations are below. Each entry in the table stands for the commutator [Column, Row].

	a_6	y_7	b_{10}	y_{11}	c_{12}	y_{13}	y_{14}	e_{26}
y_7	0	0	0	0	0	0	0	0
y_{11}	0	0	0	0	0	0	0	0
y_{13}	0	0	0	0	0	0	0	0
y_{14}	b_{10}^2	0	c_{12}^2	0	$e_{26} + a_6 b_{10}^2$	0	0	b_{10}^4

As a_6 and y_7 are both primitive, so is their commutator and so $[a_6, y_7] = Ay_{13}$, for some $A \in \mathbb{F}_2$. Applying Sq^2 to both side, we see that $A = 0$, and so a_6 and y_7 commute. Thus it follows that $[a_6, y_{14}]$ is primitive, and so must be a multiple of b_{10}^2. Applying successive dual Steenrod squares

$$[a_6, y_{14}] \overset{Sq^1_*}{\longmapsto} [a_6, y_{13}] \overset{Sq^2_*}{\longmapsto} [a_6, y_{11}] \overset{Sq^4_*}{\longmapsto} [a_6, y_7],$$

we conclude that all these commutators, with the possible exception of $[a_6, y_{14}]$ itself, vanish.

The class y_7 clearly commutes with itself, and its commutators with all other y_i are primitive. This implies at once that $[y_7, y_{11}]$ and $[y_7, y_{14}]$ vanish and that $[y_7, y_{13}] = Sq^1_*[y_7, y_{14}] = 0$ as well.

The class b_{10} commutes with y_7 for dimension reasons, and so $[b_{10}, y_{14}]$ is primitive and is therefore a multiple of c_{12}^2. Applying dual Steenrod squares, we have

$$[b_{10}, y_{14}] \overset{Sq^1_*}{\longmapsto} [b_{10}, y_{13}] \overset{Sq^2_*}{\longmapsto} [b_{10}, y_{11}],$$

which shows that $[b_{10}, y_{13}]$ and $[b_{10}, y_{11}]$ vanish.

Since y_{11} commutes with y_7, the commutator $[y_{11}, y_{14}]$ is primitive and one has $Sq^1_*[y_{11}, y_{14}] = [y_{11}, y_{13}]$. But there are no nontrivial primitives in dimension 25, and so both commutators vanish.

The classes c_{12} and y_7 commute since there are no 19 dimensional nonzero primitives, and so $[y_{14}, c_{12}]$ is primitive. Thus $[y_{14}, c_{12}]$ is a multiple of e_{26}. As before, we have

$$[c_{12}, y_{14}] \overset{Sq^1_*}{\longmapsto} [c_{12}, y_{13}] \overset{Sq^2_*}{\longmapsto} [c_{12}, y_{11}],$$

which shows that all commutators involving c_{12}, except possibly $[c_{12}, y_{14}]$, vanish.

We already established that y_{13} commutes with y_7 and y_{11}, and it commutes with y_{14} as well since the commutator $[y_{13}, y_{14}]$ is primitive. This also shows that all commutators with y_{14} with other y_i vanish.

Finally, $[e_{26}, y_7]$ vanishes for lack of primitives in dimension 33. Thus $[e_{26}, y_{14}]$ is primitive and one has a chain of operations

$$[e_{26}, y_{14}] \overset{Sq^1_*}{\longmapsto} [e_{26}, y_{13}] \overset{Sq^2_*}{\longmapsto} [e_{26}, y_{11}].$$

The only nonzero primitive in dimension 40 is b_{10}^4, and so $[e_{26}, y_{14}] = Ab_{10}^4$ for some $A \in \mathbb{F}_2$. Applying Sq^1_* and $Sq^{2,1}_*$ to b_{10}^4, we conclude that $[e_{26}, y_{13}]$ and $[e_{26}, y_{11}]$ vanish.

It remains to evaluate the commutators of the class y_{14} with the algebra generators of $H_*(\Omega\mathrm{DI}(4))$. To do that, we consider the cobar spectral sequence for $H_*(\Omega B\mathrm{Sol}(q))$, with

$$E^2 = \mathrm{Cotor}^{H_*(B\mathrm{Sol}(q))}(\mathbb{F}_2, \mathbb{F}_2) \cong H_*(T(\Sigma^{-1}\overline{H}_*(B\mathrm{Sol}(q))), d_E),$$

where d_E is the external differential on the cobar construction, induced by the reduced diagonal in $H_*(B\mathrm{Sol}(q))$.

