Boundedness of solutions for a bouncing ball model with quasiperiodic potential

Denghui Li[a], Xiaoming Zhang[b], Xianbin Liu[b] [1], Jianhua Xie[c] Celso Grebogi[d]

[a] *School Of Mathematics And Statistics, Hexi University, Zhangye, Gansu, China, 734000*

[b] *State Key Lab of Mechanics and Control for Mechanical Structures, College of Aerospace Engineering, Nanjing University of Aeronautics and Astronautics, Jiangsu, Nanjing, China, 210016*

[c] *School Of Mechanics And Aerospace Engineering, Southwest Jiaotong University, Chengdu, Sichuan, China, 610031*

[d] *Institute For Complex Systems And Mathematical Biology King'S College, University Of Aberdeen, Aberdeen Ab24 3Ue, United Kingdom*

Abstract

In this work, we consider the model of a free falling ball on a wall that is elastically reflected. The wall is supposed to move in the vertical direction according to a given quasiperiodic function. Based on the invariant curve theorem of smooth quasiperiodic twist map, we prove the boundedness of all solutions and the existence of quasi-periodic solutions for the system.

Keywords: bouncing ball, invariant curve, quasiperiodic motion, boundedness.

1. Introduction

Mechanical systems with impacts appear widely in science and engineering [1, 2]. This kind of system also appears in theoretical physics, such as the Fermi-Ulam model which describes the motion of a particle between two moving walls. This system was introduced by Fermi in an attempt to explain the origin of the high-energy cosmic radiation. The main question about this model is whether the particle will attain unlimited energy (called Fermi acceleration)? When the motion of wall is periodic, it has been shown that the existence of Fermi acceleration depends on the smoothness. If the motion of boundary is real analytic or at least C^6, then all solutions are bounded in phase space [3, 4]. While the Fermi acceleration occurs when the motion of the wall is only piecewise smooth [5–7]. See [8, 9] for more results about this model.

This paper concentrates on an impact oscillator of one degree of freedom. Consider the model of a free falling ball on a moving wall. The wall moves in the vertical direction according to a function $f(t)$ and the ball is reflected according to the law of elastic bouncing when hitting the wall. When the initial velocity of the ball is sufficiently large, a good strategy to describe the motion of the ball is to establish a map that sends (t_0, v_0) representing the time of impact and the velocity immediately after it to the next impact

[1]Corresponding Author, Email Address: xiaoming_zhang@nuaa.edu.cn

time and corresponding velocity (t_1, v_1). The implicit map is given by

$$\begin{cases} t_1 = t_0 + \frac{2}{g}v_0 - \frac{2}{g}f[t_1, t_0], \\ v_1 = v_0 + 2\dot{f}(t_1) - 2f[t_1, t_0], \end{cases} \tag{1}$$

where

$$f[t_1, t_0] = \frac{f(t_1) - f(t_0)}{t_1 - t_0}.$$

The map is a twist map preserving the symplectic form $dv^2 \wedge dt$ on the cylinder. This model has inspired many authors as it represents a simple mechanical model exhibiting complex dynamics. Marò [10] proved the existence of periodic and quasiperiodic solutions by Aubry-Mather theory. The destruction of invariant curves is strictly correlated with the presence of chaotic motion [11, 12]. Moreover, if f also has some singularity it is possible to study its statistical and ergodic properties [13].

In order to study the chaotic dynamics of the bouncing ball model, Holmes [14] considered an approximation of the map (1), which is given by the assumption of a large amplitude of the motion of the ball with respect to the amplitude of the motion of the wall. As an example, he considered the case $f(t) = \beta \sin(\omega t)$. In this case the model is described by the so-called standard map

$$\begin{cases} t_1 = t_0 + \frac{2}{g}v_0, \\ v_1 = v_0 + 2\beta\omega\cos(\omega t_0). \end{cases} \tag{2}$$

The existence of chaotic motion of the map (2) is widely studied both theoretically and numerically, see e.g. [15–17].

For the piecewise smooth Hamiltonian systems with periodic perturbations, Moser's invariant curve theorem plays a fundamental role to address the stability of the solutions [18–22]. The invariant curves separating the phase space into two invariant parts prevent the trajectories from escaping to infinity. When the perturbations of the twist map are quasiperiodic analytic functions, Zharnitsky [23] proved the existence of invariant curves and the result was applied to the stability of motions in the Fermi-Ulam problem. Huang et al extend the invariant curves theorem to smooth quasiperiodic twist maps [24], and proved the boundedness of all solutions and the existence of quasi-periodic solutions for a asymmetric oscillator [25].

Inspired by the work mentioned above, in this work we investigate the dynamics of the bouncing ball model when $f(t)$ is a quasiperiodic function. Our purpose is to prove the existence of quasiperiodic motion and the boundedness of all motion for the model based on the quasiperiodic version of the invariant curve theorem in [24]. Instead of considering the map (1) defined implicitly, we use action-angle variables to establish a twist map. The remaining of the paper is organized as follows. In Section 2, we will recall some notations that will be used in what follows. Section 3 gives precise statement of the model and of the result we want to prove. In Section 4, we reduce the system into action-angle variables and establish the twist map. Section 5 devotes to prove the existence of invariant curves. In Section 6, we present the conclusions.

