Multiplicative reasoning task design in teacher education with student teachers in Scottish schools: valuing diversity, developing flexibility and making connections

This paper considers the development of synthetic landscapes of multiplicative reasoning constructed by student teachers (pre-service teachers). It explores the implications of working in a way that integrates literature from maths recovery, cognitively guided instruction and realistic mathematics education approaches. The aim is to better understand how working in this way affects how student teachers interact with learners, the questions they ask, the tasks they design: their capacity to value diversity in literature and develop their flexibility in practice. The findings indicate that this embodied approach to task design within teacher education led to all student teachers engaging in a participatory view of learning mathematics, discussing a range of strategies and models used by the children. Although many student teachers experimented with different pedagogical approaches and were beginning to design their own tasks, very few understood the underlying relational structures between a series of connected tasks within number strings or investigations.

Keywords: teacher education; professional development; multiplicative reasoning; task design; inclusive practice

Introduction

One of the concerns within Scottish Education, as elsewhere, is the pace and scale of change that has been increasing rapidly over the past two decades. These changes, whilst well intentioned, tend to take the form of short-lived interventions that rarely produce the positive effects expected as many teachers are picking up the product of a process they have not been a part of thereby missing the understanding developed through the process of constructing. It assumes that a product can produce the positive impact needed to solve a problem, for example, ‘raising attainment’ or ‘closing the gap’ (Sosu and Ellis 2014). Often a bureaucratic approach to solving problems is instigated: purchasing a system or product, enforcing its implementation across the board. This can be an extremely costly and short-term reaction whereby the time another bureaucrat decides that there should be some evidence, some ‘value added’, suddenly we are already onto the next initiative. This leads to a pattern of superficial adoption of research findings (Black and Wiliam 2009) or mutation-in-practice (Watson and Ohtani 2015) in Scottish schools that relies on a one-size-fits-all approach to the natural dilemmas of teaching.

Given this environment, what can we use within teacher education to inform ourselves, our student teachers, our professional colleagues that accepts the natural dilemmas of teaching? What might improve professional and ethical decision-making in a politicalised world of curriculum design and change?

To address this question, this educational design research project (Gravemeijer and Cobb 2006) focuses on the first of the mathematics education courses within a four-year undergraduate degree programme. The author, alongside a teaching colleague, have developed and redeveloped this course over the past four years to consider how to better support student teachers entering this environment. The aim of the project is to reduce the number of teachers who have been ‘trained’ in one system and to develop teachers’ pedagogical design flexibility (Brown 2009). The course values the diversity of teaching approaches and their ability to make connections between what has gone before and changes that will happen throughout their career. It recognises that they need to be able to adapt quickly when in-service to changing priorities using informed professional decision-making.

So rather than taking one piece of major research and focusing on this to the exclusion of others we looked at using several major studies. The task design requires student teachers to build a synthetic landscape that would allow us, student teachers and colleagues, to consider the benefits and limitations of each in a complementary rather than a competitive manner. It is by synthesizing ideas from different perspectives that have different foci that student teachers can become aware of the similarities and differences; the complementary strengths and non-overlapping weaknesses of each system in turn.

In order to support this, the course is designed to provide the time for student teachers to construct their own understanding. To create the space and time to think more slowly the course focuses on the key shift in thinking from additive to multiplicative reasoning. The first semester explores additive reasoning working with children aged 4−7 years old and the second semester focuses on multiplicative reasoning working with children aged 8−12 years old.

The focus of this paper is on the implications of constructing landscapes of multiplicative reasoning based on a process of synthesizing ideas from three different perspectives: Maths Recovery (Wright et al. 2012, 2015), Cognitively Guided Instruction (Carpenter et al. 2015) and an American adaptation of Realistic Mathematics Education (Fosnot and Dolk 2001). Juxtaposing different approaches considering the similarities and differences is a pedagogical use of variation theory, and in particular 'example spaces' (Watson and Mason 2005; Watson and Ohtani 2015), to consider task design within a teacher education classroom.

