Coupled analytical solutions for circular tunnels considering rock creep effects and time-dependent anchoring forces in prestressed bolts

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Abstract

The instability of the surrounding rock in underground tunnels often induces major engineering disasters. The application of prestressed bolts is an effective reinforcement method for underground structures. In this work, we consider the interaction between the rock creep and the time-dependent anchoring forces in prestressed bolts. We derive theoretical solutions for rock creep displacement caused by excavation and for the anchoring forces of the prestressed bolts, and verify the solutions using a numerical simulation and an engineering example. First, based on the coordinated deformation between the prestressed bolts and the creeping rock mass, we establish a coupled model that takes into account the rock creep and the evolving anchoring forces. We then use the superposition principle to derive elastic solutions for rock displacement and anchoring force. Second, to reflect the effect of rock creep and time-dependent anchoring force, the Burgers model is used for the rock mass and the elastic model is used for prestressed bolts. According to the coordinated deformation between rock and bolts, we obtain the analytical solutions under the coupled actions in the Laplace space. The viscoelastic solutions for rock displacement and anchoring force considering the coupling effect are then solved by using the inverse Laplace transform. Finally, we compare the analytical solutions with numerical simulation results from FLAC^{3D} and monitoring results from an engineering example to verify the accuracy of the analytical solutions. The theoretical model provides a reference for studying tunnel reinforcement, analyzing rock creep behavior and long-term stability of the reinforcement structure.

Keywords: Tunnel; prestressed anchor bolt; rock creep; coupling effect; tunnel reinforcement; viscoelasticity

1 Introduction

Underground rocks usually exhibit time-dependent behavior, especially for soft rocks where creep can account for more than half of the total deformation (Sabitova et al., 2021; Zhu et al., 2022;). Hard rocks under high stress can also show significant time-dependent behavior or rheological characteristics. In tunneling at a great depth, excavation changes the stress state in rocks that may induce a time-dependent squeezing behavior. The instability and failure of tunnels is closely related to time, which has long been observed in many field examples and experimental studies (Forlati et al., 2001; Yang et al., 2014; Masoud et al., 2020;).-Therefore, the creep deformation of the surrounding rock in tunnels cannot be ignored. Creep has a significant impact on the safety and long-term stability of underground engineering structures.

Prestressed anchor bolt is the primary measure for reinforcing underground structures, and the long-term evolution of the anchoring force in the bolt plays an important role in the overall safety and stability of the rocks (Shreedharan and Kulatilake, 2016; Fan et al., 2018; Hu et al., 2020;). In this work, we study the coupled effect between rock creep and the evolving anchoring forces of prestressed bolts, and we aim to provide a theoretical basis for optimizing underground reinforcement design and improving engineering safety and reliability.

The use of prestressed bolts to reinforce creeping rock masses has attracted much attention of researchers around the world, and important research progress has been made. Many studies using field monitoring data have shown that the time-dependent effect exists in the loss of prestress of anchor cables in reinforcing applications (Zhu et al., 2002; Liu et al., 2012;). To study the stability of the surrounding rock of the tunnel during construction, Liu et al. (2007) explored the distribution and evolution of rock displacement and bolt axial force using field monitored data. Charlie et al. (2010) used observations in mines and revealed the loading conditions and failure modes of bolts under high-stress conditions. Zhang and Liu (2014) monitored the axial forces in reinforcing bolts over time and studied the evolution of the internal forces of different bolts. By monitoring the anchoring forces at several cross-sections of a roadway during its excavation, Zhang et al. (2015) found that the anchoring forces in the bolts changed, which affected the reinforcing capacity.

In terms of experimental studies and numerical simulation, Chen et al. (2011) used physical model experiments to study the bolt-reinforced tunnel and analyzed the evolution of rock stress and bolt axial force over time. Based on the numerical analysis using the finite difference method FLAC^{3D}, Du et al. (2016) explored the effect of prestressed bolts on the stress redistribution of the surrounding rock of the tunnel. Using the finite element method, Qian and Zhou (2018) examined the deformation and failure of rocks under high in-situ stress in the underground cavern group of the Jinping I Hydropower Station, and discussed the mechanisms for the failure and overrun of some anchor bolts. Considering the interaction between the reinforcement system and the rock, Cai et al. (2020) and Sun et al. (2021) studied the evolutions of bolt anchoring force and rock stress and displacement through numerical simulations.

In terms of theoretical research, scholars have conducted numerous studies on the viscoelastic and viscoplastic solutions of stress and displacement after excavation for rheological rock masses (Fritz, 2010; Wang et al., 2015; Thanh-Canh and Jeong-Tae, 2018; Wu et al., 2020; Gao et al., 2021;). However, few theoretical studies have considered the reinforcement effect of bolts on creeping rock masses (Oreste, 2003; Park and Kim, 2006; Nomikos et al., 2011; Do et al., 2020; Do et al., 2021), especially considering the coupling between rock creep and the anchoring forces. Based on the elasto-viscoplastic constitutive model proposed by Cristesc, Roatesi (2014) conducted theoretical time-effect analysis and numerical simulations on the reinforcing system under static water condition. Wang et al. (2015) used the Kelvin model for rock masses and the Maxwell model for anchored bolts to provide the viscoelastic solutions for circular tunnels by taking into account the coupling effect. Considering the stability and safety of tunnels, Wang et al. (2017; 2018) used the generalized Kelvin model for rock masses and the elastic model for anchored bolts to derive the viscoelastic solution of stress and displacement of non-circular tunnels and twin-tunnels. Zeng et al. (2020) analyzed the time-dependent displacement of two viscoelastic models with time and obtained analytical solutions for displacement and stress induced by continuous excavation in viscoelastic rocks.

In summary, most studies have used the generalized Kelvin model to characterize rock masses and conducted theoretical analysis for tunnel deformation. In the derivation of viscoelastic problems, the number of creep parameters greatly affects the difficulty of solving the problems. Each additional parameter further increases the difficulty. Therefore, in previous studies, the generalized Kelvin model is used to characterize the creep properties of rocks. However, the generalized Kelvin model with three parameters cannot fully characterize rock masses with steady-state creep properties (or secondary creep with a constant stain rate), especially not suitable for weak rocks with long-term deformation at constant strain rate. And in previous research, few scholars analyzed the coupling time-dependent effect between rocks and bolts. In this work, we have established a theoretical model to reveal the interaction between rock creep and the time-dependent anchoring force of prestressed anchor bolts. When the tunnel contacts with the prestressed anchor bolt, the surrounding rock of the tunnel will deform under the initial geo-stress and the anchor bolt anchoring force. Considering the compatible displacements between bolts and rocks (equal-strain assumption), the rock creep will cause the corresponding deformation of the anchor bolt. Therefore, the anchoring force of the prestressed bolt are affected, then the change of the anchoring force of the bolts will affect the creep of the rock. Based on the interaction between the creep of rock mass and the anchoring force of prestressed anchor bolt, we derived the viscoelastic theoretical solution of the coupled model, which can intuitively reflect the interaction. Finally, we compare and analyze the analytical solutions, numerical solutions, and monitoring results of rock displacement and anchoring forces of prestressed bolts to verify the fidelity of the proposed model.

2 Coupled mechanical analysis of prestressed bolts and tunnel rock masses

2.1 Coupled mechanical model for prestressed anchor bolts and tunnel rock masses

Considering the creep characteristics, the rock displacement increases over time due to the excavation effect. Once locked, the bolts and the anchored rock mass can be regarded as an integral unit bearing the forces. Therefore, the anchoring forces of the prestressed bolts are affected by the displacement of the rock mass. The time-dependent deformation of the rock mass occurs simultaneously with the deformation of the anchored bolts. As the rock mass and the prestressed bolts have deformation compatibility, the rock creep induces a corresponding deformation to the prestressed bolts, which causes the axial forces of the prestressed bolts to change accordingly. Therefore, the creep of the rock mass interacts with the anchoring forces of the prestressed bolts, based on which we establish a mechanistic model that couples the rock creep and the anchoring forces of the bolts.

Anchor bolts can be considered as springs with load proportional to the elongation (Bobet, 2006). The anchor bolt can be replaced by a pair of concentrated loads of equal magnitude and opposite direction, one acting on the anchor head and the other acting on the anchoring point. In tunnel excavation and reinforcement, the anchor bolt transmits the force through the structure of the anchoring section. Therefore, anchor bolts can be considered as springs with load proportional to the elongation. Both ends of the anchor bolt exert a concentrated force of equal magnitude and opposite direction. Fig. 1 shows the coupled mechanical model of bolts and the rock mass. σ_0 is the initial rock stress, and P is the anchoring force of the prestressed bolt. r is the excavation radius of the circular tunnel, L is the length of the tension section of the prestressed bolt, R is the length from the center of the tunnel to the end of the bolt, L_θ and L_z are the radial and circumferential spacing of the bolts, respectively.

The following assumptions are made in this work: (1) The cross-section of the tunnel is circular under plane strain condition; (2) The rock mass is a homogeneous, isotropic, and viscoelastic medium with infinitesimal deformation; (3) The in-situ stress is a hydrostatic stress σ_0 acting on the far field boundaries; (4) The excavation and reinforcement of the tunnel are carried out simultaneously; and (5) The deformation of the prestressed bolts and the deformation of the rock mass are coordinated (compatible).

Based on these assumptions, the theoretical solutions of the coupling model for prestressed bolts and the rock mass are linear elasticity with small deformation. Thus, the equations of elasticity, including the Lame equation, the Equilibrium differential equation, the Beltrami-Michel equation, and the Fourier series equation, as well as the displacement boundary conditions and stress boundary conditions, are all linear. Therefore, we use the superposition principle to solve the elastic theoretical solutions for the coupled model. Fig. 2 shows the four basic problems that are superimposed.

Because of elasticity, the four basic problems correspond to the following:

(i) a concentrated force P applied at the anchor head at the tunnel perimeter (Fig. 2a);

(ii) a concentrated force P in an infinite medium (without the tunnel) applied at a distance R (Fig. 2b);

(iii) a stress field applied at the tunnel perimeter, the magnitude of which is equal to that of problem (ii) at the same location, but in the opposite direction (Fig. 2c). A superposition of problems (ii) and (iii) will solve the problem of a concentrated force in a medium with a circular tunnel, i.e., zero normal and shear stresses at the tunnel perimeter;

(iv) a far-field stress σ_0 acting on the surrounding rock of the tunnel (Fig. 2d).

We superimpose the solutions of problem (i) to problem (iv) to obtain the coupled elastic theoretical solutions for tunnels considering rock creep and evolving anchoring forces in prestressed bolts. We assume that the excavation of the tunnel and the installation of the prestressed bolts are carried out at the same time. Once the circular tunnel is excavated, the deformation between the surrounding rock of the tunnel and the prestressed bolts is coordinated. Therefore, the displacement of the prestressed bolt is equal to the displacement of the surrounding rock, and the two ends of the prestressed bolt are subjected to concentrated forces (Bobet, 2006; Wang et al., 2015),

$$\Delta u_1^i + \Delta u_2^i + \Delta u_3^i + \Delta u_4^i = \left(\frac{4 L_Z L_\theta L}{\pi E_b d_b^2} P \right)^i = \left(\frac{P}{k} \right)^i \tag{1}$$

where $\Delta u^i = u_\rho^i |_{\rho=R} - u_\rho^i |_{\rho=r}$; $\Delta \boldsymbol{u}$ is the displacement of the rock mass; i is the number of the prestressed bolts; E_b is the elastic modulus of the rock mass; d_b is the diameter of the prestressed bolt; P is the anchoring force of the prestressed bolt; L_θ and L_z are radial and circumferential spacing of the prestressed bolts, respectively; L is the length of the tension section of the prestressed bolt; k is the stiffness of the reinforcement system, and $k=\frac{\pi E_b d_b^2}{4L_Z L_\theta L}$.

The left side of Eq. (1) represents the elongation of prestressed bolt i caused by far-field stresses and all prestressed bolts, and the right side represents the elongation of the prestressed bolt shaft due to force P. Assuming that the prestressed bolt loads are distributed by the length of the tunnel, the product of the load and the longitudinal prestressed bolt spacing is the actual load carried by the prestressed bolts. Δu_1^i, Δu_2^i, Δu_3^i, and Δu_4^i represent the deformation of the prestressed bolts in problem 1 to problem 4, respectively. The stress and displacement boundary conditions are as follows.

The initial stress boundary conditions:

$$\sigma_\rho |_{\rho=R,\theta=0} = \sigma_0 \tag{2}$$

$$\sigma_\rho |_{\rho=r,\theta=0} = \frac{P}{L_\theta L_z} \tag{3}$$

The displacement compatibility conditions:

$$u_\rho |_{\rho=R} = u_b |_{\rho=R} \tag{4}$$

$$u_\rho |_{\rho=r} = u_b |_{\rho=r} \tag{5}$$

where σ_ρ is the radial stress of the surrounding rock; σ_0 is the initial rock stress; u_ρ is the radial displacement of the surrounding rock; u_b is the displacement of the prestressed bolt; and ρ represents the radial coordinate distance; R is the length from the center of the tunnel to the end of the bolt; r is the Excavation radius of the circular tunnel.