2. Some preliminaries

2.1. Quasiperiodic function

Definition 2.1. *$f : \mathbb{R}^1 \to \mathbb{R}^1$ is called a C^p quasiperiodic function with frequency $\omega = (\omega_1, \omega_2, \cdots, \omega_n)$, if there exists a C^p function $F(\theta_1, \theta_2, \cdots, \theta_n)$ which has period 2π in each* *variable such that*

$$f(t) = F(\omega_1 t, \omega_2 t, \cdots, \omega_n t).$$

F is called a shell function of f.

A continuous quasiperiodic function can be represented by a Fourier series:

$$f(t) = \sum_k f_k e^{i\langle k,\omega\rangle t},$$

where $k = (k_1, k_2, \cdots, k_n) \in \mathbb{Z}^n$, $\omega = (\omega_1, \omega_2, \cdots, \omega_n)$, $\langle k, \omega\rangle = k_1\omega_1 + \cdots + k_n\omega_n \neq 0$ if $k \neq 0$.

Denote by $Q(\omega)$ the space of quasiperiodic functions with frequency ω. $Q(\omega)$ forms a vector space over the real numbers and is closed under forming products and quotients, in the latter case provided that the denominator is bounded away from 0. To each quasiperiodic function $f(t)$, the mean value

$$\lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} f(t)dt$$

exists. (c.f. Chapter 3 of [26]).

2.2. Quasiperiodic twist map

Let $\mathcal{S} = \mathbb{R} \times [a, b]$. Consider the planar map

$$M : \begin{cases} \theta_1 = \theta_0 + r_0 + f(\theta_0, r_0), \\ r_1 = r_0 + g(\theta_0, r_0), \end{cases} \quad (\theta_0, r_0) \in \mathcal{S}, \tag{3}$$

defined on the strip $\mathcal{S}$, where the perturbations $f(r_0, \theta_0)$ and $g(r_0), \theta_0)$ are quasiperiodic in θ_0 with frequency $\omega = (\omega_1, \omega_2, \cdots, \omega_n)$, and C^p smooth. Then M is called a quasiperiodic twist map of class C^p.

Definition 2.2. *For the map M given by (3), we say that it satisfies the intersection property if*

$$M(\Gamma) \cap \Gamma \neq \varnothing$$

for every curve $\Gamma : \theta = \xi + \varphi(\xi), r = \psi(\xi)$ in $\mathcal{S}$, where the continuous functions φ and ψ are quasiperiodic in ξ with frequency $\omega = (\omega_1, \omega_2, \cdots, \omega_n)$.

Let $p \geqslant 0$ be an integer. For a function $h : \mathcal{S} \to \mathbb{R}$ of class C^p, let

$$\|h\|_p = \sum_{0\leq k_1+k_2\leq p} \sup_{(\theta,r)\in\mathcal{S}} \left|\frac{\partial^{k_1+k_2} h(\theta, r)}{\partial\theta^{k_1}\partial r^{k_2}}\right|.$$

Choose a rotation number α satisfying the inequalities

$$\begin{cases} a + 12^{-3}\gamma \leqslant \alpha \leqslant b - 12^{-3}\gamma, \\ \left|\langle k, \omega\rangle \dfrac{\alpha}{2\pi} - j\right| \geqslant \dfrac{\gamma}{|k|^\tau}, \text{ for all } k \in \mathbb{Z}^n \setminus \{0\}, j \in \mathbb{Z} \end{cases} \tag{4}$$

with constants γ, τ satisfying

$$0 < \gamma < \frac{1}{2}\min\{1, 12^3(b - a)\}, \quad \tau > n. \tag{5}$$

If $\tau > n$, then for suitably small γ, the set of α satisfying (4) has a positive measure.

Theorem 2.1. *(**[24]**) Suppose that the quasiperiodic twist map M given by (3) is of class C^p with $p > 2\tau + 1$, satisfies the intersection property and the following smoothness conditions*

$$\|f\|_p + \|g\|_p \leqslant \varepsilon_0$$

where ε_0 is a small positive constant. Then for any number α satisfying the inequalities (4), M has an invariant curve Γ_0 having the form

$$\theta = \mu + \xi(\mu), \quad r = \eta(\mu),$$

where ξ, η are quasiperiodic functions with frequency $\omega = (\omega_1, \omega_2, \cdots, \omega_n)$, and the invariant curve Γ_0 is continuous. Moreover, the restriction of M onto Γ_0 is the rigid rotation

$$\mu_1 = \mu_0 + \alpha.$$

Definition 2.3. *For the quasiperiodic twist map M given by (3), we say that it is an exact symplectic if $dr_0 \wedge d\theta_0 = dr_1 \wedge d\theta_1$ and for every curve $\Gamma : \theta_0 = \xi + \phi(\xi), r_0 = \psi(\xi)$ we have*

$$\lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} r_0 \mathrm{d}\theta_0 = \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} r_1 \mathrm{d}\theta_1,$$

where the continuous functions φ and ψ are quasiperiodic in ξ with frequency $\omega = (\omega_1, \omega_2, \cdots, \omega_n)$.