Background

In Scotland, there are some professional development opportunities for in-service teachers to learn about Maths Recovery (MR) approaches, Cognitively Guided Instruction (CGI) and, to a lesser extent, Realistic Mathematics Education (RME) drawing on the work of Fosnot and Dolk, in various geographical areas and under particular local authorities. However, there is little research around the implementation of these different approaches. Elsewhere there are reports on some of these such as

Moffett and Corcoran (2011) who evaluated the impact of using RME in primary schools in Ireland and Dickinson and Hough (2012) in England. As such, this paper is focused on developing a better understanding of how task design in a teacher education classroom impacts on task design in school classrooms.

In this course, student teachers are asked to design tasks that value the diversity of strategies that the children work with and help develop the children's flexibility in how they solve problems to rebalance time spent on activities that resonate with Dewey's description:

> Sheer imitation, dictation of steps to be taken, mechanical drill, may give results most quickly and yet strengthen traits likely to be fatal to reflective power. The pupil is enjoined to do this and that specific thing, with no knowledge of any reason except that by doing so he gets his result most speedily; his mistakes are pointed out and corrected for him; he is kept at pure repetition of certain acts till they become automatic. (Dewey 1910, 51)

Similarly, for student teachers, the capacity to change habits and to understand the reasoning of others needs to be more important than 'sheer imitation' (Martin 2014).

Research-informed task design can support student teachers in developing their mathematical reasoning more effectively where it is embedded within their experiences of *learning mathematics for teaching* (Hill et al. 2004; Martin 2014). This in turn involves both specialised mathematical knowledge and mathematical pedagogical content knowledge (Ball and Bass 2003). This embodied approach to task design within teacher education incorporates tasks focused on student teachers as learners of mathematics, learning mathematics for teaching and becoming teachers of mathematics to build an awareness of themselves as mathematicians, researchers and teachers. When working with student teachers we emphasise mathematical practices (Ollerton and Watson 2001; Mason 2002; Swan 2006; Boaler 2009) to provide a more balanced and participatory view of doing mathematics, which stresses learning as a process of becoming a member of a community: promoting an interest in people-in-action and ‘being’ in constant flux (Sfard 1998).

As such, constructs emerging from the literature are interpreted, oftentimes transformed, to develop a coherent synthetic landscape, a dynamic framework comprising a network of constructs. Labels representing sets of synthetic constructs are used with particular attention being paid to understanding the many labels others have used. This interpretative process is highlighted by Mason and Johnston-Wilder who stress that

> …becoming aware of and adapting effective frameworks enables teachers to discern phenomena more closely, to distinguish phenomena more sensitively, to notice more than they would be able to without the framework labels. (Mason & Johnston-Wilder 2004, 3)

It is the contention here that this process of considering different perspectives in constructing their own landscape allows student teachers to notice more and develop the capacity to think through new initiatives, synthesizing different ideas in an informed manner: creating a way of working that might sustain professional decision-making in a politicalised world of curriculum design and change.

Theoretical background to task design in teacher education

Beneath the individual styles of a teacher there are beliefs about what mathematics is and what children and young people doing mathematics means (Askew et.al. 1997). These beliefs affect the way we interact with learners, the questions we ask, the tasks we design.

One of the key aspects when synthesising different views is being explicit about what comes from where so that the core of the curricular structure recognizes the similarities of approaches taken but also values the differences. The design of the course applies the idea of mathematical ‘example spaces’ in school classrooms (Watson and Mason 2005) to the context of teacher education i.e. mathematics pedagogical example spaces. Student teachers had to experience a wide enough set of examples of task types, strategies and models to allow them to develop a deep conceptual understanding of children’s multiplicative reasoning. In this instance the design favoured the notion of landscapes and hypothetical learning trajectories (Treffers 1991) to consciously avoid repeating traditional linear taxonomies and pathways. The key here was for

(1) students to construct their landscape based on other’s analysis of children’s work (a priori) and
(2) this landscape to be used to analyse their own children’s work (a posteriori), thereby assessing children’s thinking by movement across the landscape.

The latter point balances the prevalent discourse in Scottish schools in relation to assessing children based on what they cannot do, finding gaps and filling them, seeing how many questions are correct within a timed period. As such, it was important to focus consciously on what learners can do, to find the edges of their thinking.