2.2 Elastic solutions of the coupled mechanical model

By superimposing the solutions of problem (i) to problem (iv), we can obtain the coupled elastic theoretical solutions for tunnels considering rock creep effects and evolving anchoring forces in prestressed bolts.

2.2.1 Concentrated force P applied at the tunnel perimeter

The mechanical model of concentrated force P applied at the anchor head at the tunnel perimeter is shown in Fig. 3. In this section, we use the inverse solution in elasticity to solve the problem (i). First, the Airy stress function Φ that satisfies the compatibility equation is selected. Considering the compatibility equation in polar coordinates, we obtain stress components with unknown constants. Second, according to the stress boundary and displacement boundary conditions, we find the unknown constants in the Airy stress function. Finally, we obtain the displacement analytical solutions for the surrounding rock mass by geometric equations and physical equations.

For the mechanical model of concentrated force P applied at the anchor head at the tunnel perimeter, the Airy stress function with a structure similar to that for plane stress condition is chosen as (Timoshenko and Goodier, 1970)

$$\Phi = a_1 \varphi \rho \sin\theta + a_2 \ln\rho + a_3 \rho\theta \sin\theta + a_4 \rho \ln\rho \cos\theta + \frac{a_5}{\rho}\cos\theta \tag{6}$$

where Φ is the Airy stress function; a_1, a_2, a_3, a_4, and a_5 are constants obtained from boundary conditions; and θ is an auxiliary angle in polar coordinates.

From the Airy stress function, stresses can be obtained as

$$\sigma_\rho = \frac{1}{\rho}\frac{\partial\Phi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\theta^2} \tag{7}$$

$$\sigma_\theta = \frac{\partial^2\Phi}{\partial\rho^2} \tag{8}$$

$$\tau_{\rho\theta} = -\frac{\partial}{\partial\rho}\left(\frac{1}{\rho}\frac{\partial\Phi}{\partial\theta}\right) \tag{9}$$

where σ_ρ is the radial stress of the rock; σ_θ is the tangential stress of the rock; and $\tau_{\rho\theta}$ is the shear stress of the rock.

This Airy stress function satisfies equilibrium, strain compatibility, and boundary conditions. Thus, the Airy stress function Φ for a concentrated force in plane strain condition in an elastic infinite medium is written as (Bobet, 2006)

$$\Phi = \frac{P}{\pi}\varphi\rho\sin\theta + \frac{Pr}{2\pi}\ln\rho - \frac{P}{2\pi}\rho\theta\sin\theta - \frac{P(1-2\nu)}{4\pi(1-\nu)}\rho\ln\rho\cos\theta - \frac{Pr^2(3-4\nu)}{8\pi(1-\nu)}\frac{1}{\rho}\cos\theta \tag{10}$$

Combining Eq. (10) with Eqs. (7), (8), and (9) yields the stress components as

$$\sigma_\rho = \frac{P}{\pi}\left\{\begin{aligned} &\frac{2\rho\cos\theta - r\left(1+\cos^2\theta\right)}{(\rho - r\cos\theta)^2 + r^2\sin^2\theta} + \frac{\left(r^2-\rho^2\right)\sin^2\theta}{\left[(\rho - r\cos\theta)^2 + r^2\sin^2\theta\right]^2} \\ &+\frac{r}{2\rho^2} - \frac{(5-6\nu)\cos\theta}{4\rho(1-\nu)} + \frac{(3-4\nu)r^2\cos\theta}{4(1-\nu)\rho^3} \end{aligned}\right\} \tag{11}$$

$$\sigma_\theta = -\frac{P}{\pi}\left\{\begin{array}{l} \dfrac{2(r-\rho\cos\theta)r^2\sin^2\theta}{\left[(\rho-r\cos\theta)^2+r^2\sin^2\theta\right]^2}+\dfrac{r}{2\rho^2}-\dfrac{(1-2\nu)\cos\theta}{4\rho(1-\nu)} \\ +\dfrac{(3-4\nu)r^2\cos\theta}{4(1-\nu)\rho^3} \end{array}\right\} \tag{12}$$

$$\tau_{\rho\theta} = \frac{P}{\pi}\left\{\begin{array}{l} \dfrac{2r\sin\theta\cos\theta}{(\rho-r\cos\theta)^2+r^2\sin^2\theta}+\dfrac{2r^2\rho\sin^3\theta}{\left[(\rho-r\cos\theta)^2+r^2\sin^2\theta\right]^2} \\ -\dfrac{(1-2\nu)\sin\theta}{4\rho(1-\nu)}+\dfrac{(3-4\nu)r^2\sin\theta}{4(1-\nu)\rho^3} \end{array}\right\} \tag{13}$$

Based on the stress components, the displacement components are solved by geometric and physical equations, and the solutions are written as

$$\begin{aligned} \varepsilon_\rho &= \frac{\partial u_\theta}{\partial \rho} \\ \varepsilon_\theta &= \frac{u_\rho}{\rho}+\frac{1}{\rho}\frac{\partial u_\theta}{\partial \theta} \end{aligned} \tag{14}$$

$$\begin{aligned} \varepsilon_\rho &= \frac{1}{E_r}\left(\sigma_\rho-\nu\sigma_\theta\right) \\ \varepsilon_\theta &= \frac{1}{E_r}\left(\sigma_\theta-\nu\sigma_\rho\right) \end{aligned} \tag{15}$$

where u_θ is the tangential displacement of the surrounding rock, ε_ρ is the Radial strain of the surrounding rock, ε_θ is the tangential strain of the surrounding rock, E_r is the elastic modulus of the rock mass.

Therefore, combining Eqs. (11), (12), (14), and (15) yields the radial and tangential displacements of concentrated force P applied at the anchor head at the tunnel perimeter as

$$u_\rho = -\frac{P(1+\nu)}{\pi E_r}\left\{\begin{array}{l} (1-\nu)\cos\theta\ln\left[(\rho-r\cos\theta)^2+r^2\sin^2\theta\right]-\dfrac{5-12\nu+8\nu^2}{4(1-\nu)}\cos\theta\ln\rho \\ -(1-2\nu)\sin\theta\left[\tan^{-1}\left(\dfrac{\rho-r\cos\theta}{r\sin\theta}\right)-\dfrac{\pi}{2}sign(\theta)\right] \\ +\dfrac{r\rho\sin^2\theta}{(\rho-r\cos\theta)^2+r^2\sin^2\theta}-\dfrac{r}{2\rho}-\dfrac{(3-4\nu)r^2\cos\theta}{8(1-\nu)\rho^2} \end{array}\right\} \tag{16}$$

$$u_\theta = \frac{P(1+\nu)}{\pi E_r}\left\{\begin{array}{l}(1-\nu)\sin\theta\ln\left[(\rho - r\cos\theta)^2 + r^2\sin^2\theta\right] - \dfrac{5-12\nu+8\nu^2}{4(1-\nu)}\sin\theta\ln\rho \\ +(1-2\nu)\sin\theta\left[\tan^{-1}\left(\dfrac{\rho - r\cos\theta}{r\sin\theta}\right) - \dfrac{\pi}{2}sign(\theta)\right] \\ +\dfrac{(\rho^2 - r^2)\sin^2\theta}{2\left[(\rho - r\cos\theta)^2 + r^2\sin^2\theta\right]} - \dfrac{(1-2\nu)\sin\theta}{4(1-\nu)} - \dfrac{(3-4\nu)r^2\sin\theta}{8(1-\nu)\rho^2}\end{array}\right\} \tag{17}$$

2.2.2 Concentrated force *P* in an infinite medium

The concentrated force *P* is assumed to act at a point in the infinite medium, and the coordinates are shown in Fig. 4a. We derive the stress and displacement equations without gravity, which is the Kelvin's problem (Boresi et al., 2011). In the following discussion, we assume that the concentrated force *P* in an infinite medium is spatially axisymmetric.

In this section, we define a biharmonic Love displacement function to obtain the stress and displacement components under the special form of the Galerkin vector. We transform the spatially axisymmetric problem into a two-dimensional axisymmetric problem in the process, as shown in Fig. 4b. The Galerkin vector is written as

$$U = -\frac{1}{2(1-\nu)}\nabla\frac{\partial\theta_3}{\partial z} + e_3\nabla^2\theta_3 \tag{18}$$

where *U* is the function in a special form of the Galerkin vector; ν is the Poisson's ratio of the rock; θ_3 is the Love displacement function; e_3 is the direction unit vector; and ∇ is the Laplace operator, and $\nabla = \frac{\partial^2}{\partial\lambda^2} + \frac{1}{\lambda}\frac{\partial^2}{\partial\lambda} + \frac{\partial^2}{\partial z^2}$.

The stress and displacement components in polar coordinates under the special form of the Galerkin vector are expressed as

$$\sigma_\rho = -\frac{P}{8\pi(1-\nu)}\left\{\begin{array}{l}\dfrac{-2(r+L)+(7-4\nu)\rho\cos\theta - 4(1-\nu)(r+L)\cos 2\theta - \rho\cos 3\theta}{(\rho-(r+L)\cos\theta)^2 + (r+L)^2\sin^2\theta} \\ -\rho^2\dfrac{(r+L)+\rho\cos\theta - 2(r+L)\cos 2\theta - \rho\cos 3\theta + (r+L)\cos 4\theta}{\left[(\rho-(r+L)\cos\theta)^2 + (r+L)^2\sin^2\theta\right]^2}\end{array}\right\} \tag{19}$$

$$\sigma_\theta = -\frac{P}{8\pi(1-\nu)}\left\{\begin{array}{l}\dfrac{-2(r+L)-(3-4\nu)\rho\cos\theta+4(1-\nu)(r+L)\cos 2\theta+\rho\cos 3\theta}{\left(\rho-(r+L)\cos\theta\right)^2+(r+L)^2\sin^2\theta} \\ +\rho^2\dfrac{(r+L)+\rho\cos\theta-2(r+L)\cos 2\theta-\rho\cos 3\theta+(r+L)\cos 4\theta}{\left[\left(\rho-(r+L)\cos\theta\right)^2+(r+L)^2\sin^2\theta\right]^2}\end{array}\right\} \tag{20}$$

$$\tau_{\rho\theta} = -\frac{P}{8\pi(1-\nu)}\left\{\begin{array}{l}\dfrac{-(5-4\nu)\rho\sin\theta+4(1-\nu)(r+L)\sin 2\theta+\rho\sin 3\theta}{\left(\rho-(r+L)\cos\theta\right)^2+(r+L)^2\sin^2\theta} \\ +\rho^2\dfrac{3\rho\cos\theta-2(r+L)\sin 2\theta-\rho\sin 3\theta+(r+L)\sin 4\theta}{\left[\left(\rho-(r+L)\cos\theta\right)^2+(r+L)^2\sin^2\theta\right]^2}\end{array}\right\} \tag{21}$$

$$u_\rho = \frac{P(1+\nu)}{8\pi E_r(1-\nu)}\left\{\begin{array}{l}(3-4\nu)\cos\theta\ln\left[\left(\rho\cos\theta-(r+L)\right)^2+\rho^2\sin^2\theta\right] \\ +\dfrac{2(r+L)\rho\sin^2\theta}{\left(\rho\cos\theta-(r+L)\right)^2+r^2\sin^2\theta}\end{array}\right\} \tag{22}$$

$$u_\theta = -\frac{P(1+\nu)}{8\pi E_r(1-\nu)}\left\{\begin{array}{l}(3-4\nu)\sin\theta\ln\left[\left(\rho\cos\theta-(r+L)\right)^2+\rho^2\sin^2\theta\right] \\ +\dfrac{2\left(\rho-(r+L)\cos\theta\right)\rho\sin\theta}{\left(\rho\cos\theta-(r+L)\right)^2+r^2\sin^2\theta}\end{array}\right\} \tag{23}$$

2.2.3 Stress field at the tunnel perimeter

In Section 2.2.2, the elastic solutions of the stress and displacement under the concentrated force in an infinite medium are obtained. However, this is not the case, because there is an opening with a radius of r. The stress field in problem (ii) creates non-zero radial and shear stresses at the tunnel perimeter. The correct solution is obtained by applying radial and shear stresses of the same magnitude and opposite signs as problem (ii) at the tunnel perimeter, as shown in Fig. 5a. Problem (iii) is transformed into solving the uniformly distributed stress field of a circular opening in an infinite medium. An approximate solution can be found by expressing the radial and shear stresses at the tunnel perimeter in the form of Fourier series and then using the Michell's solution. The formulas are expressed as (Soutas-Little, 1999)

$$\sigma_\rho = \frac{d_0}{\rho^2}-\frac{2d_1}{\rho^3}\cos\theta-\sum_{n=2}^{\infty}\left[n(n+1)\frac{d_n}{\rho^{n+2}}+(n+2)(n-1)\frac{f_n}{\rho^n}\right]\cos n\theta \tag{24}$$

$$\sigma_\theta = -\frac{d_0}{\rho^2}+\frac{2d_1}{\rho^3}\cos\theta+\sum_{n=2}^{\infty}\left[n(n+1)\frac{d_n}{\rho^{n+2}}+(n-2)(n-1)\frac{f_n}{\rho^n}\right]\cos n\theta \tag{25}$$

$$\tau_{\rho\theta} = -\frac{2d_1}{\rho^3}\sin\theta - \sum_{n=2}^{\infty}\left[n(n+1)\frac{d_n}{\rho^{n+2}} + n(n-1)\frac{f_n}{\rho^n}\right]\sin n\theta \tag{26}$$

$$u_\rho = -\frac{(1+\nu)}{E_r}\left\{-\frac{d_0}{\rho} + \frac{d_1}{\rho^2}\cos\theta + \sum_{n=2}^{\infty}\left[n\frac{d_n}{\rho^{n+1}} + (n+2-4\nu)\frac{f_n}{\rho^{n-1}}\right]\cos n\theta\right\} \tag{27}$$

$$u_\theta = -\frac{(1+\nu)}{E_r}\left\{\frac{d_1}{\rho^2}\sin\theta + \sum_{n=2}^{\infty}\left[n\frac{d_n}{\rho^{n+1}} + (n-4+4\nu)\frac{f_n}{\rho^{n-1}}\right]\sin n\theta\right\} \tag{28}$$