Remark 2.1. *If the map M is exact symplectic, then it has the intersection property, see [25, Lemma 2.9].*

3. The model

Consider the model of a ball with unit mass bouncing on a horizontal wall that is moving in the vertical direction according to a given regular quasiperiodic function $f(t)$. Suppose that the gravity force is acting on the ball. At the impact, the change of velocity is assumed to be elastic. See Figure 1.

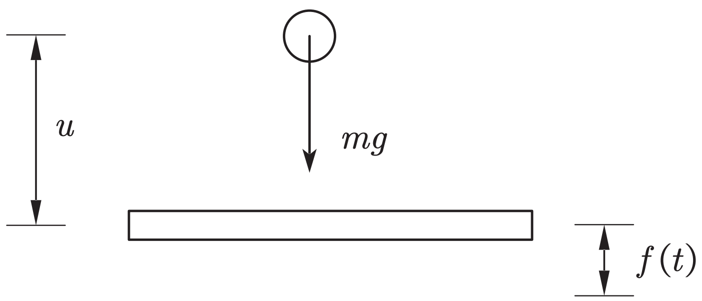

Figure 1: The bouncing ball model

In the coordinate system fixed at the wall, the equation of motion of the bouncing ball is given by

$$\begin{aligned} &\ddot{u} = -(g + \ddot{f}(t)), \\ &u(t) \geq 0, \\ &u(\tau) = 0 \Rightarrow \dot{u}\left(\tau^{+}\right) = -\dot{u}\left(\tau^{-}\right), \end{aligned} \tag{6}$$

where f is a C^ν quasiperiodic function with frequency $\omega = (\omega_1, \omega_2, \cdots, \omega_n)$.

The main result of this work is the following

Theorem 3.1. *Suppose that $\nu > 2n+4$ and $f(t)$ satisfies $\|f\|_\nu \le \varepsilon_0$, where $\varepsilon_0 > 0$ is a constant sufficiently small. Then the system (6) has infinitely many quasiperiodic solutions. Moreover, all solutions of the system are bounded, i.e., for every solution $u(t)$,*

$$\sup_{t\in\mathbb{R}} \left(|u(t)| + |\dot{u}(t)|\right) < +\infty.$$

Remark 3.1. *Note that $n = 1$ corresponds to the periodic case, the results of Theorem 3.1 is consistent with that obtained by the Moser's twist theorem.*

4. Reduction to a twist map

Set $\ddot{f}(t) = \varepsilon p(t)$. Then $p(t)$ is a quasiperiodic function of class $C^{\nu-2}$. The Hamiltonian of the system (6) is

$$H(u, v, t) = \frac{1}{2}v^2 + gu + \varepsilon p(t)u, \qquad u \ge 0,$$

where $v = \dot{u}$.

We will express the equation of the system in terms of action-angle variables I and θ. As usual (c.f. [27]), the action I is defined as the area bounded by the level curve γ in region $\{(u,v) \in \mathbb{R}^2 | u \ge 0\}$ defined by

$$H_0(u, v) = \frac{1}{2}v^2 + gu = h(I),$$

that is

$$I(h) = 2\int_0^{\frac{h}{g}} \sqrt{2h - 2gu}\,du = \frac{4\sqrt{2}h^{\frac{3}{2}}}{3g}. \tag{7}$$

It follows from (7) that

$$h(I) = \frac{(3gI)^{\frac{2}{3}}}{2\cdot 4^{\frac{1}{3}}}. \tag{8}$$

We define now the generating function as the area (shaded in Figure 2):

$$\begin{aligned} S(I, v) &= \int_v^{\sqrt{2h}} u(s)ds = \int_v^{\sqrt{2h}} \frac{1}{g}\left(h - \frac{1}{2}s^2\right) ds \\ &= \frac{2\sqrt{2}h^{\frac{3}{2}}}{3g} - \frac{hv}{g} + \frac{v^3}{6g}, \end{aligned}$$

where h is given by the equation (8).

We obtain the map $(\phi, I) \mapsto (u, v)$ via

$$\phi(I, v) = \frac{\partial S}{\partial I}(I, v), \qquad u(I, v) = -\frac{\partial S}{\partial v}(I, v). \tag{9}$$

This map is symplectic. Indeed, using (9) gives

$$\begin{aligned} &du \wedge dv = (-S_{vI}dI - S_{vv}dv) \wedge dv = -S_{vI}dI \wedge dv = S_{vI}dv \wedge dI; \\ &d\phi \wedge dI = (S_{II}dI + S_{Iv}dv) \wedge dI = S_{Iv}dv \wedge dI \end{aligned}$$

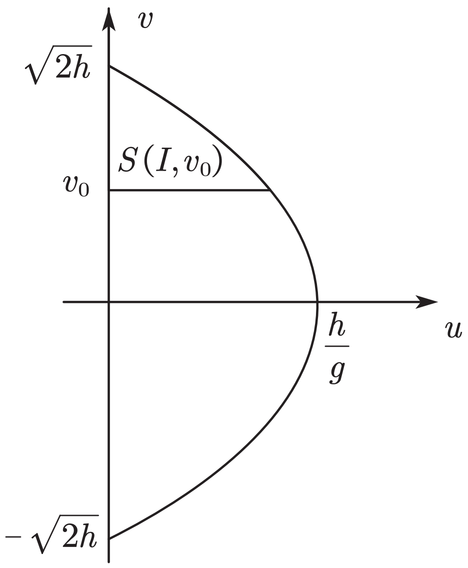

Figure 2: The definition of action and angle variables

so that $du \wedge dv = d\phi \wedge dI$.