One of the key findings from Martin (2014) was the importance of the use of child-writing within teacher education. This was described as a teacher educator version of multiple representations in mathematics (Swan 2006). Within this course it was important that the student teacher’s constructions remain handwritten and re-written with text, symbols, drawings and so on. This was to open up different ways that the student teachers could annotate their understandings and to avoid being drawn into a linear or hierarchical structure; to retain the metaphor of a landscape with various learning trajectories. Student teachers were also encouraged to handwrite when planning for learning – to avoid reducing a thoughtful process to a mechanical exercise.

As noted above, the three different perspectives are Realistic Mathematics Education (using Fosnot and Dolk 2001), Cognitively Guided Instruction (Carpenter et al. 2015) and Maths Recovery (Wright et al. 2012, 2015). They sit alongside the formal curricular guidelines in Scotland (Scottish Government 2009, 2017). Each approach will be discussed in turn, identifying their value and their limitations before discussion turns to the issue of synthesising approaches in the classroom.

Formal Curriculum

In order to realise the student teachers' sense of agency (Florian and Spratt 2013) they have to recognize the formal curriculum, both what is there and more importantly what is not. However, it is also important to move beyond the curricular guidelines (Scottish Government 2009) to consider different perspectives, different organising structures with relative valorization of different aspects of multiplicative reasoning.

The formal curriculum in Scotland is encapsulated within the Curriculum for Excellence (CfE) guidelines; a relevant extract of the experiences and outcomes is shown below (figure 1). These guidelines are relatively vague compared to other countries and is a non-legislative document which encourages teacher autonomy in how it is enacted in schools and classrooms. At the request of teachers, benchmarks were later produced (Scottish Government 2017) to add detail to the experiences and outcomes. However, the manner in which these documents are tabulated and coded emphasises a linear and disconnected set of objectives.

Number, money and measure

	Early	First	Second	Third	Fourth
Estimation and rounding	*I am developing a sense of size and amount by observing, exploring, using and communicating with others about things in the world around me[1].* **MNU 0-01a**	*I can share ideas with others to develop ways of estimating the answer to a calculation or problem, work out the actual answer, then check my solution by comparing it with the estimate.* **MNU 1-01a**	*I can use my knowledge of rounding to routinely estimate the answer to a problem then, after calculating, decide if my answer is reasonable, sharing my solution with others.* **MNU 2-01a**	*I can round a number using an appropriate degree of accuracy, having taken into account the context of the problem.* **MNU 3-01a**	*Having investigated the practical impact of inaccuracy and error, I can use my knowledge of tolerance when choosing the required degree of accuracy to make real-life calculations.* **MNU 4-01a**
Number and number processes including addition, subtraction, multiplication, division and negative numbers	*I have explored numbers, understanding that they represent quantities, and I can use them to count, create sequences and describe order.* **MNU 0-02a** *I use practical materials and can 'count on and back' to help me to understand addition and subtraction, recording my ideas and solutions in different ways.* **MNU 0-03a**	*I have investigated how whole numbers are constructed, can understand the importance of zero within the system and can use my knowledge to explain the link between a digit, its place and its value.* **MNU 1-02a** *I can use addition, subtraction, multiplication and division when solving problems, making best use of the mental strategies and written skills I have developed.* **MNU 1-03a**	*I have extended the range of whole numbers I can work with and having explored how decimal fractions are constructed, can explain the link between a digit, its place and its value.* **MNU 2-02a** *Having determined which calculations are needed, I can solve problems involving whole numbers using a range of methods, sharing my approaches and solutions with others.* **MNU 2-03a**	*I can use a variety of methods to solve number problems in familiar contexts, clearly communicating my processes and solutions.* **MNU 3-03a**	*Having recognised similarities between new problems and problems I have solved before, I can carry out the necessary calculations to solve problems set in unfamiliar contexts.* **MNU 4-03a**

Figure 1: Excerpt from CfE (Scottish Government, 2009)