Finally, the coefficients in Eqs. (24) - (28) can be found in the Fourier series terms Eq. (29). Note that the stresses and displacements in Eqs. (24) - (28) consists of a series of terms. As we expected, the more terms the more accurate the result. For practical purposes, a good approximation can be found with only a few terms as

$$\begin{aligned}
d_0 &= \frac{2Pr^2}{8\pi(1-\nu)}\frac{1}{(r^2-R^2)^3}\left[\frac{1}{R}(R^6-r^6) - 3Rr^2(R^2-r^2)\right] \\
d_1 &= -\frac{r^2}{4R}d_0 \\
d_n &= \frac{Pr^{n+2}}{8\pi(1-\nu)}\frac{1}{(r^2-R^2)^3}\left(\frac{r}{R}\right)^{n-2} \\
&\left[\begin{aligned}&-\frac{n}{n+1}\frac{1}{R^3}(R^8-r^8) + \frac{1}{n}\left(4n-4\nu-\frac{n-2}{n+1}\right)\frac{1}{R}(R^6-r^6) \\ &-\frac{3}{n}\left(2n-4\nu+\frac{n+2}{n+1}\right)R(R^4-r^4) + \frac{1}{n}\left(2n-12\nu+\frac{5n+6}{n+1}\right)R^3(R^2-r^2)\end{aligned}\right] \\
f_n &= \frac{Pr^n}{8\pi(1-\nu)}\frac{1}{(r^2-R^2)^3}\left(\frac{r}{R}\right)^{n-2} \\
&\left[\begin{aligned}&\frac{1}{R^3}(R^8-r^8) - \frac{4n-4\nu-1}{n-1}\frac{1}{R}(R^6-r^6) + \frac{3(2n-4\nu+1)}{n-1}R(R^4-r^4) \\ &-\frac{4n-12\nu+5}{n-1}R^3(R^2-r^2)\end{aligned}\right]
\end{aligned} \tag{29}$$

2.2.4 Tunnel with a far-field stress

The stress and displacement solutions of the surrounding rock in problem (iv) can be solved according to the plane strain problem (Cai, 2013), and the stress analysis diagram is shown in Fig. 5b. In addition, Fig. 5c shows a schematic diagram of the force on the representative elementary volume (REV) at a certain distance from the center of the tunnel in Fig. 5b. The stress and displacement fields under the original stress under the plane strain condition are

$$\sigma_\rho = \sigma_0\left(1-\frac{r^2}{\rho^2}\right) \tag{30}$$

$$\sigma_\theta = \sigma_0\left(1+\frac{r^2}{\rho^2}\right) \tag{31}$$

$$u_\rho = \frac{\sigma_0(1+\nu)}{E_r}\left[(1-2\nu)\rho + \frac{r^2}{\rho}\right] \tag{32}$$

2.2.5 Elastic solution of the coupled mechanical model

By superimposing the above problems (i), (ii), (iii), and (iv), we obtain the elastic solution of the displacement component u_ρ of the rock mass as

$$\begin{aligned} u_\rho = &-\frac{P(1+\nu)}{\pi E_r}\left\{\begin{array}{l} (1-\nu)\cos\theta\ln\left[(\rho-r\cos\theta)^2+r^2\sin^2\theta\right]-\dfrac{5-12\nu+8\nu^2}{4(1-\nu)}\cos\theta\ln\rho \\ -(1-2\nu)\sin\theta\left[\tan^{-1}\left(\dfrac{\rho-r\cos\theta}{r\sin\theta}-\dfrac{\pi}{2}sign(\theta)\right)\right] \\ +\dfrac{r\rho\sin^2\theta}{(\rho-r\cos\theta)^2+r^2\sin^2\theta}-\dfrac{r}{2\rho}-\dfrac{(3-4\nu)r^2\cos\theta}{8(1-\nu)\rho^2} \end{array}\right\} \\ &+\frac{P(1+\nu)}{8\pi E_r(1-\nu)}\left\{\begin{array}{l} (3-4\nu)\cos\theta\ln\left[(\rho\cos\theta-(r+L))^2+\rho^2\sin^2\theta\right] \\ +\dfrac{2(r+L)\rho\sin^2\theta}{(\rho\cos\theta-(r+L))^2+r^2\sin^2\theta} \end{array}\right\}+\frac{\sigma_0(1+\nu)}{E_r}\left[(1-2\nu)\rho+\frac{r^2}{\rho}\right] \\ &+\frac{P(1+\nu)}{8\pi(1-\nu)E_r}\left\{\left(\frac{1}{\rho}+\frac{r^2\cos\theta}{4\rho^2R}\right)\frac{2r^2}{(r^2-R^2)^3}\left[\frac{1}{R}(R^6-r^6)-3Rr^2(R^2-r^2)\right]\right\} \end{aligned} \tag{33}$$

Then, Eq. (33) is further simplified as

$$u_\rho = -\frac{P(1+\nu)}{\pi E_r(1-\nu)}\left[\begin{array}{l} f_1(1-\nu)^2 - f_2(5-12\nu+8\nu^2) \\ -f_3(1-2\nu)(1-\nu)+f_4(1-\nu) \\ -(f_5+f_6)(3-4\nu)-f_7 \end{array}\right]+\frac{\sigma_0(1+\nu)}{E_r}\left[(1-2\nu)\rho+\frac{r^2}{\rho}\right] \tag{34}$$

where $f_1 = \cos\theta\ln\left[(\rho-r\cos\theta)^2+r^2\sin^2\theta\right]$, $f_2 = \dfrac{\cos\theta}{4}\ln\rho$,

$f_3 = \sin\theta\left[\tan^{-1}\left(\dfrac{\rho-r\cos\theta}{r\sin\theta}-\dfrac{\pi}{2}sign(\theta)\right)\right]$, $f_4 = \dfrac{r\rho\sin^2\theta}{(\rho-r\cos\theta)^2+r^2\sin^2\theta}-\dfrac{r}{2\rho}$,

$$f_5 = \frac{r^2 \cos\theta}{8\rho^2}, \quad f_6 = \frac{1}{8}\cos\theta \ln\left[\left(\rho\cos\theta - (r+L)\right)^2 + \rho^2 \sin^2\theta\right],$$

$$f_7 = \frac{1}{4}\left(\begin{array}{l} \dfrac{(r+L)\rho\sin^2\theta}{\left(\rho\cos\theta-(r+L)\right)^2 + r^2\sin^2\theta} \\ +\left(\dfrac{1}{\rho}+\dfrac{r^2\cos\theta}{4\rho^2 R}\right)\dfrac{r^2}{\left(r^2-R^2\right)^3}\left[\dfrac{1}{R}\left(R^6-r^6\right)-3Rr^2\left(R^2-r^2\right)\right] \end{array} \right).$$

The radial displacements at the top and tail ends of the prestressed bolt are marked as u_r and u_R, respectively. Then the radial displacements are expressed as

$$\begin{aligned} u_r = &-\frac{P(1+\nu)}{\pi E_r}\left\{ \begin{array}{l} (1-\nu)\cos\theta\ln\left[(r-r\cos\theta)^2+r^2\sin^2\theta\right]-\dfrac{5-12\nu+8\nu^2}{4(1-\nu)}\cos\theta\ln r \\ -(1-2\nu)\sin\theta\left[\tan^{-1}\left(\dfrac{1-\cos\theta}{\sin\theta}-\dfrac{\pi}{2}sign(\theta)\right)\right] \\ +\dfrac{\sin^2\theta}{(1-\cos\theta)^2+\sin^2\theta}-\dfrac{1}{2}-\dfrac{(3-4\nu)\cos\theta}{8(1-\nu)} \end{array} \right\} \\ &+\frac{P(1+\nu)}{8\pi E_r(1-\nu)}\left\{ \begin{array}{l} (3-4\nu)\cos\theta\ln\left[\left(r\cos\theta-(r+L)\right)^2+r^2\sin^2\theta\right] \\ +\dfrac{2(r+L)r\sin^2\theta}{\left(r\cos\theta-(r+L)\right)^2+r^2\sin^2\theta} \end{array} \right\}+\frac{\sigma_0(1+\nu)}{E_r}\left[(1-2\nu)r+r\right] \\ &+\frac{P(1+\nu)}{8\pi E_r(1-\nu)}\left\{\left(\frac{1}{r}+\frac{\cos\theta}{4R}\right)\frac{2r^2}{\left(r^2-R^2\right)^3}\left[\frac{1}{R}\left(R^6-r^6\right)-3Rr^2\left(R^2-r^2\right)\right]\right\} \end{aligned} \tag{35}$$

$$\begin{aligned} u_R = &-\frac{P(1+\nu)}{\pi E_r}\left\{ \begin{array}{l} (1-\nu)\cos\theta\ln\left[(R-r\cos\theta)^2+r^2\sin^2\theta\right]-\dfrac{5-12\nu+8\nu^2}{4(1-\nu)}\cos\theta\ln R \\ -(1-2\nu)\sin\theta\left[\tan^{-1}\left(\dfrac{R-r\cos\theta}{r\sin\theta}-\dfrac{\pi}{2}sign(\theta)\right)\right] \\ +\dfrac{rR\sin^2\theta}{(R-r\cos\theta)^2+r^2\sin^2\theta}-\dfrac{r}{2R}-\dfrac{(3-4\nu)r^2\cos\theta}{8(1-\nu)R^2} \end{array} \right\} \\ &+\frac{P(1+\nu)}{8\pi E_r(1-\nu)}\left\{ \begin{array}{l} (3-4\nu)\cos\theta\ln\left[\left(R\cos\theta-(r+L)\right)^2+R^2\sin^2\theta\right] \\ +\dfrac{2(r+L)R\sin^2\theta}{\left(R\cos\theta-(r+L)\right)^2+r^2\sin^2\theta} \end{array} \right\}+\frac{\sigma_0(1+\nu)}{E_r}\left[(1-2\nu)R+\frac{r^2}{R}\right] \\ &+\frac{P(1+\nu)}{8\pi E_r(1-\nu)}\left\{\left(\frac{1}{R}+\frac{r^2\cos\theta}{4R^3}\right)\frac{2r^2}{\left(r^2-R^2\right)^3}\left[\frac{1}{R}\left(R^6-r^6\right)-3Rr^2\left(R^2-r^2\right)\right]\right\} \end{aligned} \tag{36}$$

Without considering the effects of rock gravity and other environmental stresses, we assume that the prestress of the prestressed bolt acts uniformly on the rock. Then, the anchoring force of the prestressed bolt is

$$P=\sigma_b A_b=\varepsilon_b E_b A_b=\frac{\delta_0-\Delta L}{L}E_b A_b=P_0-\frac{\Delta L}{L}E_b A_b \tag{37}$$

where σ_b is the stress of prestressed anchor bolt; P_0 is the initial prestress of anchor bolt; ε_b is the total strain of the prestressed bolt; E_b is the elastic modulus of the prestressed bolt; A_b is the cross-sectional area of the prestressed bolt; δ_0 is the pre-tension length of the prestressed bolt; and ΔL is the deformation of the prestressed bolt during coordinated deformation.

In addition, because of the coordinated deformation between the prestressed bolt and the rock mass, the axial deformation ΔL of the anchor bolt is equal to the deformation of the rock mass Δu_ρ as

$$\Delta L=\Delta u_\rho=u_R-u_r \tag{38}$$

where Δu_ρ is the deformation of the rock mass.