A simple computation shows that

$$\phi(I,v) = \frac{1}{2} - \frac{v}{(12gI)^{\frac{1}{3}}},$$

$$v = \frac{1}{2}(1-2\phi)(12gI)^{\frac{1}{3}},$$

$$u(I,\phi) = \left(h(I) - \frac{1}{2}v(I,\phi)^2\right)/g = \frac{(12gI)^{\frac{2}{3}}\phi(1-\phi)}{2g}.$$

The Hamiltonian of the system (6) is transformed into

$$\begin{aligned} H(\phi,I,t) &= h(I) + \varepsilon p(t)u(I,\phi) \\ &= \frac{(3gI)^{\frac{2}{3}}}{2\cdot 4^{\frac{1}{3}}} + \frac{(12gI)^{\frac{2}{3}}\phi(1-\phi)}{2g}\varepsilon p(t). \end{aligned}$$

where $\phi \in [0,1]$.

Without loss of generality, it suffices to study the system with Hamiltonian

$$H(\phi,I,t) = I^{\frac{2}{3}} + \frac{4}{g}I^{\frac{2}{3}}\phi(1-\phi)\varepsilon p(t), \quad \phi \in [0,1]. \tag{10}$$

Clearly, the system (10) is near integrable in the region where I is finite provided that ε is small enough. However, using some tricks based on the fact that the flow of the Hamiltonian system and the 1-form $Id\phi - Hdt$ are invariantly related, we will show that the system (10) is also near integrable in the region with arbitrarily high energy.

Rescale the system (10) via

$$\kappa I = P, \quad \kappa H = K,$$

where $\kappa > 0$ is a small constant. Based on $Pd\phi - Kdt = \kappa(Id\phi - Hdt)$, the system (10) in the new coordinates has the Hamiltonian

$$
\begin{aligned}
K(\phi, P, t) &= \kappa \left(\left(\frac{P}{\kappa}\right)^{\frac{2}{3}} + \left(\frac{P}{\kappa}\right)^{\frac{2}{3}} \phi(1-\phi)\varepsilon p(t) \right) \\
&= \kappa^{\frac{1}{3}} \left(P^{\frac{2}{3}} + P^{\frac{2}{3}} \frac{4}{g} \phi(1-\phi)\varepsilon p(t) \right).
\end{aligned}
$$

Since the integral curves of the Hamiltonian system are invariantly associated with the differential form

$$
\kappa I d\phi - \kappa H dt = -(Kdt - Pd\phi),
$$

we choose P as the new Hamiltonian, the variables t, K and ϕ as the new position, momentum and time, respectively. Then we obtain the system

$$
\frac{dt}{d\phi} = \frac{\partial P}{\partial K}, \quad \frac{dK}{d\phi} = -\frac{\partial P}{\partial t}, \tag{11}
$$

where

$$
P = \left(\frac{K\kappa^{-\frac{1}{3}}}{1 + \frac{4}{g}\phi(1-\phi)\varepsilon p(t)} \right)^{\frac{3}{2}}, \quad \phi \in [0,1].
$$

The system (11) is equivalent to the one with Hamiltonian

$$
F(K, t, \phi, \varepsilon) = K^{3/2} \left(\frac{1}{1 + \frac{4}{g}\phi(1-\phi)\varepsilon p(t)} \right)^{\frac{3}{2}}, \quad \phi \in [0,1].
$$

It is clear that F is well defined for small ε.

Set

$$
w(x) = (\frac{1}{1+x})^{3/2}.
$$

Since we assume that ε is small, expanding the term $w(\frac{4}{g}\phi(1-\phi)\varepsilon p(t))$ within F in Taylor series, we get:

$$
F(K, t, \phi, \varepsilon) = K^{3/2} + K^{3/2} m(\phi, t, \varepsilon), \tag{12}
$$

where $m(\phi, t, \varepsilon)$ is the remainder of the Taylor series in integral form:

$$
m(\phi, t) = \int_0^x w'(s) ds, \ x = \frac{4}{g}\phi(1-\phi)\varepsilon p(t). \tag{13}
$$

It follows from (13) that $m(\phi, t, \varepsilon)$ is quasiperiodic about t and as smooth as $\frac{4}{g}\phi(1-\phi)\varepsilon p(t)$. Moreover, $\|m(\phi, t, \varepsilon)\|_{\nu-2} = O(\varepsilon)$.

