

Parametric instability induced scalar gravitational waves from a model pulsating neutron star

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Abstract

We identify a new dynamical mechanism for a strong scalar gravitational field effect. To illustrate this mechanism, we investigate the parametric excitation and emission of scalar gravitational waves by a radially pulsating model neutron star.

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1. Introduction

Within the metric description of gravitation, scalar-tensor (ST) theory of gravity provides a class of natural extensions to general relativity (GR), by including a scalar field as part of gravity. Though a multiplet of scalar fields can be considered, here we shall focus only on a single real-valued scalar field and denote it by ϕ . It modifies gravity in the following way: In the absence of this field, the motion of matter fields is determined using the metric g_{ab}^* , referred to as the “Einstein metric” as per GR. In the presence of the scalar field ϕ , the motion of matter fields are affected by the “physical metric” g_{ab} obtained from the Einstein metric through a conformal mapping $g_{ab} = A(\phi)^2 g_{ab}^*$, where $A(\phi)$ is a theory-dependent “coupling function” [1]. Since $A(\phi)$ effectively re-scales the mass of particles, it is also known as the “mass function” [2, 3]. The coupling strength between matter and scale fields ϕ is proportional to the field derivative $\alpha(\phi) := \partial a(\phi)/\partial \phi$ of the natural logarithm $a(\phi) := \ln A(\phi)$ of the coupling function. (See the field equations below.)

The pioneering scalar-tensor theory by Brans and Dicke [4, 5] is equivalent to the simple choice of a linear function $a(\phi) = \alpha_0 \phi$ where α_0 is a constant and is related to the Brans-Dicke parameter ω_{BD} by $\alpha_0^2 = (2\omega_{\text{BD}} + 3)^{-1}$. Thus the larger ω_{BD} , the weaker scalar field coupling. The most stringent test to date of the Brans-Dicke theory is provided by measuring the frequency shift of the radio signals to and from the Cassini-Huygens spacecraft, suggesting that $\omega_{\text{BD}} > 40000$ [6]. Further tests could be provided by the measurements of (tensor) gravitational wave forms from neutron stars and black holes using advanced laser interferometers [7, 8].

The weakness of the Brans-Dicke type scalar coupling can be explained in terms of the relaxation of a more general nonlinear function $a(\phi)$ to its local minimum during the cosmological evolution [9]. The effect amounts to a local attractor ϕ_0 around which $a(\phi) \approx \beta(\phi - \phi_0)^2/2$ with some constant $\beta > 0$. On the other hand, if ϕ_0 corresponds to a local maximum, then $\beta < 0$ and “spontaneous scalarization” could occur [10, 11]. Indeed, for $\beta \lesssim -4$ large deviation of the scalar field from its cosmological value may be developed inside a neutron star [10, 11]. The implications of this strong-field effect have attracted considerable recent attention e.g. [12–14].

The purpose of this paper is to point out a new strong field effect for $\beta > 0$. The scenario involves dynamical parametric instability for the scalar field inside a rapidly pulsating strong source of gravity, such as a young neutron star. In order to gain some understanding of this new effect, an idealized neutron star model is analyzed. We show that for some sufficiently large β together with certain physically reasonable conditions on the star, the pulsating model neutron star behaves like an optical cavity in which resonant scalar waves are parametrically amplified. The surface of the star then acts like a partial (anti-phase) reflector that releases travelling scalar waves analogous to an optical laser. This work therefore provides an initial estimate of the effect that can be extended for further investigation with realistic stars, including the possible energy transfer from a collapsed star core to stalled shock waves in supernova formations and other astrophysical problems [15]. A further motivation of this work is to seek a possible source of conformal fluctuations of spacetime as a result of background scalar gravitational waves [16].

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2. The scalar gravitational field

The general action for scalar-tensor gravity including matter can be expressed as [2, 5]:

$$S = \frac{c^4}{16\pi G_*} \int \frac{d^4x}{c} g_*^{1/2} R_* + S_\phi + S_m \quad (1)$$

where $a, b, \dots = 0, 1, 2, 3$, $d^4x = dx^0 d^3x$ and $x^0 = ct$, and

$$S_\phi = -\frac{c^4}{4\pi G_*} \int \frac{d^4x}{c} g_*^{1/2} \left[\frac{1}{2} g_*^{ab} \phi_{,a} \phi_{,b} + V(\phi) \right] \quad (2)$$

is the action for the scalar field ϕ in terms of a potential function $V(\phi)$ and

$$S_m = S_m[\psi, A^2(\phi)g_{ab}^*]$$

is the action for matter fields in terms of a coupling function $A(\phi)$.