After calculating Eq. (38), we obtain

$$\Delta L=\frac{P(1+\nu)}{\pi E_r(1-\nu)}\begin{bmatrix} g_1(1-\nu)^2-g_2\left(5-12\nu+8\nu^2\right) \\ -g_3(1-2\nu)(1-\nu)+g_4(1-\nu) \\ -g_5(1-\nu)-g_6(3-4\nu) \\ -g_7(3-4\nu)-g_8 \end{bmatrix}-\frac{\sigma_0(1+\nu)}{E_r}\left[(1-2\nu)(r-R)+r-\frac{r^2}{R}\right] \tag{39}$$

Subsequently, combining Eq. (38) with Eq. (37), the resultant force on the prestressed bolt unit can be expressed as

$$P=P_0-\frac{\Delta L}{L}E_b A_b=P_0-\frac{\Delta u_\rho}{L}E_b A_b \tag{40}$$

Therefore, Δu_ρ can be easily obtained by combining Eqs. (35) - (40). Finally, the elastic solution of the radial deformation Δu_ρ of the rock mass under the coupled effect is obtained as

$$\Delta u_\rho=\frac{P_0 L(1+\nu)\begin{bmatrix} g_1(1-\nu)^2-g_2\left(5-12\nu+8\nu^2\right) \\ -g_3(1-2\nu)(1-\nu)+g_4(1-\nu) \\ -g_5(1-\nu)-g_6(3-4\nu) \\ -g_7(3-4\nu)-g_8 \end{bmatrix}-\pi L\sigma_0\left(1-\nu^2\right)\left[(1-2\nu)(r-R)+r-\frac{r^2}{R}\right]}{\pi L E_r(1-\nu)+E_b A_b\dfrac{(1+\nu)}{(1-\nu)}\begin{bmatrix} g_1(1-\nu)^2-g_2\left(5-12\nu+8\nu^2\right) \\ -g_3(1-2\nu)(1-\nu)+g_4(1-\nu)-g_5(1-\nu) \\ -g_6(3-4\nu)-g_7(3-4\nu)-g_8 \end{bmatrix}} \tag{41}$$

Based on the deformation coordination between the bolt and the rock mass, the elastic solution of the anchoring force T of the bolt under the coupled effect is obtained as

$$\begin{aligned} T &= k\Delta u_\rho A_m = k\Delta u_\rho \cdot \frac{\pi d_g^2}{4} \\ &= P_0 - \frac{\pi d_g^2 E_b A_b}{4 L_Z L_\theta} \left(\frac{P_0\left(1+\nu\right)\begin{bmatrix} g_1\left(1-\nu\right)^2 - g_2\left(5-12\nu+8\nu^2\right) \\ -g_3\left(1-2\nu\right)\left(1-\nu\right)+g_4\left(1-\nu\right) \\ -g_5\left(1-\nu\right)-g_6\left(3-4\nu\right) \\ -g_7\left(3-4\nu\right)-g_8 \end{bmatrix} - \pi\sigma_0\left(1-\nu^2\right)\left[\left(1-2\nu\right)\left(r-R\right)+r-\frac{r^2}{R}\right]}{\pi L E_r\left(1-\nu\right)+E_b A_b \frac{\left(1+\nu\right)}{\left(1-\nu\right)}\begin{bmatrix} g_1\left(1-\nu\right)^2 - g_2\left(5-12\nu+8\nu^2\right) \\ -g_3\left(1-2\nu\right)\left(1-\nu\right)+g_4\left(1-\nu\right)-g_5\left(1-\nu\right) \\ -g_6\left(3-4\nu\right)-g_7\left(3-4\nu\right)-g_8 \end{bmatrix}} \right) \end{aligned} \tag{42}$$

where A_m is the cross-sectional area of the anchoring section of the bolt, and $A_m = \frac{\pi d_g^2}{4}$; d_g is the grouting diameter of the anchor rod bonding surface; $g_1 = \cos\theta \ln\frac{\left[\left(r-r\cos\theta\right)^2+r^2\sin^2\theta\right]}{\left[\left(R-r\cos\theta\right)^2+r^2\sin^2\theta\right]}$, $g_2 = \frac{\cos\theta}{4}\ln\frac{r}{R}$,

$$g_3 = \sin\theta\left[\left(\tan^{-1}\left(\left(\frac{1-\cos\theta}{\sin\theta}\right)-\frac{\pi}{2}sign\left(\theta\right)\right)\right)-\left(\tan^{-1}\left(\left(\frac{R-r\cos\theta}{r\sin\theta}\right)-\frac{\pi}{2}sign\left(\theta\right)\right)\right)\right],$$

$$g_4 = \left[\frac{\sin^2\theta}{\left(1-\cos\theta\right)^2+\sin^2\theta}-\frac{rR\sin^2\theta}{\left(R-r\cos\theta\right)^2+r^2\sin^2\theta}\right],\quad g_5 = \frac{\left(R-r\right)}{2R},$$

$$g_6 = \frac{\left(R^2-r^2\right)\cos\theta}{8R^2},\quad g_7 = \frac{\cos\theta}{8}\ln\frac{\left[\left(r\cos\theta-R\right)^2+r^2\sin^2\theta\right]}{\left[\left(R\cos\theta-R\right)^2+R^2\sin^2\theta\right]},$$

$$g_8 = \frac{1}{4}\left(\begin{aligned} &\frac{Rr\sin^2\theta}{\left(r\cos\theta-R\right)^2+r^2\sin^2\theta}-\frac{R^2\sin^2\theta}{\left(R\cos\theta-R\right)^2+r^2\sin^2\theta} \\ &+\left(\frac{1}{r}+\frac{\cos\theta}{4R}\right)\frac{2r^2}{\left(r^2-R^2\right)^3}\left[\frac{1}{R}\left(R^6-r^6\right)-3Rr^2\left(R^2-r^2\right)\right] \\ &-\left(\frac{1}{R}+\frac{r^2\cos\theta}{4R^3}\right)\frac{2r^2}{\left(r^2-R^2\right)^3}\left[\frac{1}{R}\left(R^6-r^6\right)-3Rr^2\left(R^2-r^2\right)\right] \end{aligned}\right).$$

3 Viscoelastic analytical solutions of the coupled model

In the small deformation range of rock materials, the viscoelastic problem and the elastic problem differ only in constitutive relations; and the equilibrium equations, geometric equations, and boundary conditions are exactly the same. According to the principles of elasticity-viscoelasticity, the viscoelastic problems can be solved through the following procedure. First, elastic parameters in the elastic solution of the theoretical model are replaced by viscoelastic parameters. Then, the operator function of the creep model is substituted to obtain the analytical solutions of the problem in the Laplace space. Finally, the inverse Laplace transform is applied to the analytical solutions to obtain the viscoelastic solution of the problem (Mogilevskaya, 2018).

3.1 Selection and definition of the coupled model

In this work, the linear elastic model is used to describe the mechanical behavior of the bolt, and the Burgers model is used to describe the mechanical behavior of the rock. The component diagrams of the two models are shown in Fig. 6.

3.1.1 Selection of creep model for the prestressed bolt and definition of operator functions

The prestressed bolt is described by the elastic model, whose constitutive equation satisfies

$$\sigma = E_b \varepsilon \tag{43}$$

where σ and ε are total stress and total strain, respectively.

Because the axial stiffness of the prestressed bolt is much larger than the tangential stiffness, the prestressed bolt can be regarded as an ideal one-dimensional (1D) elastic material. The axial stress is represented by σ_b and the axial strain is represented by ε_b, and the generalized one-dimensional elastic constitutive equation of the prestressed bolt can then be written as

$$\sigma_b = \frac{Q_b(D)}{P_b(D)} \varepsilon_b \tag{44}$$

$$D = \frac{\partial}{\partial t}, P_b(D) = D = \frac{\partial}{\partial t}, P_b(D) = \sum_{k=0}^{m} p_k \frac{\partial^k}{\partial t^k}, Q_b(D) = \sum_{k=0}^{m} q_k \frac{\partial^k}{\partial t^k} \tag{45}$$

where D is the differential operator; $P_b(D)$ and $Q_b(D)$ are operator functions for the 1D constitutive equation of the prestressed bolt in Eq. (44); and p_k and q_k are constants of the prestressed bolt material.

Hence, the parameter transformation in the Laplace domain is given by

$$E_b(s) = \frac{\bar{Q}_b(s)}{\bar{P}_b(s)} \tag{46}$$

where s is the Laplace variable; $\bar{P}_b(s)$ and $\bar{Q}_b(s)$ are operator functions of the 1D constitutive equation of the prestressed bolt after the Laplace transform. Specifically, the operator functions are written as

$$\begin{aligned} \bar{P}_b(s) &= 1 \\ \bar{Q}_b(s) &= E_b \end{aligned} \tag{47}$$

3.1.2 Selection of creep model for the rock mass and definition of operator functions

The Burgers model is used to describe the creep properties of the rock mass. The one-dimensional constitutive equation of the rock mass satisfies

$$\frac{\eta_{1r}\eta_{2r}}{E_{1r}E_{2r}}\ddot{\sigma} + \left(\frac{\eta_{1r}}{E_{1r}} + \frac{\eta_{1r}+\eta_{2r}}{E_{2r}}\right)\dot{\sigma} + \sigma = \frac{\eta_{1r}\eta_{2r}}{E_{2r}}\ddot{\varepsilon} + \eta_{1r}\dot{\varepsilon} \tag{48}$$

where $\dot{\sigma}$ and $\dot{\varepsilon}$ are the derivatives of σ and ε, respectively, and $\ddot{\sigma}$ and $\ddot{\varepsilon}$ are the second derivatives of σ and ε, respectively. E_{1r} and E_{2r} are the visco-elastic parameters.

It is well known that rock mechanics and engineering problems are often three-dimensional. The rock mass in the tunnel should be considered as three-dimensional viscoelastic material, therefore the one-dimensional constitutive equation should be expanded to three-dimensional. From the perspective of elastic theory, the one-dimensional form of the elastic constitutive relationship is $\sigma = E_r\varepsilon$, and the three-dimensional tensor form is expressed as

$$S_{ij} = 2Ge_{ij}, \sigma_{ij} = 3K\varepsilon_{ij} \tag{49}$$

where G and K are the bulk modulus and shear modulus, respectively. S_{ij} and e_{ij} are the deviatoric stress and strain tensors, respectively. σ_{ij} and ε_{ij} are the stress tensor and strain tensor, respectively.

The three-dimensional constitutive models for elastic and viscoelastic materials can be expressed as

$$S_{ij} = 2Ge_{ij} = 2\frac{Q'(D)}{P'(D)}e_{ij}, \sigma_{ij} = 3K\varepsilon_{ij} = 3\frac{Q''(D)}{P''(D)}\varepsilon_{ij} \tag{50}$$

where $P'(D), Q'(D), P''(D), Q''(D)$ are the operator functions of the viscoelastic constitutive model.

Therefore, the parameter transformation in the Laplace domain is given by

$$G(s) = \frac{\bar{Q}'(s)}{\bar{P}'(s)}, K(s) = \frac{\bar{Q}''(s)}{\bar{P}''(s)} \tag{51}$$

where $\bar{P}'(s), \bar{Q}'(s), \bar{P}''(s), \bar{Q}''(s)$ are the operator functions of the viscoelastic constitutive model after the Laplace transformation. In addition, the operator functions of the Burgers model are

$$
\begin{aligned}
&\bar{P}'(s)=1+\left(\frac{\eta_{1r}+\eta_{2r}}{G_{1r}}+\frac{\eta_{2r}}{G_{2r}}\right)s+\frac{\eta_{1r}\eta_{2r}}{G_{1r}G_{2r}}s^2=1+p_{1r}s+p_{2r}s^2\\
&\bar{Q}'(s)=\eta_{2r}s+\frac{\eta_{1r}\eta_{2r}}{G_{1r}}s^2=q_{1r}s+q_{2r}s^2\\
&\bar{P}''(s)=1\\
&\bar{Q}''(s)=K
\end{aligned}
\tag{52}
$$

where η_{1r} and η_{2r} are the viscosity coefficients of the rock mass; G_{1r} and G_{2r} are the elastic shear modulus and visco-elastic shear modulus of the rock mass, respectively.