The system with the Hamiltonian given by (12) is

$$
\begin{cases}
\dfrac{dt}{d\phi} = \dfrac{\partial F}{\partial K} = \dfrac{3}{2} K^{1/2} + \dfrac{3}{2} K^{1/2} m(\phi, t, \varepsilon), \\[2ex]
\dfrac{dK}{d\phi} = -\dfrac{\partial F}{\partial t} = -K^{3/2} \dfrac{\partial m}{\partial t}(\phi, t, \varepsilon).
\end{cases} \tag{14}
$$

Introducing the new coordinates $(r,t) = (\frac{3}{2}K^{1/2}, t)$, then the system (14) in the new coordinates becomes

$$\begin{cases} \dfrac{dt}{d\phi} = r + rm(\phi, t, \varepsilon), \\ \dfrac{dr}{d\phi} = -\dfrac{1}{3}r^2\dfrac{\partial m}{\partial t}(\phi, t, \varepsilon). \end{cases} \tag{15}$$

Integrating the system (15) on the interval $[0,1]$ formally, the time-1-map has the following expression:

$$\mathcal{M} : \begin{cases} t_1 = t_0 + r_0 + R_1(t_0, r_0, \varepsilon), \\ r_1 = r_0 + R_2(t_0, r_0, \varepsilon), \end{cases} \tag{16}$$

where $R_1(t_0, r_0, \varepsilon)$ and $R_2(t_0, r_0, \varepsilon)$ are the $O(\varepsilon)$ terms of the time-1-map. In the following subsection, we will show that R_1 and R_2 are $C^{\nu-3}$ smooth and quasiperiodic functions about t_0.

Remark 4.1. *Note that the time-1-map of the system* (15) *corresponds to the usual impact map of the system* (6)*. In fact, by the definition of ϕ, the orbits of* (15) *from $\phi = 0$ to $\phi = 1$ correspond to the orbits of* (6) *from $\{(u, \dot{u}, t) : u = 0, \dot{u} > 0\}$ to $\{(u, \dot{u}, t) : u = 0, \dot{u} < 0\}$. Moreover, since the impact does not change the Hamiltonian, the time-1-map of the system* (15) *is just the impact map in the coordinates with the impact time and the rescaled energy.*

5. Proof of the Theorem 3.1

In this Section, we show that the map (16) on $\mathbb{R} \times [a,b]$ satisfies the conditions of Theorem 2.1 provided that ε is sufficiently small, where $0 < a < b$. The proof is divided into three parts. First we show that the functions $R_1(t_0, r_0, \varepsilon)$ and $R_2(t_0, r_0, \varepsilon)$ are quasiperiodic about t_0. Then a simple argument, based on the estimate of the remainders of time-1-map on differential equations, implies that the $C^{\nu-3}$ norm of R_1 and R_2 tends to zero uniformly as ε tends to zero. Finally, we use the Poincaré-Cartan integral invariant to show that the map (16) has the intersection property.

5.1. Quasiperiodicity of the remainders

To show that $R_1(t_0, r_0, \varepsilon)$ and $R_2(t_0, r_0, \varepsilon)$ are quasiperiodic about t_0, we rewrite equations (15) in the form:

$$\begin{cases} \dfrac{dt}{d\phi} = r + M_1(\omega_1 t, \cdots, \omega_n t, r, \phi, \varepsilon), \\ \dfrac{dr}{d\phi} = M_2(\omega_1 t, \cdots, \omega_n t, r, \phi, \varepsilon), \end{cases} \tag{17}$$

where M_1 and M_2 are the shell functions of $rm(\phi, t, \varepsilon)$ and $-\frac{1}{3}\frac{\partial m}{\partial t}(\phi, t, \varepsilon)$ about t, respectively. Let $\theta_i = \omega_i t$. The system (17) is equivalent to

$$\begin{cases} \dfrac{d\theta_i}{d\phi} = \omega_i\big(r + M_1(\theta_1, \cdots, \theta_n, r, \phi, \varepsilon)\big), \ i = 1, \cdots, n, \\ \dfrac{dr}{d\phi} = M_2(\theta_1, \cdots, \theta_n, r, \phi, \varepsilon). \end{cases} \tag{18}$$

Denoted by

$$\theta_i(\phi;\theta_{10},\cdots,\theta_{n0},r_0,\varepsilon),\quad r(\phi;\theta_{10},\cdots,\theta_{n0},r_0,\varepsilon),\qquad i=1,2,\cdots,n$$

the solution of the system (18) with initial condition $(\theta_{10},\cdots,\theta_{n0},r_0)$ at $\phi=0$, where $\theta_{i0}=\omega_i t_0$. Then it is clear that $\theta_i(\phi;\theta_{10},\cdots,\theta_{n0},r_0,\varepsilon)-\theta_{i0}$ and $r(\phi;\theta_{10},\cdots,\theta_{n0},r_0,\varepsilon)$ are 2π-periodic about θ_{i0}. Since

$$t(\phi;t_0,r_0,\varepsilon)=\frac{1}{\omega_i}\theta_i(\phi;\theta_{10},\cdots,\theta_{n0},r_0,\varepsilon)=\frac{1}{\omega_i}\theta_i(\phi;\omega_1 t_0,\cdots,\omega_n t_0,r_0,\varepsilon),$$

$$r(\phi;t_0,r_0,\varepsilon)=r(\phi;\theta_{10},\cdots,\theta_{n0},r_0,\varepsilon)=r(\phi;\omega_1 t_0,\cdots,\omega_n t_0,r_0,\varepsilon),$$

it follows that $R_1(t_0,r_0,\varepsilon)$ and $R_2(t_0,r_0,\varepsilon)$ are quasiperiodic about t_0 with frequency ω, where $t(\phi;t_0,r_0,\varepsilon),r(\phi;t_0,r_0,\varepsilon)$ is the solution of (15) with initial condition (t_0,r_0) at $\phi=0$,