The Minkowski metric is given by $\eta_{ab} = \text{diag}(-1, 1, 1, 1)$. The effective stress tensor for ϕ follows from (2) as

$$T_\phi^{ab} := 2c g_*^{-1/2} \frac{\delta S_\phi}{\delta g_{ab}^*} = \frac{c^4}{8\pi G_*} \left[2g_*^{ac} g_*^{bd} \phi_{,c} \phi_{,d} - g_*^{ab} (g_*^{cd} \phi_{,c} \phi_{,d} + 2V(\phi)) \right]. \quad (3)$$

The field equation for ϕ follows from varying the total action S in (1) as

$$\square_* \phi - \frac{\partial V(\phi)}{\partial \phi} = -\frac{4\pi G_*}{c^4} \alpha(\phi) T_* \quad (4)$$

where \square_* is the Laplace-Beltrami operator and T_* is the contracted stress tensor of the matter field with respect to g_{ab}^* .

We shall be concerned with the values of ϕ near a local minimum of $a(\phi)$. Thus up to an additive constant, equivalent to a re-scaling constant for the metric g_{ab} , we have approximately

$$a(\phi) = \frac{1}{2} \beta \phi^2 \quad (5)$$

for some constant $\beta > 0$. For simplicity we shall consider here the quadratic potential

$$V(\phi) = \frac{1}{2} \mu_0^2 \phi^2 \quad (6)$$

that gives rise to an effective mass

$$m_0 = \mu_0 \hbar / c$$

of the scalar field ϕ in vacuum.

Then (4) becomes

$$\square_* \phi - \mu_0^2 \phi = U \phi \quad (7)$$

where

$$U := -\frac{4\pi G_* \beta}{c^4} T_* \quad (8)$$

3. The model neutron star

It is interesting to note that (7) is a homogeneous wave equation for ϕ . As such, parametric excitation of ϕ through a time varying function U could occur. However, generation of large amplitude scalar wave does not necessarily follow unless parametric instability takes place. Such a condition would require rapid variations of a high density gravitational source with a sufficiently large value for β . In addition, damping of the parametric excitation through radiating scalar waves must also be sufficiently small, otherwise the parametric excitation would be stabilized.

In this paper, we investigate a particular scenario by means of a model pulsating neutron star to illustrate how these conditions can be satisfied through the parametric instability of the quasi-normal modes of the scalar field inside the star. We envisage that similar mechanism could be identified in other violent events in astrophysics and the early universe.

The radiation of scalar gravitational waves by a radially pulsating star using the Brans-Dicke theory was studied in [17]. (See also [2].) The emission mechanism in that case is essentially the same as that for the conventional tensor gravitational waves but monopole radiation is possible. In our present case of quadratic scalar coupling using (5), the dynamical structure is completely different, as we will see below.

Following [17], we adopt

$$T_* = -c^2 \rho + 3p$$

as the contracted stress tensor for matter inside a spherically symmetric star model with radius R , density ρ and pressure p .

Outside the star we have simply $\rho = 0 = p$ and hence $T_* = 0$. For this model, we shall further assume $c^2 \rho \gg p$ so that (8) can be approximated by

$$U = \frac{4\pi G_* \beta}{c^2} \rho. \quad (9)$$

For simplicity, we shall also ignore the curvature of g_{ab}^* by approximating it with the Minkowski metric η_{ab} . Therefore the scalar field equation (7) reduces:

$$\partial_0^2 \phi - \Delta \phi + \mu_0^2 \phi + U \phi = 0 \quad (10)$$

where Δ is the 3-dimensional Laplace operator.