3.2 General viscoelastic solution of the coupled model

In the three-dimensional space, the relationships between elastic modulus E, Poisson ratio μ, elastic shear modulus G, and bulk modulus K are

$$E=\frac{9GK}{3K+G} \tag{53}$$

$$\mu=\frac{3K-2G}{2(3K+G)} \tag{54}$$

Substituting the expressions of E and μ into Eqs. (41) and (42), the spatial solutions of the radial deformation Δu_ρ of the rock mass and the anchoring force T of the bolt can then be obtained as

$$
\Delta u_\rho=\frac{P_0L\left(1+\frac{3K-2G}{2(3K+G)}\right)\left[\begin{array}{l}g_1\left(1-\frac{3K-2G}{2(3K+G)}\right)^2-g_2\left(5-12\frac{3K-2G}{2(3K+G)}+8\left(\frac{3K-2G}{2(3K+G)}\right)^2\right)\\-g_3\left(1-2\frac{3K-2G}{2(3K+G)}\right)\left(1-\frac{3K-2G}{2(3K+G)}\right)+g_4\left(1-\frac{3K-2G}{2(3K+G)}\right)\\-g_5\left(1-\frac{3K-2G}{2(3K+G)}\right)-g_6\left(3-4\frac{3K-2G}{2(3K+G)}\right)\\-g_7\left(3-4\frac{3K-2G}{2(3K+G)}\right)-g_8\end{array}\right]-\pi L\sigma_0\left(1-\left(\frac{3K-2G}{2(3K+G)}\right)^2\right)\left[\left(1-2\frac{3K-2G}{2(3K+G)}\right)(r-R)+r-\frac{r^2}{R}\right]}{\pi L\frac{9GK}{(3K+G)}\left(1-\frac{3K-2G}{2(3K+G)}\right)+E_bA_b\frac{\left(1+\frac{3K-2G}{2(3K+G)}\right)}{\left(1-\frac{3K-2G}{2(3K+G)}\right)}\left[\begin{array}{l}g_1\left(1-\frac{3K-2G}{2(3K+G)}\right)^2-g_2\left(5-12\frac{3K-2G}{2(3K+G)}+8\left(\frac{3K-2G}{2(3K+G)}\right)^2\right)\\-g_3\left(1-2\frac{3K-2G}{2(3K+G)}\right)\left(1-\frac{3K-2G}{2(3K+G)}\right)+g_4\left(1-\frac{3K-2G}{2(3K+G)}\right)\\-g_5\left(1-\frac{3K-2G}{2(3K+G)}\right)-g_6\left(3-4\frac{3K-2G}{2(3K+G)}\right)\\-g_7\left(3-4\frac{3K-2G}{2(3K+G)}\right)-g_8\end{array}\right]}
\tag{55}
$$

$$T = P_0 - \frac{\pi d_g^2 E_b A_b}{4 L_z L_\theta}\left(\frac{P_0 L\left(1+\frac{3K-2G}{2(3K+G)}\right)\left[\begin{array}{l} g_1\left(1-\frac{3K-2G}{2(3K+G)}\right)^2 - g_2\left(5-12\frac{3K-2G}{2(3K+G)}+8\left(\frac{3K-2G}{2(3K+G)}\right)^2\right) \\ -g_3\left(1-2\frac{3K-2G}{2(3K+G)}\right)\left(1-\frac{3K-2G}{2(3K+G)}\right)+g_4\left(1-\frac{3K-2G}{2(3K+G)}\right) \\ -g_5\left(1-\frac{3K-2G}{2(3K+G)}\right)-g_6\left(3-4\frac{3K-2G}{2(3K+G)}\right) \\ -g_7\left(3-4\frac{3K-2G}{2(3K+G)}\right)-g_8 \end{array}\right] - \pi L \sigma_0\left(1-\left(\frac{3K-2G}{2(3K+G)}\right)^2\right)\left[\left(1-2\frac{3K-2G}{2(3K+G)}\right)(r-R)+r-\frac{r^2}{R}\right]}{\pi L\frac{9GK}{(3K+G)}\left(1-\frac{3K-2G}{2(3K+G)}\right)+E_b A_b \frac{\left(1+\frac{3K-2G}{2(3K+G)}\right)}{\left(1-\frac{3K-2G}{2(3K+G)}\right)}\left[\begin{array}{l} g_1\left(1-\frac{3K-2G}{2(3K+G)}\right)^2 - g_2\left(5-12\frac{3K-2G}{2(3K+G)}+8\left(\frac{3K-2G}{2(3K+G)}\right)^2\right) \\ -g_3\left(1-2\frac{3K-2G}{2(3K+G)}\right)\left(1-\frac{3K-2G}{2(3K+G)}\right)+g_4\left(1-\frac{3K-2G}{2(3K+G)}\right) \\ -g_5\left(1-\frac{3K-2G}{2(3K+G)}\right)-g_6\left(3-4\frac{3K-2G}{2(3K+G)}\right) \\ -g_7\left(3-4\frac{3K-2G}{2(3K+G)}\right)-g_8 \end{array}\right]}\right) \quad (56)$$

Next, the equations for the spatial solutions, (i.e., Eqs. (55) and (56)), are solved using the Laplace transform as follows. In the viscoelastic case, P_0 is replaced by its Laplace transform $\frac{P_0}{s}$, G is replaced by $\frac{\bar{Q}'(s)}{\bar{P}'(s)}$, K is replaced by $\frac{\bar{Q}''(s)}{\bar{P}''(s)}$, and E_b is replaced by $\frac{\bar{Q}_b(s)}{\bar{P}_b(s)}$, the general solutions of the radial deformation Δu_ρ of the rock mass and the anchoring force *T* of the bolt in the Laplace domain are obtained.

3.3 Viscoelastic analytical solution of the coupled model

Based on the theoretical models, the Burgers model is used for the rock mass, and the elastic model is used for the prestressed bolt. Substituting the differential operators into the general solutions of Δu_ρ and *T* yields the analytical solutions of the deformation $\Delta \bar{u}_\rho$ of the rock mass and the anchoring force $\bar{T}(s)$ of the prestressed bolt in the Laplace domain. Here, we first combine Eqs. (47) and (52) with the general solutions of the rock radial deformation Δu_ρ and the anchoring force *T*, and then the corresponding expressions of $\Delta \bar{u}_\rho$ and $\bar{T}(s)$ can be obtained as

$$\Delta \bar{u}_\rho(s) = \frac{h_1 s^8 + h_2 s^7 + h_3 s^6 + h_4 s^5 + h_5 s^4 + h_6 s^3 + h_7 s^2 + h_8 s + h_9}{s\left(h_{10} s^8 + h_{11} s^7 + h_{12} s^6 + h_{13} s^5 + h_{14} s^4 + h_{15} s^3 + h_{16} s^2 + h_{17} s + h_{18}\right)} \quad (57)$$

$$\bar{T}(s) = \frac{j_1 s^8 + j_2 s^7 + j_3 s^6 + j_4 s^5 + j_5 s^4 + j_6 s^3 + j_7 s^2 + j_8 s + j_9}{s\left(j_{10} s^8 + j_{11} s^7 + j_{12} s^6 + j_{13} s^5 + j_{14} s^4 + j_{15} s^3 + j_{16} s^2 + j_{17} s + j_{18}\right)} \quad (58)$$

where $h_1 \sim h_{18}$, $j_1 \sim j_{18}$ can be found in Appendix A.

Then, Eq. (57) is further simplified as

$$\Delta\bar{u}_\rho(s) = \frac{h_1 s^8 + h_2 s^7 + h_3 s^6 + h_4 s^5 + h_5 s^4 + h_6 s^3 + h_7 s^2 + h_8 s + h_9}{s\left(h_{10} s^8 + h_{11} s^7 + h_{12} s^6 + h_{13} s^5 + h_{14} s^4 + h_{15} s^3 + h_{16} s^2 + h_{17} s + h_{18}\right)}$$
$$= \frac{r_1}{(s-s_1)} + \frac{r_2}{(s-s_2)} + \frac{r_3}{(s-s_3)} + \frac{r_4}{(s-s_4)} + \frac{r_5}{(s-s_5)} + \frac{r_6}{(s-s_6)} + \frac{r_7}{(s-s_7)} + \frac{r_8}{(s-s_8)} + \frac{r_9}{(s-s_9)} \quad (59)$$

In Eq. (59), s_1 ~ s_9 are the roots of the characteristic equation $h_{10}s^9 + h_{11}s^8 + h_{12}s^7 + h_{13}s^6 + h_{14}s^5 + h_{15}s^4 + h_{16}s^3 + h_{17}s^2 + h_{18}s = 0$. r_1 ~ r_9 are coefficients to be determined, which are called residues of Eq. (59) at s_1 ~ s_9, and can be calculated according to the following formula

$$r_i = \lim_{s\to s_i}(s-s_i)\Delta\bar{u}_\rho(s) \quad (60)$$

Finally, these analytical solutions can be inverted back into the time domain using the inverse Laplace transform. Hence, by using the inverse Laplace transform on Eq. (59), the viscoelastic analytical solutions of the rock radial deformation Δu_ρ under the coupled effect can be obtained as

$$\Delta u_\rho(t) = r_1 e^{s_1 t} + r_2 e^{s_2 t} + r_3 e^{s_3 t} + r_4 e^{s_4 t} + r_5 e^{s_5 t} + r_6 e^{s_6 t} + r_7 e^{s_7 t} + r_8 e^{s_8 t} + r_9 e^{s_9 t} \quad (61)$$

where s is the Laplace variable; t is the time.

Subsequently, Eq. (58) is further simplified as,

$$\bar{T}(s) = \frac{j_1 s^8 + j_2 s^7 + j_3 s^6 + j_4 s^5 + j_5 s^4 + j_6 s^3 + j_7 s^2 + j_8 s + j_9}{s\left(j_{10} s^8 + j_{11} s^7 + j_{12} s^6 + j_{13} s^5 + j_{14} s^4 + j_{15} s^3 + j_{16} s^2 + j_{17} s + j_{18}\right)}$$
$$= \frac{r_1'}{(s-s_1)} + \frac{r_2'}{(s-s_2)} + \frac{r_3'}{(s-s_3)} + \frac{r_4'}{(s-s_4)} + \frac{r_5'}{(s-s_5)} + \frac{r_6'}{(s-s_6)} + \frac{r_7'}{(s-s_7)} + \frac{r_8'}{(s-s_8)} \quad (62)$$

In Eq. (62), s_1 - s_9 are the roots of the characteristic equation $j_{10}s^9 + j_{11}s^8 + j_{12}s^7 + j_{13}s^6 + j_{14}s^5 + j_{15}s^4 + j_{16}s^3 + j_{17}s^2 + j_{18}s = 0$. r_1' ~ r_9' are coefficients to be determined, and are the residues of Eq. (62) at s_1 ~ s_9, and can be calculated according to the following formula

$$r_i' = \lim_{s\to s_i}(s-s_i)\bar{T}(s) \quad (63)$$

Finally, we perform the inverse Laplace transform on Eq. (62) to obtain the viscoelastic analytical solution of the anchoring force T of the bolt in the time domain under the coupled effect as

$$T(t) = r_1' e^{s_1 t} + r_2' e^{s_2 t} + r_3' e^{s_3 t} + r_4' e^{s_4 t} + r_5' e^{s_5 t} + r_6' e^{s_6 t} + r_7' e^{s_7 t} + r_8' e^{s_8 t} + r_9' e^{s_9 t} \quad (64)$$

4 Verification of the theoretical model

In this section, we use the finite difference software FLAC^{3D} to verify the fidelity of the analytical solutions proposed in this work.

4.1 Establishment of the numerical model

FLAC^{3D} is used to numerically simulate the coupled effect of the rock creep and the evolving anchoring force. Since the theoretical model is symmetrical, we simplify the numerical model to a quarter circular tunnel. The mesh of the model is shown in Fig. 7. The size of the numerical model, divided into 1230 grids and 2750 nodes, is 40m in length (X direction), 0.5m in width (Y direction), and 40m in height (Z direction) with a tunnel radius of 4m. In addition, the length of the anchor bolt in the numerical model is 5m, and the length of the tension segment is 4.5m. The prestressed bolts are set at different anchoring angles of 0°, 15°, 30°, 45°, 60°, 75°, and 90°. To comply with the plane strain assumption, the displacements of the nodes in the direction perpendicular to the plane are fixed, and that along the plane direction are variable. The boundaries are fixed and set far enough to eliminate the boundary condition effects.

In numerical simulations, the rock mass is described by the Burgers model, and the prestressed bolt is described by the Elastic model. To make the numerical simulation consistent with the analytical solution, it is necessary to ensure that the anchoring force of the bolt changes with the creep effect of the rock at each time step. Accordingly, the Fish function is used to calculate the deformation of the rock at each time step, and the anchoring force of the bolt in the current creep state is calculated by using Eq. (37) to meet the requirement of the coupled creep effect. Rock mechanical parameters used in the numerical model are listed in Table 1. Parameters of the prestressed bolt used in the numerical model are provided in Table 2.

4.2 Comparison of analytical solutions and simulation results

Fig. 8 shows the tunnel total displacement nephogram after FLAC^{3D} calculation. Fig. 9 shows the comparison of the analytical and numerical solutions of the rock radial displacement and the anchoring force of the bolt in the coupled model, the monitoring point is at the anchor head of the anchor bolt, as shown in Fig. 7. The simulation results show that the radial displacement $\Delta\mu$ of the rock and the anchoring force T of the bolt increase with time.

Fig. 9a shows the increasing rock radial displacement with time. Clearly, both numerical simulation results and analytical solutions of the rock radial displacement exhibit time dependence. Specifically, the rock mass deforms along the radial direction of the bolt towards the center of the tunnel. The radial displacement increases with time and eventually converges to a stable value. Right after the tunnel excavation and reinforcement, the rock mass is unstable, the initial deformation rate is large, and then it gradually decreases. Fig. 9b shows the evolution of bolt anchoring force T with time. Similarly, both the numerical simulation results and analytical solutions show time dependence. Specifically, the anchoring force of the bolt gradually increases over time and eventually converges to a stable value.

When considering the coupling effect, the evolutions of the rock radial displacement and the bolt anchoring force in the numerical simulation are similar to that of the analytical solutions. In addition, the magnitudes associated with the two solutions are also very consistent, suggesting the fidelity of the solution procedure and the results of the coupled model. In general, the final rock radial displacement from the numerical simulation is slightly larger than that from the analytical solution. Similarly, the anchoring force from the numerical simulation is also slightly larger than that from the analytical solution. The main reasons for the difference between the analytical solution and the numerical simulation are related to the size and the geometric distribution of the numerical model, and the simplification of the analytical model of the rock mass.