5.2. Regularity of the remainders

Take $\Delta>0$ and a compact interval $[a,b]$ with $0<a<b$. First note that the vector field of system (15) is $C^{\nu-3}$ about (t,r,ϕ), C^∞ about ε, and uniformly bounded on $(t,r,\phi,\varepsilon)\in\mathbb{R}\times[a,b]\times[0,1]\times[0,\Delta]$. By general theory of differential equations, see e.g. [28], for $(t_0,r_0,\varepsilon)\in\mathbb{R}\times[a,b]\times[0,\Delta]:=\mathcal{D}$, the solution $(t(\phi;t_0,r_0,\varepsilon),r(\phi;t_0,r_0,\varepsilon))$ of the system (15) is well defined in $[0,1]$ and keeps the regularity $C^{\nu-3}$ and C^∞ about (t_0,r_0) and ε, respectively.

For each $(t_0,r_0,\varepsilon)\in\mathcal{D}$, by the mean value theorem,

$$\begin{cases} R_1(t_0,r_0,\varepsilon)=t(1;t_0,r_0,\varepsilon)-t(1;t_0,r_0,0)=\dfrac{\partial t}{\partial\varepsilon}(1;t_0,r_0,\zeta_1)\varepsilon,\\ R_2(t_0,r_0,\varepsilon)=r(1;t_0,r_0,\varepsilon)-r(1;t_0,r_0,0)=\dfrac{\partial r}{\partial\varepsilon}(1;t_0,r_0,\zeta_2)\varepsilon,\end{cases}\tag{19}$$

where $\zeta_1,\zeta_2\in(0,\varepsilon)$. It follows from (19) that $R_1(t_0,r_0,\varepsilon),R_2(t_0,r_0,\varepsilon)$ are $C^{\nu-3}$ about (t_0,r_0) and C^∞ about ε. We now show that the functions $\frac{\partial t}{\partial\varepsilon}(1;t_0,r_0,\alpha)$, $\frac{\partial r}{\partial\varepsilon}(1;t_0,r_0,\alpha)$ are bounded on $(t_0,r_0,\alpha)\in\mathcal{D}$. First, note that a quasiperiodic function depending on parameters continuously is uniformly bounded if the parameters are within a compact set. We have $\frac{\partial}{\partial\varepsilon}R_1(t_0,r_0,\varepsilon)=\frac{\partial t}{\partial\varepsilon}(1;t_0,r_0,\varepsilon)$, it follows that the function $\frac{\partial t}{\partial\varepsilon}(1;t_0,r_0,\alpha)$ is quasiperiodic about t_0. Besides, $(r_0,\alpha)\in[a,b]\times[0,\Delta]$. Therefore, $\frac{\partial t}{\partial\varepsilon}(1;t_0,r_0,\alpha)$ is bounded on $(t_0,r_0,\alpha)\in\mathcal{D}$. Similarly, we can show that $\frac{\partial r}{\partial\varepsilon}(1;t_0,r_0,\alpha)$ is bounded on $(t_0,r_0,\alpha)\in\mathcal{D}$. Thus, by (19) we obtain that $R_i(t_0,r_0,\varepsilon)\to0$ as $\varepsilon\to0$, $i=1,2$. Note that the derivative of a quasiperiodic function is also a quasiperiodic function with the same frequency. Repeating the above procedure, we can prove inductively that $\frac{\partial^{k+l}R_i}{\partial t_0^k\partial r_0^l}(t_0,r_0,\varepsilon)\to0$ as $\varepsilon\to0$, $i=1,2$, where $0\le k+l\le\nu-3$. Therefore,

$$\|R_1(t_0,r_0,\varepsilon)\|_{\nu-3}+\|R_2(t_0,r_0,\varepsilon)\|_{\nu-3}\to0,\quad\text{as}\quad\varepsilon\to0,$$

where $\|\cdot\|_{\nu-3}$ is taken about t_0 and r_0.

5.3. Intersection property of the twist map $\mathcal{M}_0$

Since the transformation $(t_0, r_0) = (t_0, \frac{3}{2}K_0^{1/2})$ is a homeomorphism when $K > 0$, to prove that the map $\mathcal{M}$ has the intersection property, it suffices to deal with

$$\mathcal{M}_0 : (t_0, K_0) \mapsto (t_1, K_1),$$

which is the time-1-map of the system (14). We verify next that $\mathcal{M}_0$ satisfies the exact symplectic condition which implies the intersection property. To this end, we fix a strip

$$\mathcal{A} = \{(t_0, K_0) : t_0 \in \mathbb{R}, K_0 \in [4, 16]\}.$$

Now, $r_0 \in [3, 6]$.

$\mathcal{M}_0$ is the time-1-map of the Hamiltonian system (14), so it preserves the symplectic structure $dt \wedge dK$. It remain to prove that $\mathcal{M}_0$ is exact. Let $\gamma_1 : t = \xi + \alpha(\xi), K = \beta(\xi)$ be a parametric curve on the surface $\phi = 0$, which lies in the region $\mathcal{A}$, where $\xi \in [-T, T]$ for some $T > 0$. The continuous functions α and β are quasiperiodic in ξ with frequency $\omega = (\omega_1, \omega_2, \cdots, \omega_n)$, γ_2 the image of γ_1 under the map $\mathcal{M}_0$. γ_3 and γ_4 are the integral curves of the system (14), which connect the corresponding endpoints of γ_1 and γ_2. See Figure 3.