One of the first tasks the student teachers engage with is to deconstruct the CfE guidelines and reconstruct all aspects relevant to multiplicative reasoning in a way that makes sense to them. By the end of this process they have a clearer idea of what is there and a realisation that certain aspects are prominent such as the extending range of numbers from the natural numbers, whole numbers, integers to rational numbers. However, others are spread across the tables such as contexts which use multiplicative reasoning: money, time, measurement, probability and statistics. Even the operations themselves appear in unexpected places: for example, partitive division appears in the section on fractions before it appears in number processes (MNU 0-07a and MNU 1-03a in figure 2). Similarly, mathematical structures such as the associative, commutative and distributive laws only appear later in the algebra section. Student teachers need to look carefully across all 137 statements and not rely on pre-existing headings and sub headings.

CfE: Experiences and Outcomes and benchmarks: **blue emphasises operations**; **green the strategies**; **red the big ideas** (mathematical structures); **purple the skills** eg explaining, justifying etc
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I can **share out a group of items by making smaller groups** and can **split a whole object into smaller parts**. MNU 0-07a

I can use addition, subtraction, **multiplication and division** when solving problems, **making best use of the mental strategies and written skills** I have developed. (MNU 1-03a)

-- Demonstrates understanding of the **commutative law**

-- **Applies strategies** to determine multiplication facts, for example, **repeated addition**, grouping, arrays and **multiplication facts.**

-- **Applies strategies** to determine division facts, for example, **repeated subtraction**, equal groups, **sharing equally**, arrays and **multiplication facts.**

-- **Uses multiplication and division facts** to solve problems within the number range 0 to 1000.

-- Applies knowledge of **inverse operations** (addition and subtraction; multiplication and division).

Having determined which calculations are needed, I can solve problems involving whole numbers **using a range of methods, sharing my approaches and solutions with others.** (MNU 2-03a)
-- **Uses multiplication and division facts** to the 10th multiplication table.
-- **Multiplies and divides** whole numbers by multiples of 10, 100 and 1000.
-- **Multiplies** whole numbers by two-digit numbers.

Describes, continues and creates number patterns using addition, subtraction, **doubling, halving, counting in jumps (skip counting) and known multiples.** (MTH 1-13b)

I can use a variety of methods to solve number problems in familiar contexts, **clearly communicating my processes and solutions**. (MNU 3-03a)
-- **Recalls quickly multiplication and division facts** to the 10th multiplication table.
-- **Uses multiplication and division facts** to the 12th multiplication table.
-- Solves multiplication and division problems working with integers.

I can continue to **recall number facts quickly and use them** accurately when making calculations. (MNU 3-03b)

Figure 2: Landscape of relevant experiences and outcomes

The other aim in this course is to give them explicit permission to go beyond an 'age and stage' perspective for example if working with a class of 8-year-olds (first level) then they include consideration of those statements coded as 2nd or 3rd level. As can be seen in figure 2 there is an emphasis on recall of number facts, speed and accuracy, with a commensurate lack of emphasis on skills such as explaining, justifying. In fact, prior to the benchmarks being published in 2017, there was no mention of specific solution strategies beyond ‘use a variety of methods’ and even now most of those strategies referred to sit within additive reasoning and not multiplicative reasoning. The provisional colour coding in figure 2 was added by student teachers after having considered the other perspectives.

Maths Recovery

Maths Recovery approaches developed in Australia were used in the initial development of the course four years ago as these were common in Scottish primary schools. At that time there was a strong contingent of teachers trained in MR and although the formal professional development opportunities have declined the interest in the Stages for Early Arithmetical Learning has increased (Wright et al. 2012, 2015). Using individual scripted assessments to develop a profile of pupil knowledge leading to individual teaching plans is a powerful approach to using data to drive instructional decision-making. Although this work was developed in the context of intensive intervention with low-attaining pupils there is some anecdotal evidence from student teachers in schools of this being used successfully with small groups. Another useful aspect of the original training was the use of video-taped interviews with individual children to be analysed by the teacher and trainer to consider professional decision-making as well as pupil knowledge.