The star is modelled as a solid sphere with a constant equilibrium density ρ_0 . Its density fluctuations are described by the wave equation

$$\partial_0^2 \rho - (v/c)^2 \Delta \rho = 0 \quad (11)$$

where v is the speed of the density/pressure wave. For simplicity we shall consider a single mode radial oscillation of the density subject to zero boundary condition at the surface $r = R$ so that

$$\rho = \rho_0 [1 - \epsilon \chi_m(r) \cos(\Omega_m t)] \quad (12)$$

for some positive integer m as a mode index, where

$$\Omega_m = \frac{m\pi v}{R} \quad (13)$$

is the oscillation frequency, ϵ is a dimensionless amplitude parameter, and

$$\chi_n(r) := \frac{R}{r} \sin(\kappa_n r) \quad (14)$$

with the wave number

$$\kappa_n = n\pi/R$$

for $n = 1, 2, \dots$. The following orthogonality relation holds:

$$\int_0^R dr r^2 \chi_n(r) \chi_m(r) = \frac{R^3}{2} \delta_{nm} \quad (15)$$

for any $m, n = 1, 2, \dots$.

Substituting (14) into (9) we have

$$U = U_0 [1 - \epsilon \chi_m(r) \cos(\Omega_m t)] \quad (16)$$

where

$$U_0 := \frac{4\pi G_* \beta}{c^2} \rho_0 \geq 0. \quad (17)$$

Then (10) becomes

$$\partial_0^2 \phi - \Delta \phi + \mu^2 \phi - \epsilon U_0 \chi_m(r) \cos(\Omega_m t) \phi = 0 \quad (18)$$

where

$$\mu^2 := \mu_0^2 + U_0. \quad (19)$$

The energy density and flux of ϕ can then be evaluated from (3) using $g_{ab}^* = \eta_{ab}$ to be

$$u := T_\phi^{00} = \frac{c^4}{8\pi G_*} [\phi_{,0} \phi_{,0} + \eta^{\alpha\beta} \phi_{,\alpha} \phi_{,\beta} + 2V(\phi)] \quad (20)$$

$$f^\beta := c T_\phi^{0\beta} = -\frac{c^5}{4\pi G_*} \phi_{,0} \phi_{,\beta} \quad (21)$$

respectively, where $\alpha, \beta = 1, 2, 3$. The potential takes the forms

$$V(\phi) = \frac{1}{2} \mu^2 \phi^2$$

and

$$V(\phi) = \frac{1}{2} \mu_0^2 \phi^2$$

inside and outside the model star respectively.

4. Approximate normal modes of the scalar field inside the model star

In the absence of the density oscillation, i.e. $\epsilon = 0$, the model star has a constant density ρ_0 , inside which ϕ satisfies the Klein-Gordon equation with mass parameter $\mu > \mu_0$ given in (19).

For $\mu/\mu_0 \gg 1$ the star surface at $r = R$ behaves like a perfect (anti-phase) reflector for outgoing ϕ . As such the scalar field inside the star can be approximated by a standing wave subject to the boundary conditions $\phi(R, t) = 0$ as follows:

$$\phi \approx \sum_n \phi_n := \sum_n \varphi_n(t) \chi_n(r) \quad (22)$$

where $n = 1, 2, \dots$. Each ϕ_n denotes a normal mode with

$$\varphi_n(t) = \Re \varphi_{n0} e^{-i\omega_n t} \quad (23)$$

where φ_{n0} is a modal amplitude constant and

$$\frac{\omega_n^2}{c^2} = \kappa_n^2 + \mu^2. \quad (24)$$

To evaluate the energy associated with these normal modes, we first obtain the energy density of ϕ inside the model star

$$u = \frac{c^4}{8\pi G_*} [(\phi_{,0})^2 + (\phi_{,r})^2 + \mu^2 \phi^2] \quad (25)$$

by using (20) in spherical coordinates.

Using (14), (15), (22), (23) and (24), this yields the following energy for ϕ_n :

$$E_n = 4\pi \int_0^R dr r^2 u = \frac{c^2}{2G_*} R^3 \varphi_{n0}^2 \omega_n^2. \quad (26)$$

5. Quasi-normal modes and damping due to transmitted scalar waves

For finite $\mu/\mu_0 > 1$, the star surface does allow some scalar wave to propagate across it.