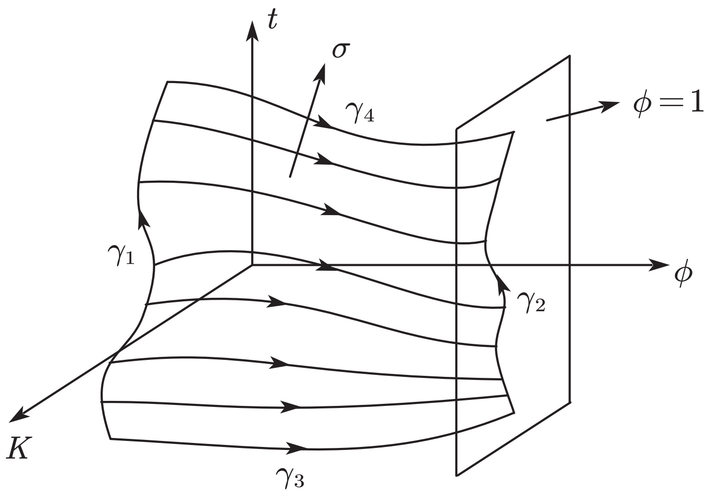

Figure 3: The definition of $\gamma_1, \gamma_2, \gamma_3$ and γ_4

For the system (14), consider the integral

$$\oint_{\gamma_1+\gamma_4-\gamma_2-\gamma_3} Kdt - Fd\phi = \iint_\sigma dK \wedge dt - dF \wedge d\phi,$$

where σ is the surface generated by piecewise of integral curves, which is enclosed by curves $\gamma_1, \gamma_2, \gamma_3$ and γ_4. Since $Kdt - Fd\phi$ is the Poincaré-Cartan integral invariant of (14), we have $\oint_{\gamma_1+\gamma_4-\gamma_2-\gamma_3} Kdt - Fd\phi = 0$. Thus,

$$\int_{\gamma_1-\gamma_2} Kdt - Fd\phi = \int_{\gamma_3-\gamma_4} Kdt - Fd\phi. \tag{20}$$

Note that $\phi = 0$ on γ_1, and $\phi = 1$ on γ_2. We have

$$\int_{\gamma_1-\gamma_2} Kdt - Fd\phi = \int_{\gamma_1-\gamma_2} Kdt, \tag{21}$$

and

$$\int_{\gamma_3-\gamma_4} Kdt - Fd\phi = \int_{\gamma_3-\gamma_4} \left(K\frac{\partial F}{\partial K} - F \right) d\phi = \frac{1}{2}\int_{\gamma_3-\gamma_4} Fd\phi. \tag{22}$$

The solutions of (14) starting from $\gamma_1 : t_0 = \xi + \alpha(\xi), K_0 = \beta(\xi)$ can be written as $t(\phi;\xi), K(\phi;\xi)$. Consider the differential equation

$$\frac{dK}{d\phi} = -K^{3/2}\frac{\partial m}{\partial t}(\phi, t(\phi;\xi), \varepsilon), \tag{23}$$

where $\phi \in [0,1], \xi \in [-T,T]$, $t(0;\xi) = \xi + \alpha(\xi), K(0;\xi) = \beta(\xi)$. Since $m(\phi,t,\varepsilon)$ is quasiperiodic about t, the vector field of (23) is bounded. Besides, the initial condition and the interval of the integral are also bounded. Therefore, $K(\phi;\xi)$ is bounded on $(\phi,\xi) \in [0,1] \times [-T,T]$ independent of T, and hence, so does to $F(\phi;\xi)$. Thus, by (22), we have

$$\lim_{T\to+\infty} \frac{1}{2T}\int_{\gamma_3-\gamma_4} Kdt - Fd\phi = 0. \tag{24}$$

Now, it follows from (20), (21) and (24) that

$$\lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} K_0 dt_0 = \lim_{T\to+\infty} \frac{1}{2T}\int_{-T}^{T} K_1 dt_1.$$

This proves that $\mathcal{M}_0$ satisfies the exact symplectic condition. By Remark 2.1, it has the intersection property.

For $\nu - 3 > 2n+1$, i.e., $\nu > 2n+4$, we have verified that the map $\mathcal{M}$ satisfies all conditions of Theorem 2.1. Thus, it turns out that $\mathcal{M}_0$ has invariant curves on $\mathcal{A}$. By the definitions of K, H, it follows that $K \in [4,16]$ and $H \in [\frac{4}{\kappa}, \frac{16}{\kappa}]$. Thanks to the form of the Hamiltonian (10), the rescaled parameter $\kappa > 0$ has no influence on the dynamics. Therefore, if the map (16) has invariant curves in the region $\mathbb{R} \times [3,6]$ for small ε, then the system (10) possesses invariant tori in $H \in [\frac{4}{\kappa}, \frac{16}{\kappa}]$ for any $\kappa > 0$. Take κ sufficiently small, there exist invariant curves in the region with arbitrary high energy, which provide barriers for any solution of the system (6), i.e., any solution of (6) is bounded.