Maths recovery has a landscape with six domains of knowledge: number words and numerals; structuring numbers 1 to 20; conceptual place value; addition and subtraction to 100; multiplication and division; and written computation where the focus here is on the last two. Within this there are

(1) four aspects of development: knowledge of multiples, distancing the settings, sophistication of unitizing numbers and development of non-counting strategies;

(2) six phases: building on pupils' emergent strategies for multiplying and dividing; sequences of multiples; structuring numbers multiplicatively; developing multiplicative strategies; habituation of basic facts and extending to multi-digit factors and beyond 100 and

(3) five levels: initial grouping, perceptual counting in multiples, figurative composite counting, repeated abstract composite grouping and multiplication and division as operators.

The strong emphasis on assessment provides student teachers with a tight structure which early on in the additive reasoning course can be a benefit. However, the instructional designs do not offer the same possibilities. This approach, whilst it refers to progressive mathematization (Freudenthal 1991; Treffers and Beishuizen 1999) uses word problems rather than realisable rich contexts such as those within RME. It has a very tight focus on the mathematics itself that can provide a level of detail which is perhaps not as apparent in the other approaches in the early levels. However, this also produced some tension as it is so tightly structured that student teachers rarely 'tinkered' (Gravemeijer 1994). As in the formal curriculum, there is a foregrounding of size of numbers and the direction of travel moves quickly towards bare number tasks and formal algorithms.

Cognitively Guided Instruction

This approach has a strong focus on sense-making and a recognition that children can often solve problems beyond what might be expected by a formal curriculum (Carpenter et al. 2015). It emphasises teacher sense-making in parallel with that of children based on principles of respect and questioning. These last two are again considered both in relation to the children and the teachers simultaneously and this supports student teachers in strengthening connections between their work on campus and in schools. Moscardini (2014), working mainly within the additional support needs sector, has used and developed this approach in Scotland.

Table 1. Example of CGI mapping

Task Types		Multiplication (grouping)	Division (**Grouping**, quotative, measurement)	Division (**Sharing**, partitive)
Countable Objects (without remainders): Discrete		*Maggie has 5 bags of cookies. There are 3 cookies in each bag. How many cookies does Maggie have altogether?*	*Maggie has 15 cookies. She puts 3 cookies in each bag, how many bags can she fill?*	*Maggie has 15 cookies. She puts the cookies into 5 bags with the same number of cookies in each bag. How many cookies are there in each bag?*
Countable Objects (with remainders): Discrete	More		*13 people are going to the cinema. 3 people can ride in each car. How many cars are needed?*	*Example needed?*
	Less		*3 eggs are needed to bake a cake. How many cakes can you bake with 17 eggs?*	*Beth has 21 marbles. She wants to share them equally among the 5 children. How many marbles does each child get?*
	Remainders		*David has 16 tennis balls. Pack 3 balls in each box. How many will be left over?*	*Amy has 17 cakes. She gives each of the 3 children the same number of cakes. How many are left over?*
	Fractions		*Example needed?*	*Amy shares all 17 cakes among the 3 children. How many does each child get?*
Measurable Quantities: Continuous	Rates (time)	*A baby elephant gains 3 lbs. each day. How many pounds did it gain in 5 days?*	*A baby elephant gains 3 lbs. each day. How many days will it take to gain 15 pounds?*	*A baby elephant gains 15 lbs. in 5 days. If she gains the same weight each day, how much does she gain in 1 day?*
	Rates (cost)	*Pies cost £3 each. How much do 5 pies cost?*	*Pies cost £3 each. How many pies can you buy for £15?*	*Maggie bought 5 pies. She spent a total of £15. if each pie costs the same amount, how much did one pie cost?*
	Comparisons (ratio)	*The P1 class have a hamster and a gerbil. The hamster weighs 5 times as much as the gerbil. The gerbil weighs 3 ounces. How much does the hamster weigh?*	*The giraffe is 15 ft tall and the kangaroo is 3 ft tall. How many times taller is the giraffe than the kangaroo?*	*The giraffe is 15 ft tall. She is 5 times as tall as the antelope. How tall is the antelope?*
Symmetric Problems: (factors play an equivalent role)	Arrays (discrete)			
	Areas (continuous)	*A farmer plants a rectangular garden 3 m long and 5 m wide. How many square meters are there?*	*A farmer has enough space to plant a garden 5 m long. How wide does it have to be to have 15 square meters?*	
	Combinations (discrete)	*A café has 3 different types of ice cream cone and 5 different flavours of ice cream. How many different combinations could they make?*		