5 Engineering application

5.1 Project overview

Qingdao Metro Line 6 is located in the Huangdao District, Qingdao City. The project is a typical shallow-buried large cross-section tunnel with a main section of 27.5 m. Most of the tunnel is in a slightly weathered granite formation, which is relatively stable. Its engineering geological structure of the tunnel project is shown in Fig. 10. However, some sections of the tunnel pass through multiple fractures and fractured zones, and local joints are well developed, leading to the deformation of the surrounding rock and large surface subsidence. The long-term stability of the surrounding rock is the key to the safe operation of the tunnel. If the reinforcing system has safety hazards during construction, it is easy to cause large rock deformation, fracture of the reinforcing components, and even large-scale landslides in subway stations. Therefore, to ensure the long-term safety of the Qingdao Metro Line 6 project, we carried out long-term monitoring on the anchoring force of the bolts and the displacement of the rock. The monitoring data are used to analyze the evolutions of rock creep and anchoring force of prestressed bolts in the tunnel. The location and field application of prestressed bolts in the tunnel of the Qingdao Metro Line 6 are shown in Figs. 11 and 12 (Wang et al., 2022).

5.2 Model validation

Taking the Qingdao Metro Line 6 project as an example, we compare the measured displacement of the rock mass and the anchoring force of the bolts with the analytical solutions to verify the applicability of the theoretical model.

Constant resistance and energy absorption bolts are used for reinforcement in this area. The parameters of the rock mass and prestressed bolts are given in Table 3, and the comparison results are shown in Fig. 13. The bolt prestress design value for this area is 130 kN, and the design length is 2.4 m. The diameter of the bolt is 18mm, and the tensile strength is 906 MPa.

As can be seen from Fig. 13, the theoretical model results are consistent with the field monitoring data, indicating the validity of the theoretical model. Specifically, the tunnel vault settled. Affected by rock properties and in-situ stress, the initial rate of the displacement is relatively large and tends to stabilize after 600 hours. The anchoring force of the bolt increases, and the prestress change rates in the analytical solution result and the monitoring data are 2.29% and 2.12%, respectively. Therefore, the theoretical model can be used to analyze the interaction between the rock mass and prestressed anchor bolts.

6 Discussion

6.1 Application analysis

Based on the study of the coupling effect between the rock creep and the changing anchoring force of the prestressed bolts, in this section we discuss and analyze the rock radial deformation with and without the coupling effect to gain a comprehensive understanding of the reinforcing effect of the theoretical model.

Fig. 14 shows the creep decomposing into elementary strains in different stages. If the rock creep strain reaches stability after a long enough time, it is called a stable creep. If the creep keeps increasing and cannot get stabilized, it is then called an unstable creep. Most hard rocks exhibit stable creep behavior, which can be described by the generalized Kelvin model. The Burgers model is often used to describe unstable creep for soft rocks.

In order to verify the applicability of the new theoretical model, Qingdao Metro Line 6 Project in China is taken as the research background. The actual engineering parameters are brought into the single Burgers model, the generalized Kelvin coupling model and the Burgers coupling model for calculation, and are compared with the detection data of rock mass deformation in the actual engineering. Fig. 15 shows the rock radial displacement as a function of time, with and without the coupling effect. From the analysis of Fig. 15, we can get the following:

(i) The prestressed bolts are applied to the rock after excavation, but the coupling effect between the changing anchoring force and the rock creep is not considered (red line). When the rock exhibits unstable creep behavior, the radial displacement continues to increase, which cannot fully reflect the reinforcing effect of the anchor bolts.

(ii) When the Burgers model is applied to the rock and the coupling effect is considered (black line), although the rock exhibits unstable creep behavior, the radial displacement after excavation is eventually stabilized with prestressed bolts, which can well reflect the reinforcing effect of the prestressed bolts.

(iii) When the generalized Kelvin model is used for the rock and the coupling effect is considered (blue line), the final radial displacement of the rock also reaches a stable value. In addition, the displacement is smaller than that of the Burgers model (black line), which is related to the stable creep properties of the rock and the reinforcing effect of the prestressed bolts.

(iv) The engineering observed data of tunnel surrounding rock displacement after anchoring are plotted into curves (green line) and compared with Burgers model (black line) and generalized Kelvin model (blue line) considering coupling effects, as shown in Fig. 15. It can be seen that the theoretical derivation results of the coupling model discussed in this work have certain applicability to actual projects. In addition, comparing the engineering observed data curve (green line) with the separate Burgers model (red line) in the Fig.15, it is obvious that the displacement of the surrounding rock of the tunnel is obviously constrained after the anchor bolt is applied to the rock mass, and no longer increases infinitely with time, which confirms the supporting role of the anchor bolt in the tunnel engineering.

It is worth noting that after considering the coupling effect between the rock creep and the changing anchoring force of prestressed bolts, the rock with unstable creep properties also shows stable creep behavior (black line) after being reinforced with prestressed bolts. This phenomenon fully reflects the reinforcing effect of the prestressed bolts and is consistent with the observations from engineering practices.

6.2 Research prospect

In this work, a series of assumptions are set, so the following problems still need to be solved:

The theoretical model of coupling mechanics studied in this work is applicable to actual engineering, but the model has been simplified to some extent before researching, ignoring some influencing factors, which affects the calculation accuracy. Therefore, in the follow-up study, the influence of the hydrogeological conditions and the internal structural characteristics of the surrounding rock on the time-dependent displacement of the tunnel surrounding rock will be considered to further improve the accuracy of the calculation model. In addition, the theoretical geometric model is assumed to be circular cross-section in this work, and horseshoe shaped and rectangular tunnel sections are also commonly used in engineering. As the theoretical calculation of horseshoe shaped and rectangular tunnels is more difficult, it is one of the contents that we need to study in the future.

7 Conclusions

In this work, we developed coupled analytical solutions for the rock radial displacement and the anchoring forces of prestressed bolts that consider the rock creep and the evolving anchoring forces. Subsequently, we validated the fidelity of the analytical solutions by comparing against numerical simulation results using the finite difference software FLAC^{3D} and monitoring results from an engineering example. The following conclusions can be drawn from this study.

(1) We established a theoretical model considering the coupled effect between rock creep and the time-dependent anchoring forces of prestressed bolts. We further derived the elastic and viscoelastic analytical solutions for the rock displacement and the bolt anchoring force under coupled actions.

(2) The numerical results, engineering monitoring data, and analytical solutions are all in good agreement, which suggests the fidelity of the analytical solutions considering the coupling effect. The model provides a

theoretical reference for studying the tunnel reinforcement, analyzing the creep behavior of underground rock masses and the long-term stability of the reinforcement structure.

(3) Both the displacement of rock mass and the anchoring force of anchor bolts exhibit time dependence. After excavation, the surrounding rock mass undergoes creep under the initial geo-stress and the anchoring force of prestressed anchor bolt. The creep causes corresponding deformation of anchor bolt, and the anchoring force changes accordingly, which limits the creep of rock mass. As the model considers the coupling effect, for the rock mass with unstable creep properties, the rock displacement after excavation and reinforcement also reaches to a stable value eventually, which can well reflect the reinforcing effect of prestressed bolts.

(4) Because the mathematical derivation in the theoretical analysis process is extremely complex, some assumptions are applied to simplify the research. For more complex cases, it will be further studied in future work, such as non-circular tunnels, complex hydrogeological conditions, etc.

Appendix A

The constants in the expressions of $\Delta\bar{u}_\rho$ and $\bar{T}(s)$ are as follows.

$$h_1 = K^3 P_0 L p_{2r}^4 \begin{pmatrix} 27g_1 - 108g_2 + 54g_4 - 54g_5 \\ -108g_6 - 108g_7 - 108g_8 \end{pmatrix} + K^2 P_0 L p_{2r}^3 q_{2r} \begin{pmatrix} 108g_1 - 432g_2 - 54g_3 + 162g_4 \\ -162g_5 - 432g_6 - 432g_7 - 216g_8 \end{pmatrix}$$
$$+KP_0 L p_{2r}^2 q_{2r}^2 \left(144g_1 - 684g_2 - 144g_3 + 144g_4 - 144g_5 - 468g_6 - 468g_7 - 108g_8\right)$$
$$+P_0 L p_{2r} q_{2r}^3 \left(64g_1 - 400g_2 - 96g_3 + 32g_4 - 32g_5 - 112g_6 - 112g_7 - 16g_8\right) - \pi L\sigma_0 p_{2r} q_{2r}^3 \left(96(r-R) + 32\left(r - \frac{r^2}{R}\right)\right)$$
$$-K^2 g_{10} p_{2r}^3 q_{2r} \left(54(r-R) + 162\left(r - \frac{r^2}{R}\right)\right) - 144\pi K L\sigma_0 p_{2r}^2 q_{2r}^2 \left((r-R) + \left(r - \frac{r^2}{R}\right)\right) - 54\pi K^3 L\sigma_0 \left(r - \frac{r^2}{R}\right) p_{2r}^4$$

,

$$h_2 = K^3 P_0 L p_{1r} p_{2r}^3 \begin{pmatrix} 108g_1 - 432g_2 - 216g_4 - 216g_5 \\ -432g_6 - 432g_7 - 432g_8 \end{pmatrix} + K^2 P_0 L p_{2r}^3 q_{1r} \begin{pmatrix} 108g_1 - 432g_2 - 54g_3 + 162g_4 - \\ 162g_5 - 432g_6 - 432g_7 - 216g_8 \end{pmatrix}$$
$$+K^2 P_0 L p_{1r} p_{2r}^2 q_{2r} \begin{pmatrix} 324g_1 - 1296g_2 - 162g_3 + 486g_4 \\ -486g_5 - 1296g_6 - 1296g_7 - 648g_8 \end{pmatrix} + KP_0 L\left(p_{2r}^2 q_{1r} q_{2r} + p_{1r} p_{2r} q_{2r}^2\right) \begin{pmatrix} 288g_1 - 1368g_2 - 288g_3 + 288g_4 \\ -288g_5 - 936g_6 - 936g_7 - 216g_8 \end{pmatrix}$$
$$+P_0 L p_{2r} q_{1r} q_{2r}^2 \begin{pmatrix} 192g_1 - 1200g_2 - 288g_3 + 96g_4 \\ -96g_5 - 336g_6 - 336g_7 - 48g_8 \end{pmatrix} + P_0 L p_{1r} q_{2r}^3 \begin{pmatrix} 64g_1 - 400g_2 - 96g_3 + 32g_4 \\ -32g_5 - 112g_6 - 112g_7 - 16g_8 \end{pmatrix}$$
$$-288\pi K L\sigma_0 p_{2r}^2 q_{1r} q_{2r} \left((r-R) + \left(r - \frac{r^2}{R}\right)\right) - \pi L\sigma_0 p_{2r} q_{1r} q_{2r}^2 \left(288(r-R) + 96\left(r - \frac{r^2}{R}\right)\right) - 216\pi K^2 L\sigma_0 p_{1r} p_{2r}^3 \left(r - \frac{r^2}{R}\right)$$
$$-\pi K^2 L\sigma_0 p_{2r}^3 q_{1r} \left(54(r-R) + 162\left(r - \frac{r^2}{R}\right)\right) - \pi K^2 L\sigma_0 p_{1r} p_{2r}^2 q_{2r} \left(162(r-R) + 486\left(r - \frac{r^2}{R}\right)\right) - \pi L\sigma_0 p_{1r} q_{2r}^3 \left(96(r-R) + 32\left(r - \frac{r^2}{R}\right)\right)$$

,

$$h_3 = K^3 P_0 L p_{1r}^2 p_{2r}^2 \left(162g_1 - 648g_2 + 324g_4 - 324g_5 - 648g_6 - 648g_7 - 648g_8\right) - 324\pi K^3 L\sigma_0 p_{1r}^2 p_{2r}^2 \left(r - \frac{r^2}{R}\right)$$

$$+K^3 P_0 L p_{2r}^3 \left(108g_1 - 432g_2 + 216g_4 - 216g_5 - 432g_6 - 432g_7 - 432g_8\right) - 216\pi K^3 L\sigma_0 p_{2r}^3 \left(r - \frac{r^2}{R}\right)$$

$$+\left(p_{1r} p_{2r}^2 q_{1r} + p_{1r}^2 p_{2r} q_{2r} + p_{2r}^2 q_{2r}\right)\left[K^2 P_0 L \begin{pmatrix} 324g_1 - 1296g_2 - 162g_3 + 486g_4 \\ -486g_5 - 1296g_6 - 1296g_7 - 648g_8 \end{pmatrix} - \pi K^2 L\sigma_0 \left(162\left(r-R\right) + 486\left(r - \frac{r^2}{R}\right)\right)\right]$$

$$+KP_0 L p_{1r} p_{2r}^2 q_{1r}^2 \left(144g_1 - 684g_2 - 144g_3 + 144g_4 - 144g_5 - 468g_6 - 468g_7 - 108g_8\right) - 144\pi KL\sigma_0 p_{2r}^2 q_{1r}^2 \left(\left(r-R\right) + \left(r - \frac{r^2}{R}\right)\right)$$

$$+P_0 LK p_{1r} p_{2r} q_{1r} q_{2r} \left(576g_1 - 2736g_2 - 576g_3 + 576g_4 - 576g_5 - 1872g_6 - 1872g_7 - 432g_8\right) - 576\pi KL\sigma_0 p_{1r} p_{2r} q_{1r} q_{2r} \left(\left(r-R\right) + \left(r - \frac{r^2}{R}\right)\right)$$

$$+P_0 L\left(p_{2r} q_{1r}^2 q_{2r} + p_{1r} q_{1r} q_{2r}^2\right)\begin{pmatrix} 192g_1 - 1200g_2 - 288g_3 + 96g_4 \\ -96g_5 - 336g_6 - 336g_7 - 48g_8 \end{pmatrix} - \pi L\sigma_0 \left(p_{2r} q_{1r}^2 q_{2r} + p_{1r} q_{1r} q_{2r}^2\right)\left(288\left(r-R\right) + 96\left(r - \frac{r^2}{R}\right)\right)$$

$$+P_0 L q_{2r}^3 \left(64g_1 - 400g_2 - 96g_3 + 32g_4 - 32g_5 - 112g_6 - 112g_7 - 16g_8\right) - \pi L\sigma_0 q_{2r}^3 \left(96\left(r-R\right) + 32\left(r - \frac{r^2}{R}\right)\right)$$

