UNIVERSITY OF ABERDEEN: SESSION 2022/2023

Degree Examination in PX*4012* *Statistical Physics and Stochastic Systems*

Date 7th December 2022 *Time* 9:00 – 11:00

PLEASE READ CAREFULLY
Candidates should attempt to answer TWO questions from Section A (30 marks) and THREE questions from Section B (45 marks). **Total**: 75 marks (75% of the course marks).

Rubric

You may find the following formulae useful:

Stirling's approximation

$$N! \cong N^N e^{-N} \quad \text{or} \quad \ln(N!) \cong N[\ln(N) - 1]$$

The volume of a 3*N*-dimensional sphere with radius R is given by

$$V = \frac{\pi^{3N/2}}{\Gamma(\frac{3N}{2}+1)} R^{3N} = \frac{\pi^{3N/2}}{\left(\frac{3N}{2}\right)!} R^{3N}$$

The Gaussian integral holds

$$\int_{-\infty}^{+\infty} dx\, e^{-x^2} = \sqrt{\pi}$$

The hyperbolic cosine and sine of a variable *x* are

$$\cosh(x) = \frac{e^x + e^{-x}}{2}, \quad \sinh(x) = \frac{e^x - e^{-x}}{2}$$

The geometric sums

$$\sum_{n=1}^{\infty} x^n = \frac{1}{1-x}, x < 1 \quad \sum_{n=1}^{N} n = N(N+1)/2 \quad \sum_{n=1}^{N} n\text{^}2 = N(N+1)(2N+1)/6$$

You may need to use the following constants in your calculations:

Constant	*Symbol*	*Value*
Ideal gas constant	R	$= 8.314\ J/(mol\ K)$
Boltzmann constant	k_B	$= 1.381 \times 10^{-23}\ J/K$
Planck constant	h	$= 6.626 \times 10^{-34}\ J\ s$
Avogadro constant	N_A	$= 6.626 \times 10^{-23}\ 1/mol$

SECTION A – Classical Statistical Mechanics **Total: 30 marks**
(You should attempt to solve TWO problems in this section)

1. *Consider an isolated box with adiabatic walls and volume V. The box has a removable interface that divides the volume in half, V/2. Initially, let an ideal gas with N particles be placed in one section of the container.*

(a) Write the Hamiltonian of the ideal gas and find an expression for the available microstates when the total energy is E. [7]

(b) Find the change in entropy from the free expansion of the gas after the interface has been removed (i.e., from the initial volume to the whole V). [4]

(c) Discuss how irreversibility emerges as a statistical property from the reversible microscopic dynamics if one assumes the postulate of equal a priori probabilities. Explain why the opposite phenomenon, that is, the gas particles all moving to the initial fraction of the volume, is never observed macroscopically. [4]

2. *Consider a closed system of N identical and non-interacting particles with two energy levels ($s_i = 0$ or $s_i = 1$, with $i = 1, \ldots, N$) that is in thermal equilibrium with a heat reservoir at temperature T. The single-particle energies for these two states are: $E_i(s_i = 0) = -\varepsilon/2$ and $E_i(s_i = 1) = \varepsilon/2$, where $\varepsilon > 0$.*

(a) Determine the mean energy per particle ($\langle H \rangle / N$). [5]

(b) Discuss the behaviour of the mean energy per particle at low ($k_B T \ll \varepsilon$) and high ($k_B T \gg \varepsilon$) temperatures. Describe physically what happens to the occupation levels of each state in both extreme scenarios. [5]

(c) Find Helmholtz free energy $F(T, V)$ and the entropy $S(T, V)$. [5]

3. (a) *State the classical equipartition theorem* [3]

(b) *Consider an equilibrium mixture of two monoatomic (or noble) gases at low density*: Helium (atomic mass number A = 4) and Neon (A = 20). Which of the two gases will have the highest mean speed (i.e., root mean squared velocity v_{rms}) and by how much? Briefly explain your answer. [6]

(c) *Consider an ideal gas at thermal equilibrium made of diatomic molecules.* Describe qualitatively how its specific heat at constant volume c_v depends on temperature, and which degrees of freedom are contributing to the c_v as the temperature is increased. [6]

SECTION B – Quantum Particles and Stochastic Systems **Total: 45 marks**
(You should attempt to solve THREE problems in this section)

1. *The mathematical expression of Planck's distribution* $u(\nu, T)$ *is*

$$u(\nu, T) = \frac{8\pi\, h\, \nu^3}{c^3} \frac{1}{e^{h\,\nu/\,k_B T} - 1}$$

(a) Discuss the meaning of Plank's distribution and its classical limit. What kind of physics paradox is solved by this law? Draw a sketch of the distribution. [8]

(b) Consider a black-body cavity at temperature T_0. If you heat it to a higher temperature, $T_1 > T_0$, will the number of photons per unit frequency in the cavity decrease or increase? Explain your answer. [7]

2. (a) *Discuss the differences between the Bose-Einstein statistics for bosons and the Fermi-Dirac statistics for fermions.* Focus on the properties of the mean occupation number for fermions and bosons, drawing sketches and explaining the difference between statistics for conserved and non-conserved particles. [8]

(b) Explain under which conditions the Bose-Einstein and Fermi-Dirac statistics converge to the classical Boltzmann distribution. [7]

3. *Consider a ferromagnet of dimension D = 3 and temperature T is placed in an external magnetic field B. The Hamiltonian of this spin system is*

$$H = -J \sum_{\{i,j\}} s_i\, s_j - B\mu \sum_{i=1}^{N} s_i$$

(a) Use the mean-field approximation to derive an equation for the magnetization M (**Hint:** proceed as discussed in the lectures for $B = 0$ and then include the external field). [7]

(b) Assume that the intrinsic magnetic moment is $\mu = 9.3\times10^{-24}$ *Joules/Tesla*, which can either be parallel or anti-parallel to the field. Let $J = 2.5\times10^{-21}$ *Joules* and $T = 300$ *K*. Discuss how the magnetization changes when the amplitude of the magnetic field is changed from -400 *Tesla* to +400 *Tesla*. [8]

4. *Consider the following stochastic process on a one-dimensional lattice with spacing a = 1. There is a walker using an icosahedron-sided fair die (i.e., a die with 20 sides) to advance randomly along the lattice, where the starting position is arbitrarily set to 0 (n = 0). That is, at every step, the walker rolls the die and moves forward according to the result of the die roll.*

(a) Find the mean position and its standard deviation (i.e., root-mean-square deviation) after N time-steps. [8]

(b) Give the probability distribution $P(n, N)$ for the walker's position after having taken a large number of N steps ($N >> 1$). [7]

THE END