UNIVERSITY OF ABERDEEN: SESSION 2022/2023

Degree Examination in

PX3020 Mathematical Methods in Physics

Monday 12 December 2022 09:00 – 11:00

PLEASE READ CAREFULLY

Attempt 2 of the 3 questions in section A, each worth 15 marks.

Attempt 2 of the 4 questions in section B, each worth 15 marks.

Total marks of this exam are 60 marks.

You may use the following formulae in your calculations:

Green theorem

$$\iint_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S} = \oint_{\partial S} \vec{F} \cdot d\vec{s}$$

Divergence theorem

$$\iiint_V \vec{\nabla} \cdot \vec{F}\, dV = \oiint_{\partial V} \vec{F} \cdot d\vec{S}$$

Integrals

$$\int_0^{2\pi} \frac{2\cos\theta - 1}{4\cos\theta - 5}\, d\theta = 0$$

$$\int e^{-x^2} dx = \frac{\sqrt{\pi}}{2}\operatorname{erf}(x) + C$$

where $\operatorname{erf}(x)$ is the error function.

SECTION A

1. Consider a path on the 2-dimensional (x, y) plane parametrised by a real number θ with a positive coefficient a as follows:

$$\begin{aligned} x &= ae^{\theta}\cos\theta \\ y &= ae^{\theta}\sin\theta \end{aligned}$$

   (a) Find the corresponding velocity. [3]

   (b) Find the corresponding acceleration. [3]

   (c) Sketch the image of this path in the (x, y) plane using the polar coordinates, i.e. $(r = \sqrt{x^2+y^2},\ \theta\ )$, for $a = 1$ over $0 \le \theta \le 2\pi$. [3]

   (d) Sketch the image of this path in the (x, y) plane using the polar coordinates for $a = e^{-2\pi}$ over $2\pi \le \theta \le 4\pi$. [3]

   (e) Compare your results in parts iii and iv and provide a brief mathematical or geometrical interpretation about the image of this path for a fixed positive coefficient a. [3]

2. In the 3-dimensional space with Cartesian coordinates (x, y, z), consider a vector field

$$\vec{F}(\vec{r}) = (x + y + z, 2x + y + z, -x - 2y + z)$$

   and a loop

$$L = (0, -2, 1) \to (0, 1, 1) \to (0, 1, -2) \to (0, -2, -2) \to (0, -2, 1)$$

   (a) Sketch this loop on the (y, z)-plane with key geometric features clearly labelled with a suitable background grid. [7]

   (b) Derive the circulation of $\vec{F}(\vec{r})$ around loop L. [8]

   Hint: Consider the 3-diamensional Green theorem. This requires getting directions and orientations of displacement and area right.

3. Consider the differential equation for $x(t)$:

$$\frac{dx(t)}{dt} = x(t)t - 5x(t) - 3t + 15$$

   (a) Obtain the general solution of this equation by the separation of variables. [8]

   (b) From your result in part (a), derive a solution satisfying the condition

$$x(10) = 0$$

   [4]

   (c) From your result in part (b), find a value of t which is not 10, i.e. $t \neq 10$, but satisfies

$$x(t) = 0$$

   [3]

SECTION B

4. On the 3-dimensional (x, y, z) space, consider a vector field

$$\vec{F}(\vec{x}) = \left(\frac{y}{x^2 + y^2}, \frac{-x}{x^2 + y^2}, 0\right)$$

(a) Calculate the curl of this vector field. [5]

(b) Calculate the closed line integral of this vector field.

$$\oint_{C_1} \vec{F}(\vec{x}) \cdot d\vec{x}$$

where C_1 is a unit circle on the (x, y) plane centred at the origin. [5]

(c) Calculate the closed line integral of this vector field.

$$\oint_{C_2} \vec{F}(\vec{x}) \cdot d\vec{x}$$

where C_2 is a unit circle on the (x, y) plane centred at $(2,0)$. [5]

5. Consider an unknown function $y(x)$ satisfying the inhomogeneous differential equation

$$y'' + 4y' + 13y = 7$$

(a) Obtain a particular solution to the above inhomogeneous differential equation. [5]

(b) Find the general real solution of the associated homogeneous equation to the above inhomogeneous differential equation. [5]

(c) Derive the general complex solution to the above inhomogeneous differential equation. [5]

6. On the 3-dimensional (x, y, z) space, find the following.

   (a) Write down the explicit expressions of the coordinate transformation to cylindrical polar coordinates (r, θ, z). Specifically give the explicit expressions of the functions $x(r, \theta, z)$, $y(r, \theta, z)$, and $z(r, \theta, z)$ as well as the explicit reciprocal relations $r(x, y, z)$, $\theta(x, y, z)$, and $z(x, y, z)$. [3]

   (b) From the result of part (a), derive the Jacobian matrix $\boldsymbol{J} = \partial(x, y, z)/\partial(r, \theta, z)$. [4]

   (c) From the result of part (b), calculate the Jacobian determinant $J = \det(\boldsymbol{J})$. [3]

   (d) From the result of part (c), determine the volume inside a witch hat, idealised as cone with a tip at (0,0,13) and a base as the unit circle (of radius 1) on the (x, y) plane centred at the origin. [5]

7. Using the decomposition into forward and backward moving waves, find the solution to the wave equation

$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f}{\partial x^2}$$

   for the function $f(x, t)$ with the initial conditions $f(x, 0) = 0$ and $\frac{\partial f}{\partial t}(x, 0) = e^{-x^2}$.
   [Hint: You may find an integration formula in the rubric near the top of this paper useful.]

   [15]

THE END