,

$$h_4 = K^3 P_0 L\left(p_{1r}^3 p_{2r} + 3p_{1r} p_{2r}^2\right)\left(108g_1 - 432g_2 + 216g_4 - 216g_5 - 432g_6 - 432g_7 - 432g_8\right) - 216\pi K^3 L\sigma_0\left(p_{1r}^3 p_{2r} + 3p_{1r} p_{1r}^2\right)\left(r - \frac{r^2}{R}\right)$$

$$+K^2 P_0 L\left(p_{1r}^2 p_{2r} q_{1r} + p_{2r}^2 q_{1r}\right)\begin{pmatrix}324g_1 - 1296g_2 - 162g_3 + 486g_4\\ -486g_5 - 1296g_6 - 1296g_7 - 648g_8\end{pmatrix} - \pi K^2 L\sigma_0\left(p_{1r}^2 p_{2r} q_{1r} + p_{2r}^2 q_{1r}\right)\left(162(r-R) + 486\left(r - \frac{r^2}{R}\right)\right)$$

$$+\left(p_{1r} p_{2r} q_{1r}^2 + p_{1r}^2 p_{2r} q_{2r} + 2p_{2r} q_{1r} q_{2r} + p_{1r} q_{2r}^2\right)\left[P_0 LK\begin{pmatrix}288g_1 - 1368g_2 - 288g_3 + 288g_4\\ -288g_5 - 936g_6 - 936g_7 - 216g_8\end{pmatrix} - 288\pi KL\sigma_0\left((r-R) + 8\left(r - \frac{r^2}{R}\right)\right)\right]$$

$$+P_0 LKp_{2r} q_{1r}^3\begin{pmatrix}64g_1 - 400g_2 - 96g_3 + 32g_4\\ -32g_5 - 112g_6 - 112g_7 - 16g_8\end{pmatrix} - \pi KL\sigma_0 p_{2r} q_{1r}^3\left(96(r-R) + 32\left(r - \frac{r^2}{R}\right)\right)$$

$$+P_0 L\left(p_{1r} q_{1r}^2 q_{2r} + 6p_{1r} p_{2r} q_{2r}\right)\begin{pmatrix}192g_1 - 1200g_2 - 288g_3 + 96g_4\\ -96g_5 - 336g_6 - 336g_7 - 48g_8\end{pmatrix} - \pi L\sigma_0\left(p_{1r} q_{1r}^2 q_{2r} + 6p_{1r} p_{2r} q_{2r}\right)\left(288(r-R) + 96\left(r - \frac{r^2}{R}\right)\right)$$

$$+P_0 L\left(q_{2r}^3 + 3q_{1r} q_{2r}^2\right)\begin{pmatrix}64g_1 - 400g_2 - 96g_3 + 32g_4\\ -32g_5 - 112g_6 - 112g_7 - 16g_8\end{pmatrix} - \pi L\sigma_0\left(q_{2r}^3 + 3q_{1r} q_{2r}^2\right)\left(96(r-R) + 32\left(r - \frac{r^2}{R}\right)\right)$$

,

$$h_5 = K^3 P_0 L\left(p_{1r}^4 + 12p_{1r}^2 p_{2r} + 6p_{2r}^2\right)\left(27g_1 - 108g_2 + 54g_4 - 54g_5 - 108g_6 - 108g_7 - 108g_8\right) - 54\pi K^3 L\sigma_0\left(p_{1r}^4 + 12p_{1r}^2 p_{2r} + 6p_{2r}^2\right)\left(r - \frac{r^2}{R}\right)$$

$$+\left(p_{1r}^3 q_{1r} + 6p_{1r} p_{2r} q_{1r} + 3p_{1r}^2 q_{2r} + 3p_{2r} q_{2r}\right)\left[K^2 P_0 L\left(108g_1 - 432g_2 - 54g_3 + 162g_4 - 162g_5 - 432g_6 - 432g_7 - 216g_8\right) - \pi K^2 L\sigma_0\left(54(r-R) + 162\left(r - \frac{r^2}{R}\right)\right)\right]$$

$$+\left(p_{1r}^2 q_{1r}^2 + 2p_{2r} q_{1r}^2 + 4p_{1r} q_{1r} q_{2r} + q_{2r}^2\right)\left[KP_0 L\left(144g_1 - 684g_2 - 144g_3 + 144g_4 - 144g_5 - 288g_6 - 288g_7 - 108g_8\right) - 144\pi KL\sigma_0\left((r-R) + \left(r - \frac{r^2}{R}\right)\right)\right]$$

$$+\left(p_{1r} q_{1r}^3 + q_{1r}^2 q_{2r}\right)\left[P_0 L\left(64g_1 - 400g_2 - 96g_3 + 32g_4 - 32g_5 - 112g_6 - 112g_7 - 16g_8\right) - \pi L\sigma_0\left(96(r-R) + 32\left(r - \frac{r^2}{R}\right)\right)\right]$$

,

$$h_6 = K^3 P_0 L\left(p_{1r}^3 + 3p_{1r}p_{2r}\right)\left(108g_1 - 432g_2 - 54g_3 + 162g_4 - 162g_5 - 432g_6 - 432g_7 - 216g_8\right) - 216\pi K^3 L\sigma_0\left(p_{1r}^3 + 3p_{1r}p_{2r}\right)\left(r - \frac{r^2}{R}\right)$$

$$+\left(p_{1r}^2 q_{1r} + p_{2r}q_{1r} + p_{1r}q_{2r}\right)\left[K^2 P_0 L\left(324g_1 - 1296g_2 - 162g_3 + 486g_4 - 486g_5 - 1296g_6 - 1296g_7 - 648g_8\right) - \pi K^2 L\sigma_0\left(162\left(r - R\right) + 486\left(r - \frac{r^2}{R}\right)\right)\right]$$

$$+\left(p_{1r}q_{2r}^2 + q_{1r}q_{2r}\right)\left[KP_0 L\left(288g_1 - 1368g_2 - 288g_3 + 288g_4 - 288g_5 - 936g_6 - 936g_7 - 216g_8\right) - 288\pi KL\sigma_0\left(\left(r - R\right) + \left(r - \frac{r^2}{R}\right)\right)\right]$$

$$+P_0 L q_{1r}^3\left(64g_1 - 400g_2 - 96g_3 + 32g_4 - 32g_5 - 112g_6 - 112g_7 - 16g_8\right) - \pi L\sigma_0 q_{1r}^3\left(96\left(r - R\right) + 32\left(r - \frac{r^2}{R}\right)\right)$$

,

$$h_7 = K^3 P_0 L\left(p_{1r}^2 + 1.5p_{2r}\right)\left(108g_1 - 432g_2 + 216g_4 - 216g_5 - 432g_6 - 432g_7 - 432g_8\right) - 216\pi K^3 L\sigma_0\left(p_{1r}^2 + 1.5p_{2r}\right)\left(r - \frac{r^2}{R}\right)$$

$$+\left(3p_{1r}q_{1r} + q_{2r}\right)\left[K^2 P_0 L\left(108g_1 - 432g_2 + 216g_4 - 216g_5 - 432g_6 - 432g_7 - 432g_8\right) - \pi K^2 L\sigma_0\left(54\left(r - R\right) + 162\left(r - \frac{r^2}{R}\right)\right)\right]$$

$$+KP_0 L q_{1r}^2\left(144g_1 - 684g_2 - 144g_3 + 144g_4 - 144g_5 - 468g_6 - 468g_7 - 108g_8\right) - 144\pi KL\sigma_0 q_{1r}^2\left(\left(r - R\right) + \left(r - \frac{r^2}{R}\right)\right)$$

,

$$h_8 = K^3 P_0 L p_{1r}\left(108g_1 - 432g_2 + 216g_4 - 216g_5 - 432g_6 - 432g_7 - 432g_8\right) - 216\pi K^3 L\sigma_0\left(p_{1r}^2 + 1.5p_{2r}\right)\left(r - \frac{r^2}{R}\right)$$

$$+K^2 P_0 L q_{1r}\left(108g_1 - 432g_2 + 216g_4 - 216g_5 - 432g_6 - 432g_7 - 432g_8\right) - \pi K^2 L\sigma_0 q_{1r}\left(54\left(r - R\right) + 162\left(r - \frac{r^2}{R}\right)\right)$$

,

$$h_9 = K^3 P_0 L\left(27g_1 - 108g_2 + 54g_4 - 54g_5 - 108g_6 - 108g_7 - 108g_8\right) - 54\pi K^3 L\sigma_0\left(r - \frac{r^2}{R}\right),$$

$$h_{10}=K^3E_bA_bp_{2r}^4\left(54g_1-216g_2+108g_4-108g_5-216g_6-216g_7-216g_8\right)+108\pi K^3Lp_{2r}^3q_{2r}$$
$$+K^2E_bA_bp_{2r}^3q_{2r}\left(162g_1-648g_2-108g_3+216g_4-216g_5-648g_6-648g_7-216g_8\right)+324\pi K^2Lp_{2r}^2q_{2r}^2$$
$$+KE_bA_bp_{2r}^2q_{2r}^2\left(144g_1-792g_2-180g_3+108g_4-1084g_5-360g_6-360g_7-72g_8\right)+288\pi KLp_{2r}q_{2r}^3$$
$$+E_bA_bq_{2r}^3\left(32g_1-200g_2-48g_3+16g_4-16g_5-56g_6-56g_7-8g_8\right)+64\pi Lp_{2r}q_{2r}^4$$

,

$$h_{11}=K^3E_bA_bp_{1r}p_{2r}^3\left(216g_1-864g_2+432g_4-432g_5-864g_6-864g_7-864g_8\right)+108\pi K^3Lp_{2r}^3q_{1r}+648\pi K^2Lp_{2r}^2q_{1r}q_{2r}$$
$$+K^2E_bA_b\left(p_{2r}^3q_{2r}+3p_{1r}p_{2r}^2q_{2r}\right)\left(162g_1-648g_2-108g_3+216g_4-216g_5-648g_6-648g_7-216g_8\right)+324\pi K^2Lp_{1r}p_{2r}^2q_{2r}$$
$$+KE_bA_bp_{2r}^2q_{1r}q_{2r}\left(288g_1-1584g_2-360g_3+216g_4-216g_5-720g_6-720g_7-144g_8\right)+648\pi KLp_{1r}p_{2r}q_{2r}^2$$
$$+E_bA_bp_{1r}p_{2r}q_{2r}^2\left(288g_1-1584g_2-360g_3+216g_4-216g_5-720g_6-720g_7-144g_8\right)+864\pi KLp_{2r}q_{1r}q_{2r}^2$$
$$+E_bA_bp_{2r}q_{1r}q_{2r}^2\left(96g_1-600g_2-144g_3+48g_4-48g_5-168g_6-168g_7-24g_8\right)+288\pi KLp_{1r}q_{2r}^3$$
$$+E_bA_bp_{1r}q_{2r}^3\left(32g_1-200g_2-48g_3+16g_4-16g_5-56g_6-56g_7-8g_8\right)+256\pi Lq_{1r}q_{2r}^3$$