By taking into account the resulting loss of energy, we can refine ϕ_n in (22) to be quasi-normal modes. To this end, we assume that (23) is valid over a few circles of oscillation at angular frequency ω_n . Exterior to the star, this yields the scalar field

$$\phi_n = \Re \varphi_{n0} \frac{\kappa_n R}{k_n r} e^{i(k_n r - \omega_n t + \theta_n)} \quad (27)$$

exterior to the model star ($r > R$), where θ_n is a constant phase and

$$k_n^2 = \frac{\omega_n^2}{c^2} - \mu_0^2. \quad (28)$$

Using (19), (24) we have

$$\frac{\kappa_n^2}{k_n^2} = \frac{\kappa_n^2}{\kappa_n^2 + U_0}. \quad (29)$$

The power carried by outgoing waves then follows from (21) as (at $r \gg R$):

$$P_n = 4\pi r^2 |f^\beta| = \frac{c^4}{G_*} \frac{\kappa_n^2}{k_n^2} R^2 \varphi_{n0}^2 \omega_n k_n \quad (30)$$

This yields the damping factor d_n :

$$d_n = \frac{P_n}{E_n} = \frac{2c^2 \kappa_n^2}{R \omega_n k_n} \quad (31)$$

for φ_n , which now describes a quasi-normal mode satisfying the damped oscillator equation:

$$\frac{d^2 \varphi_n}{dt^2} + d_n \frac{d\varphi_n}{dt} + \omega_n^2 \varphi_n = 0. \quad (32)$$

6. Parametric excitation of the normal modes

We now estimate the parametric excitation of the normal modes of the scalar field in the presence of the density oscillation described by (12) with $\epsilon \neq 0$. We shall first neglect the effect of damping as just discussed and then take this effect into account later on. In order to gain a simple rough estimate of the excitation effect we shall further neglect mode coupling through parametric excitation.

We therefore proceed by applying (22) with a single mode for some n into (18) and then use χ_n as a test function to extract the equation for φ_n , i.e.:

$$\int_0^R dr r^2 \chi_n [(\partial_0^2 \varphi_n) \chi_n - \varphi_n \Delta \chi_n + \varphi_n \mu^2 \chi_n - \epsilon U_0 \chi_m(r) \cos(\Omega_m t) \varphi_n \chi_n] = 0. \quad (33)$$

Using (15) this yields

$$\frac{d^2 \varphi_n}{dt^2} + \omega_n^2 \varphi_n - \epsilon U_0 c^2 \chi_{nm} \cos(\Omega_m t) \varphi_n = 0 \quad (34)$$

where

$$\chi_{nm} := \frac{2}{R^3} \int_0^R dr r^2 \chi_n^2 \chi_m = \text{Si}(m\pi) - \frac{1}{2} \text{Si}(2n\pi + m\pi) + \frac{1}{2} \text{Si}(2n\pi - m\pi). \quad (35)$$

7. Modal equation including damping and parametric excitation

Incorporating both damping and parametric excitation, we arrive at the following equation

$$\frac{d^2 \varphi_n}{dt^2} + d_n \frac{d\varphi_n}{dt} + \omega_n^2 \varphi_n - \epsilon c^2 U_0 \chi_{nm} \cos(\Omega_m t) \varphi_n = 0 \quad (36)$$

for each quasi-normal mode. To see the stability of these modes we cast (36) into the following canonical form for damped Mathieu equation:

$$\frac{d^2 \varphi_n}{d\tau^2} + 2\zeta \frac{d\varphi_n}{d\tau} + a\varphi_n - 2q \cos(2\tau)\varphi_n = 0 \quad (37)$$

by using (19), (24), (31), in terms of the following dimensionless quantities:

$$\begin{aligned} \tau &= \frac{\Omega_m}{2} t, & \zeta &= \frac{2c^2 \kappa_n^2}{R \Omega_m \omega_n k_n} \\ a &= \frac{4c^2}{\Omega_m^2} (\kappa_n^2 + \mu_0^2 + U_0), & q &= \frac{2\epsilon c^2 U_0 \chi_{nm}}{\Omega_m^2}. \end{aligned} \quad (38)$$

The stability domain near the principal parametric excitation frequency with $a \approx 1$ and $q \approx 0$ have been obtained in [18]. For $a = 1$, i.e. $\Omega_m = 2\omega_n$, φ_n becomes unstable if the condition

$$\left| \frac{q}{2\zeta} \right| = \left| \frac{\epsilon \chi_{nm} R U_0 \sqrt{\kappa_n^2 + U_0}}{4 \kappa_n^2} \right| \gtrsim 1 \quad (39)$$

is satisfied. From (17) we see that the above condition can be satisfied for sufficiently large β and ϵ .

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