6. Conclusion

A fundamental dynamical problem on the bouncing ball model is whether the energy of the ball can go to infinity (called Fermi acceleration) under collision. In this work, we consider the bouncing ball model in the case that the motion of the wall is quasiperiodic, and prove that when the motion of the wall is sufficiently regular and the frequency satisfies the Diophantine condition, the energy of the ball is always bounded. The processes of the proof, including the establishment of the twist map, the verifications of the intersection property and regularity conditions, are robust, which can be applied to other nonsmooth dynamical systems with quasiperiodic perturbations.

Acknowledgments

This work is supported by the National Natural Science Foundation of China (No. 12172306).

References

[1] R.A. Ibrahim. *Vibro-Impact Dynamics: Modeling, Mapping and Applications.* Springer-Verlag, Berlin, Heidelberg. 2009.

[2] M. di Bernardo, C.J. Budd, A.R. Champneys, P. Kowalczyk. *Piecewise-smooth Dynamical Systems: Theory and Applications.* Springer, London, 2007.

[3] S. Laederich, M. Levi. Invariant curves and time-dependent potentials. Ergod. Theory & Dynamical Systems, 11: 365-378, 1991.

[4] L.D. Pustyl'nikov, Existence of invariant curves for maps close to degenerate maps, and a solution of the Fermi-Ulam problem. Russ. Acad. Sci. Sb. Math., 82: 113-124, 1995.

[5] L.D. Pustyilnikov. Poincaré models, rigorous justification of the second element of thermodynamics on the basis of mechanics, and the Fermi acceleration mechanism. Russian Math. Surveys, 50: 145–189, 1995.

[6] V. Zharnitsky. Instability in Fermi-Ulam ping-pong problem. Nonlinearity, 11: 1481–1487, 1998.

[7] J.D. Simoi, D. Dolgopyat. Dynamics of some piecewise smooth Fermi-Ulam models. Chaos, 22(2): 486-488, 2012.

[8] Z. Cao, C. Grebogi, D. Li, J. Xie. The existence of Aubry-Mather sets for the Fermi-Ulam model. Qualitative Theory of Differential Equations, 20: 1–12, 2021.

[9] J. Zhou. A piecewise smooth Fermi-Ulam pingpong with potential. Ergodic Theory and Dynamical Systems, 42(5): 1847–1870, 2022.

[10] S. Marò. Coexistence of bounded and unbounded motions in a bouncing ball model. Nonlinearity, 26: 1439–1448, 2013.

[11] S. Marò. Chaotic dynamics in an impact problem. Annales Henri Poincaré, 16: 1633–1650, 2015.

[12] S. Marò. Diffusion and chaos in a bouncing ball model. Zeitschrift fur Angewandte Mathematik und Physik, 71:78, 2020.

[13] D. Dolgopyat. *Lectures on Bouncing Balls*, lecture notes for a course in Murcia, 2013; available at http://www2.math.umd.edu/ dolgop/BBNotes.pdf

[14] P.I. Holmes. The dynamics of repeated impacts with a sinusoidally vibrating table. Journal of Sound and Vibration, 84: 173–189, 1982.

[15] J. N. Mather. Non-existence of invariant circles. Ergodic Theory and Dynamical Systems, 4: 301–309, 1984.

[16] R.S. MacKay, I.C. Percival. Converse KAM: Theory and practice. Communications in Mathematical Physics, 98: 469–512, 1985.

[17] E. Sander, J. D. Meiss. Birkhoff averages and rotational invariant circles for area-preserving maps. Physica D, 411: 132569, 2020.

[18] R. Ortega. Asymmetric oscillators and twist mappings, J. London Math. Soc., 53: 325–342, 1996.

[19] M. Kunze, T. Küpper, J. You. On the application of KAM theory to discontinuous dynamical systems. Journal of Differential Equations, 139: 1–21, 1997.

[20] V. Zharnitsky. Invariant tori in Hamiltonian systems with impacts. Communications in Mathematical Physics, 211(2): 289–302, 2000.

[21] X. Zhang, Z. Cao, D. Li, C. Grebogi, J. Xie. Quasi-periodic solutions and homoclinic bifurcation in an impact inverted pendulum. Physica D, 434: 133210-1-14, 2022.

[22] X. Zhang, J. Xie, D. Li, Z. Cao, C. Grebogi. Stability analysis of the breathing circle billiard. Chaos, Solitons and Fractals, 155: 111643-1-12, 2022.

[23] V. Zharnitsky. Invariant curve theorem for quasiperiodic twist mappings and stability of motion in the Fermi-Ulam problem. Nonlinearity, 13: 1123–1136, 2000.

[24] P. Huang, X. Li, B. Liu. Invariant curves of smooth quasi-periodic mappings. Discrete Contin. Dyn. Syst., 38(1): 131-154, 2018.

[25] P. Huang, X. Li, B. Liu. Quasi-periodic solutions for an asymmetric oscillation. Nonlinearity, 29: 3006–3030, 2016.

[26] C.L. Siegel, J. Moser. *Lectures on Celestial Mechanics*. Springer-Verlag, Berlin Heidelberg, 1971.

[27] V.I. Arnold. Mathematical Methods of Classical Mechanics, 2nd edn, Springer, 1989.

[28] P. Hartman. Ordinary Differential Equations (2nd ed). Birkhèauser, 1982.