Recall from [7] that

$$H^*(B\mathrm{Sol}(q)) \cong P[u_8, u_{12}, u_{14}, u_{15}, t_7, t_{11}, t_{13}]/I,$$

where I is the ideal generated by the polynomials $r_1 = t_{11}^2 + u_8 t_7^2 + u_{15} t_7$, $r_2 = t_{13}^2 + u_{12} t_7^2 + u_{15} t_{11}$ and $r_3 = t_7^4 + u_{14} t_7^2 + u_{15} t_{13}$.

As for $BG_2(q)$, we denote classes in $H_*(B\mathrm{Sol}(q))$ by its dual cohomology class decorated by a bar. If $a \in \overline{H}^*(B\mathrm{Sol}(q))$ is any class, then the corresponding tensor algebra generator will be denoted by $[\bar{a}]$, while products of these generators will be written using the usual bar notation $[\overline{a_1}|\overline{a_2}|\cdots|\overline{a_n}]$. Thus the relation r_1 translate to the following equation in the E^1 page of the cobar spectral sequence:

$$0 = d_E([\overline{t_{11}^2}] + [\overline{u_8 t_7^2}] + [\overline{u_{15} t_7}]) = [\overline{t_{11}}]^2 + d_E([\overline{u_8 t_7^2}]) + [[\overline{u_{15}}], [\bar{t_7}]].$$

The classes $[\overline{t_{11}}]$, $[\overline{u_{15}}]$ and $[\bar{t_7}]$ are easily seen to be the permanent cycles in the spectral sequence corresponding to b_{10}, y_{14} and a_6 respectively. Hence we obtain the relation

$$[a_6, y_{14}] = b_{10}^2.$$

Next, notice that one has

$$[c_{12}, y_{14}] \overset{Sq_*^2}{\longmapsto} [b_{10}, y_{14}] \overset{Sq_*^4}{\longmapsto} [a_6, y_{14}]\,.$$

The commutator $[b_{10}, y_{14}]$ is primitive, while

$$\overline{\Delta}([c_{12}, y_{14}]) = a_6 \otimes [a_6, y_{14}] + [a_6, y_{14}] \otimes a_6 = a_6 \otimes b_{10}^2 + b_{10}^2 \otimes a_6.$$

Hence we conclude that

$$[b_{10}, y_{14}] = c_{12}^2 \quad \text{and} \quad [c_{12}, y_{14}] = e_{26} + a_6 b_{10}^2 = e_{26} + a_6[a_6, y_{14}].$$

Finally, by the previous calculations,

$$[e_{26}, y_{14}] = [[c_{12}, y_{14}], y_{14}] + [a_6[a_6, y_{14}], y_{14}] = [a_6, y_{14}]^2 = b_{10}^4.$$

This completes the calculation of $H_*(\Omega B\mathrm{Sol}(q))$ as a Hopf algebra over the dual Steenrod algebra.

Our final task is the calculation of the BSS for $\Omega B\mathrm{Sol}(q)$. Using the known structure of $H^*(B\mathrm{DI}(4))$ and $H^*(\mathrm{DI}(4))$ and the corresponding BSS, we conclude that

$$H^*(B\mathrm{DI}(4), \mathbb{Z}) \cong P[v_8, v_{12}, v_{15}, v_{28}]/(2v_{15}),$$

which allows us to conclude that $(\psi^q)^*(v_{2i}) = q^i v_{2i}$. Also, similarly to the corresponding computation for G_2,

$$H_i(\mathrm{DI}(4), \mathbb{Z}) \cong \begin{cases} \mathbb{Z}, & i = 0, 7, 11, 18, 27, 34, 38, 45, \\ \mathbb{Z}/2, & i = 13, 20, 24, 31, \end{cases}$$

while all other homology groups vanish. Denote homology classes by χ_i, where i corresponds to the dimension. By inspection of the mod 2 homology structure, it is easy to conclude that χ_7, χ_{11} and χ_{27} are the indecomposable among the torsion free classes, while the only torsion indecomposable class is χ_{13}.