,

$$h_{12}=K^3E_bA_b\left(1.5p_{1r}^2p_{2r}^2+p_{2r}^3\right)\begin{pmatrix}216g_1-864g_2+432g_4-432g_5\\-864g_6-864g_7-864g_8\end{pmatrix}+324\pi K^3L\left(p_{1r}p_{2r}^2q_{1r}+p_{1r}^2p_{2r}q_{2r}+p_{2r}^2q_{2r}\right)$$
$$+K^2E_bA_b\left(p_{1r}p_{2r}^2q_{1r}+p_{1r}^2p_{2r}q_{2r}+p_{2r}^2q_{2r}\right)\begin{pmatrix}486g_1-1944g_2-324g_3+648g_4\\-648g_5-1944g_6-1944g_7-648g_8\end{pmatrix}+324\pi K^2L\left(p_{2r}^2q_{1r}^2+p_{1r}^2q_{2r}^2+p_{2r}q_{2r}^2\right)$$
$$+1296\pi K^2Lp_{1r}p_{2r}q_{1r}q_{2r}+KE_bA_b\left(p_{2r}^2q_{1r}^2+p_{1r}p_{2r}q_{1r}q_{2r}+p_{1r}^2q_{2r}^2+2p_{2r}q_{2r}^2\right)\begin{pmatrix}144g_1-792g_2-180g_3+180g_4\\-108g_5-360g_6-360g_7-72g_8\end{pmatrix}$$
$$+864\pi KL\left(p_{2r}q_{1r}^2q_{2r}+p_{1r}q_{1r}q_{2r}^2\right)+E_bA_bp_{2r}q_{1r}^2q_{2r}\left(96g_1-600g_2-144g_3+48g_4-48g_5-168g_6-168g_7-24g_8\right)-24\pi KLp_{2r}q_{1r}^2q_{2r}$$
$$+648\pi KLp_{1r}p_{2r}q_{2r}^2+E_bA_b\left(3q_{1r}q_{2r}^2+q_{2r}^3\right)\left(32g_1-200g_2-48g_3+16g_4-16g_5-56g_6-56g_7-8g_8\right)+\pi L\left(384q_{1r}^2q_{2r}^2+288q_{2r}^3\right)$$

,

$$h_{13} = K^3 E_b A_b \left(3p_{1r}p_{2r}^2 + p_{1r}^3 p_{2r}\right)\left(216g_1 - 864g_2 + 432g_4 - 432g_5 - 864g_6 - 864g_7 - 864g_8\right) + 324\pi K^3 L \begin{pmatrix} p_{1r}^2 p_{2r} q_{1r} + p_{2r}^2 q_{1r} \\ + p_{1r}^3 q_{2r} + 2p_{1r}p_{2r}q_{2r} \end{pmatrix}$$
$$+K^2 E_b A_b \left(3p_{1r}^2 q_{1r} + 3p_{2r}^2 q_r + p_{1r}^3 q_{2r} + 6p_{1r}p_{2r}q_{2r}\right)\left(162g_1 - 648g_2 - 108g_3 + 216g_4 - 216g_5 - 648g_6 - 648g_7 - 216g_8\right)$$
$$+648\pi K^2 L\left(p_{1r}p_{2r}q_{1r}^2 + p_{1r}^2 q_{1r}q_{2r} + 3p_{2r}q_{1r}q_{2r} + 2p_{1r}p_{2r}q_{1r}q_{2r} + p_{1r}q_{2r}^2\right)$$
$$+KE_b A_b \left(p_{2r}^2 q_{1r}^2 + p_{1r}p_{2r}q_{1r}q_{2r} + p_{1r}^2 q_{2r}^2 + 2p_{2r}q_{2r}^2 + 2p_{1r}^2 q_{1r}q_{2r} + 4p_{2r}q_{1r}q_{2r} + 2p_{1r}q_{2r}^2\right)\begin{pmatrix} 144g_1 - 792g_2 - 180g_3 + 180g_4 \\ -108g_5 - 360g_6 - 360g_7 - 72g_8 \end{pmatrix}$$
$$+E_b A_b p_{1r}p_{2r}q_{1r}^2\left(288g_1 - 1584g_2 - 360g_3 + 216g_4 - 216g_5 - 720g_6 - 720g_7 - 144g_8\right) - 288\pi KLp_{2r}q_{1r}^3$$
$$+E_b A_b\left(p_{2r}q_{1r}^3 + 3p_{1r}q_{1r}^2 q_{2r} + 3q_{1r}q_{2r}^2\right)\left(32g_1 - 200g_2 - 48g_3 + 16g_4 - 16g_5 - 56g_6 - 56g_7 - 8g_8\right) + 864\pi Lq_{1r}q_{2r}^2$$

,

$$h_{14} = K^3 E_b A_b \left(p_{1r}^4 + 12p_{1r}^2 p_{2r} + 6p_{2r}^2\right)\left(54g_1 - 216g_2 + 108g_4 - 108g_5 - 216g_6 - 216g_7 - 216g_8\right)$$
$$+108\pi K^3 L\left(p_{1r}^3 q_{1r} + 3p_{2r}q_{1r} + 6p_{1r}p_{2r}q_{2r} + 3p_{1r}^2 q_{2r} + 3p_{2r}q_{2r}\right) + 324\pi K^2 L\left(q_{1r}^2 + 2p_{1r}q_{1r}^2 + p_{1r}^2 q_{1r}^2 + 2p_{2r}q_{1r}^2 + 4p_{1r}q_{1r}q_{2r} + q_{2r}^2\right)$$
$$+K^2 E_b A_b\left(p_{1r}^3 q_{1r} + 6p_{1r}p_{2r}q_{2r} + 3p_{1r}^2 q_{2r} + 3p_{2r}q_{2r}\right)\left(162g_1 - 648g_2 - 108g_3 + 216g_4 - 216g_5 - 648g_6 - 648g_7 - 216g_8\right)$$
$$+KE_b A_b\left(p_{2r}^2 q_{1r}^2 + 2p_{2r}q_{1r}^2 + 4p_{1r}p_{1r}q_{2r} + q_{2r}^2\right)\left(144g_1 - 792g_2 - 180g_3 + 180g_4 - 108g_5 - 360g_6 - 360g_7 - 72g_8\right) + 144\pi KL\left(p_{2r}q_{1r}^2 + 2p_{1r}q_{1r}^3\right)$$
$$+E_b A_b\left(p_{1r}q_{1r}^3 + 3p_{1r}q_{1r}^2 q_{2r} + 3q_{1r}q_{2r}^2 + 3q_{1r}^2 q_{2r}\right)\left(32g_1 - 200g_2 - 48g_3 + 16g_4 - 16g_5 - 56g_6 - 56g_7 - 8g_8\right) + 64\pi Lq_{1r}^4$$

,

$$h_{15} = K^3 E_b A_b\left(4p_{1r}^3 + 12p_{1r}p_{2r}\right)\left(54g_1 - 216g_2 + 108g_4 - 108g_5 - 216g_6 - 216g_7 - 216g_8\right) + 108\pi K^3 L\left(3p_{1r}^2 q_{1r} + q_{2r} + 3p_{1r}q_{2r}\right)$$
$$+K^2 E_b A_b\left(3p_{1r}^2 q_{1r} + 3p_{2r}q_{1r} + 3p_{1r}q_{2r}\right)\left(162g_1 - 648g_2 - 108g_3 + 216g_4 - 216g_5 - 648g_6 - 648g_7 - 216g_8\right) - 864\pi K^3 Lp_{1r}^3$$
$$+KE_b A_b\left(2p_{1r}q_{1r}^2 + 2q_{1r}^3 + 2q_{1r}q_{2r}\right)\left(144g_1 - 792g_2 - 180g_3 + 180g_4 - 108g_5 - 360g_6 - 360g_7 - 72g_8\right) + 144\pi KL\left(p_{2r}q_{1r}^2 + 2p_{1r}q_{1r}^3\right)$$
$$+628\pi K^2 Lp_{1r}q_{2r}$$

,

$$h_{16} = K^3 E_b A_b \left(6p_{1r}^2 + 4p_{2r}\right)\left(54g_1 - 216g_2 + 108g_4 - 108g_5 - 216g_6 - 216g_7 - 216g_8\right) + 108\pi K^3 L\left(q_{1r} + 3p_{1r}q_{1r}\right)$$
$$+K^2 E_b A_b \left(3p_{1r}q_{1r} + q_{2r}\right)\left(162g_1 - 648g_2 - 108g_3 + 216g_4 - 216g_5 - 648g_6 - 648g_7 - 216g_8\right)$$
$$+KE_b A_b q_{1r}^2 \left(144g_1 - 792g_2 - 180g_3 + 180g_4 - 108g_5 - 360g_6 - 360g_7 - 72g_8\right)$$

,

$$h_{17} = K^3 E_b A_b 4p_{1r}\left(54g_1 - 216g_2 + 108g_4 - 108g_5 - 216g_6 - 216g_7 - 216g_8\right)$$
$$+K^2 E_b A_b q_{1r}\left(162g_1 - 648g_2 - 108g_3 + 216g_4 - 216g_5 - 648g_6 - 648g_7 - 216g_8\right)$$

,

$$h_{18} = K^3 E_b A_b \left(54g_1 - 216g_2 + 108g_4 - 108g_5 - 216g_6 - 216g_7 - 216g_8\right),$$

$$j_1 = P_0, \quad j_2 = \frac{\pi d_g^2 E_b A_b}{4L_z L_\theta (R-r)} h_1, \quad j_3 = \frac{\pi d_g^2 E_b A_b}{4L_z L_\theta (R-r)} h_2, \quad j_4 = \frac{\pi d_g^2 E_b A_b}{4L_z L_\theta (R-r)} h_3, \quad j_5 = \frac{\pi d_g^2 E_b A_b}{4L_z L_\theta (R-r)} h_4, \quad j_6 = \frac{\pi d_g^2 E_b A_b}{4L_z L_\theta (R-r)} h_5,$$

$$j_7 = \frac{\pi d_g^2 E_b A_b}{4L_z L_\theta (R-r)} h_6, \; j_8 = \frac{\pi d_g^2 E_b A_b}{4L_z L_\theta (R-r)} h_7, \quad j_9 = \frac{\pi d_g^2 E_b A_b}{4L_z L_\theta (R-r)} h_8, \quad j_{10} = \frac{\pi d_g^2 E_b A_b}{4L_z L_\theta (R-r)} h_9, \quad j_{11} = \frac{\pi d_g^2 E_b A_b}{4L_z L_\theta (R-r)} h_{10},$$

$$j_{12} = \frac{\pi d_g^2 E_b A_b}{4L_z L_\theta (R-r)} h_{11}, \quad j_{13} = \frac{\pi d_g^2 E_b A_b}{4L_z L_\theta (R-r)} h_{12}, \; j_{14} = \frac{\pi d_g^2 E_b A_b}{4L_z L_\theta (R-r)} h_{13}, \quad j_{15} = \frac{\pi d_g^2 E_b A_b}{4L_z L_\theta (R-r)} h_{14}, \quad j_{16} = \frac{\pi d_g^2 E_b A_b}{4L_z L_\theta (R-r)} h_{15},$$

$$j_{17} = \frac{\pi d_g^2 E_b A_b}{4L_z L_\theta (R-r)} h_{16}, \quad j_{18} = \frac{\pi d_g^2 E_b A_b}{4L_z L_\theta (R-r)} h_{17}.$$

Table Captions

Table 1. Rock mechanical parameters used in the numerical model.

Table 2. Parameters of the prestressed bolt used in the numerical model.

Table 3. Physical and mechanical parameters of the rock mass of the Qingdao Metro Line 6 project.

Figure Captions

Figure 1. Simplified coupling mechanical model of bolts and tunnel rock mass.

Figure 2. Decomposition diagram of coupling mechanical model of bolts and tunnel rock mass. (a) Concentrated force *P* applied at the tunnel perimeter; (b) Concentrated force *P* in an infinite medium; (c) Stress field at the tunnel perimeter; (d) Tunnel with a far field stress.

Figure 3. Mechanical model of the concentrated force *P* applied to the anchor head at the tunnel perimeter.

Figure 4. 3D to 2D representations of concentrated force *P* at the anchor end in an infinite medium: (a) 3D representation of the concentrated force *P* in an infinite medium; (b) 2D representation of the concentrated force *P* in an infinite medium.

Figure 5. Stress analysis of tunnel：(a) Stress field at the tunnel perimeter; (b) Mechanical model of the tunnel under the action of the original rock stress; (c) Schematic diagram of the stress state in a representative elementary volume (REV).

Figure 6. Creep models of the bolts and rocks: (a) Elastic model for bolts; (b) Burgers model for rocks.

Figure 7. Dimension, grid and boundary conditions of the numerical model.

Figure 8. Tunnel total displacement nephogram (Unit: m).

Figure 9. Comparison between analytical solutions and numerical simulation results (The monitoring point is at the anchor head of the anchor bolt, as shown in Fig. 7): (a) Comparison between analytical solutions and numerical simulation results of rock mass displacement; (b) Comparison between analytical solutions and numerical simulation results of the bolt anchoring force.

Figure 10. Geological cross-section of the tunnel project.

Figure 11. Location and anchoring details of tunnel construction.

Figure 12. Field layout of the prestressed anchor bolts.

Figure 13. Comparisons between the theoretical solutions and the monitored data: (a) Comparison between the theoretical solutions and the monitored data of rock displacement; (b) Comparison between the theoretical solutions and the monitored data of the anchoring force.

Figure 14. Creep decomposing into elementary strains in different stages.

Figure 15. Comparison of rock radial displacement between coupled model and uncoupled model.

Declaration of competing Interest

We declare no potential conflicts of interest with respect to the research, authorship, and publication of this article.

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