Consider the fibration

$$\Omega\mathrm{DI}(4) \longrightarrow \Omega B\mathrm{Sol}(q) \longrightarrow \mathrm{DI}(4).$$

The integral homology SSS calculation of this fibration is similar to the one done for G_2. Since $H_*(\Omega\mathrm{DI}(4), \mathbb{Z})$ is torsion free, the E^2 page of the spectral sequence is the tensor product of the homologies of base and fibre. One has a commutative diagram of fibrations

$$\begin{array}{ccccc} \Omega\mathrm{DI}(4) & \longrightarrow & \Omega B\mathrm{Sol}(q) & \longrightarrow & \mathrm{DI}(4) \\ \| & & \downarrow & & \downarrow f^q \\ \Omega\mathrm{DI}(4) & \longrightarrow & * & \longrightarrow & \mathrm{DI}(4) \end{array}$$

where f^q is any map in the homotopy class of the composite

$$\mathrm{DI}(4) \xrightarrow{\Delta} \mathrm{DI}(4) \times \mathrm{DI}(4) \xrightarrow{\Omega\psi^q \times (-1)} \mathrm{DI}(4) \times \mathrm{DI}(4) \xrightarrow{\mu} \mathrm{DI}(4).$$

It is easy to verify that

$$H_n(\mathrm{DI}(4) \times \mathrm{DI}(4), \mathbb{Z}) \cong \bigoplus_{i+j=n} H_i(\mathrm{DI}(4), \mathbb{Z}) \otimes H_j(\mathrm{DI}(4), \mathbb{Z})$$

for $n \leq 32$. Hence for $i = 4, 6, 14$ one has $(f^q)_*(\chi_{2i-1}) = (q^i - 1)\chi_{2i-1}$. In the SSS for the path-loop fibration over DI(4) one has $d_7(\chi_7) = a_6$, $d_7(a_6\chi_7) = 2c_{12}$, $d_{11}(\chi_{11}) = b_{10}$, $d_{13}(\chi_{13}) = \bar{c}_{12}$ (the class of c_{12} modulo $\mathrm{Im}(d_7)$) and $d_{27}(\chi_{27}) = e_{26}$. Thus by commutativity of the diagram above and naturality of the SSS, one has in the integral homology SSS for the top row, $d_7(\chi_7) = (q^4 - 1)$, $d_{11}(\chi_{11}) = (q^6 - 1)b_{10}$ and $d_{27}(\chi_{27}) = (q^{14} - 1)e_{26}$. Setting $q = 4k \pm 1$ and $r_i = \nu_2(q^i - 1)$ and performing the necessary arithmetics, we see that $r_6 = r_{14} = \nu_2(k) + 3 = r_4 - 1$.

Finally, we compute the Bockstein spectral sequence for $H_*(\Omega B\mathrm{Sol}(q))$, which is a spectral sequence of modules over $H_*(\Omega\mathrm{DI}(4))$, and so we use the module structure given by (3.4) in the calculation. The first page of the spectral sequence is determined by $Sq^1_*(y_{14}) = y_{13}$. Thus

$$E^2 \cong P[a_6]/(a_6^2) \otimes P[b_{10}, c_{12}, e_{26}] \otimes E[y_7, y_{11}, h_{27}],$$

where h_{27} corresponds to the infinite cycle in E^1 given by $y_{13}y_{14}$. By the integral homology calculation above, the next nontrivial differential is $\beta_*^{r_4-1}$, and one has

$\beta^{r_4-1}(y_{11}) = c_{10}$, while $\beta^{r_4-1}(h_{27}) = e_{26}$. Next, we have $\beta_*^{r_4}(y_7) = a_6$, and since $\bar{\Delta}(c_{12}) = a_6 \otimes a_6$, it follows that $\beta_*^{r_4+1}(a_6 y_7)$ is defined and is equal to c_{12}, provided that $a_6 y_7 \neq 0$ in E^{r_4+1}, which is obvious since already E^2 does not contain a nonzero class in dimension 14. This completes the calculation of the BSS for $\Omega B\mathrm{Sol}(q)$, which takes the form

	a_6	y_7	b_{10}	y_{11}	c_{12}	y_{13}	y_{14}	$a_6 y_7$	$y_{13} y_{14}$
Sq_*^1	0	0	0	0	0	0	y_{13}	0	0
$\beta_*^{r_4-1}$	0	0	0	b_{10}	0	–	–	0	e_{26}
$\beta_*^{r_4}$	0	a_6	–	–	0	–	–	0	–
$\beta_*^{r_4+1}$	–	–	–	–	0	–	–	c_{12}	–

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Author information

Ran Levi, Institute of Mathematics, University of Aberdeen, Aberdeen AB24 3UE, UK.
E-mail: r.levi@abdn.ac.uk

Nora Seeliger, L.A.G.A. – Laboratoire de Mathématiques, Institut Galilée,
99 Avenue, J. B. Clément, 93430 Villetaneuse, France.
E-mail: seeliger@math.univ-paris13.fr