The intricate level of detail and methodical approach in terms of types of tasks (table 1) balances the focus on children's mathematical knowledge in the MR approach. There is also an explicit recognition and discussion of modelling although discussion of strategies is less well developed. Making sense of different task types can be particularly difficult when student teachers initially see these as simply multiplication or division. An adapted version of 'classifying mathematical objects' (Swan 2006) is used where the objects to be classified are the set of problems provided by Carpenter et al.

(2015). Then using video extracts, transcriptions and images of child-writing (Martin 2014) student teachers engage with ‘interpreting multiple representations’ (Swan 2006) where rather than representations of mathematical objects they are making connections between representations of children’s mathematical thinking.

Learning to listen to and make sense of children’s explanations and notations through the video-taped task-based interviews that accompany the book is crucial in allowing them to make sense of the differences between the various task types and their impact on how children solve a problem and how they model their solution. Table 1 gives an example of an initial mapping of their understanding of CGI.

Realistic Mathematics Education

The third perspective is based on the work of Hans Freudenthal (1968) who took a socio-cultural and constructivist approach to mathematics as a human activity: modelling reality so as to give a reason to use mathematical tools in an inquiry-based approach that values diversity. I refer to these as ‘ill-defined’ problems which Fosnot and Dolk (2001) explain as rich contexts where the teacher focuses on posing questions to stretch thinking. The use of ill-defined problems allows children to ‘come to understand’. Where commonly in Scotland word problems are used at the end of a piece of classwork as application here Fosnot and Dolk use ill-defined problems at the beginning as ‘construction’ to generate and explore mathematical ideas. As such, this perspective is the one which is most different from the most common practices within Scottish primary classrooms. This involves problem-posing as well as problem-solving where learners are posing questions, defending conjectures, listening to others. It emphasises the need for a community built on trust and respect and ‘everyone’ (Florian and Spratt 2013), values risk-taking and supports mathematical discussion i.e. a rich context is only rich in that particular environment/community – another teacher could use the same task and not experience the same level of discussion. The approach involves context-based investigations, inclusive problems that all pupils have access to and from which they learn different things while being equally challenged. These often have the structure of a set of connected narrative problems with key mathematical structures (big ideas) as the focus of the learning. This way of working requires the teacher to walk the edges between the structure of mathematics and the mathematical activity of individual learners as well as between the individual and the community.

These situations need to have the potential to be modelled; they allow children to realise what they are doing (to picture or imagine something concretely); they prompt learners to ask questions, notice patterns, wonder, ask why and what if (Fosnot and Dolk 2001, 23). Within these situations, decisions can be made to build in constraints that encourage more formal mathematical strategies (or higher-level strategies), to restructure informal intuitive ideas. Contexts become investigations when patterns in the data lead to learners proposing conjectures (relationships), where they want to explain the connections they notice. Another key aspect to this approach is the integration of size and spatial structure of the numbers involved e.g. smaller numbers on an array are likely to encourage those who count-by-ones to skip count as numbers less than 5 are more often subitised. The approach is more explicit about how to support key shifts in thinking.

In relation to multiplicative reasoning, Fosnot and Dolk (2001) focus primarily on realistic contexts that involve arrays (e.g. fruit boxes, egg cartons, windows etc.) which means that the big ideas of the associative and commutative principles seem relatively straightforward. This leap into array modelling means that there is space in the landscape that can be considered from the other two perspectives. There are examples of discrete and continuous data but these are less explicit about the shift than the CGI approach. Similarly, they reference combinations and permutations as well as ratios and proportion although not in great depth. The use of realistic contexts (contexts that can be imagined, that is realised) also allow remainders to be considered much earlier than would normally happen in Scottish classrooms.

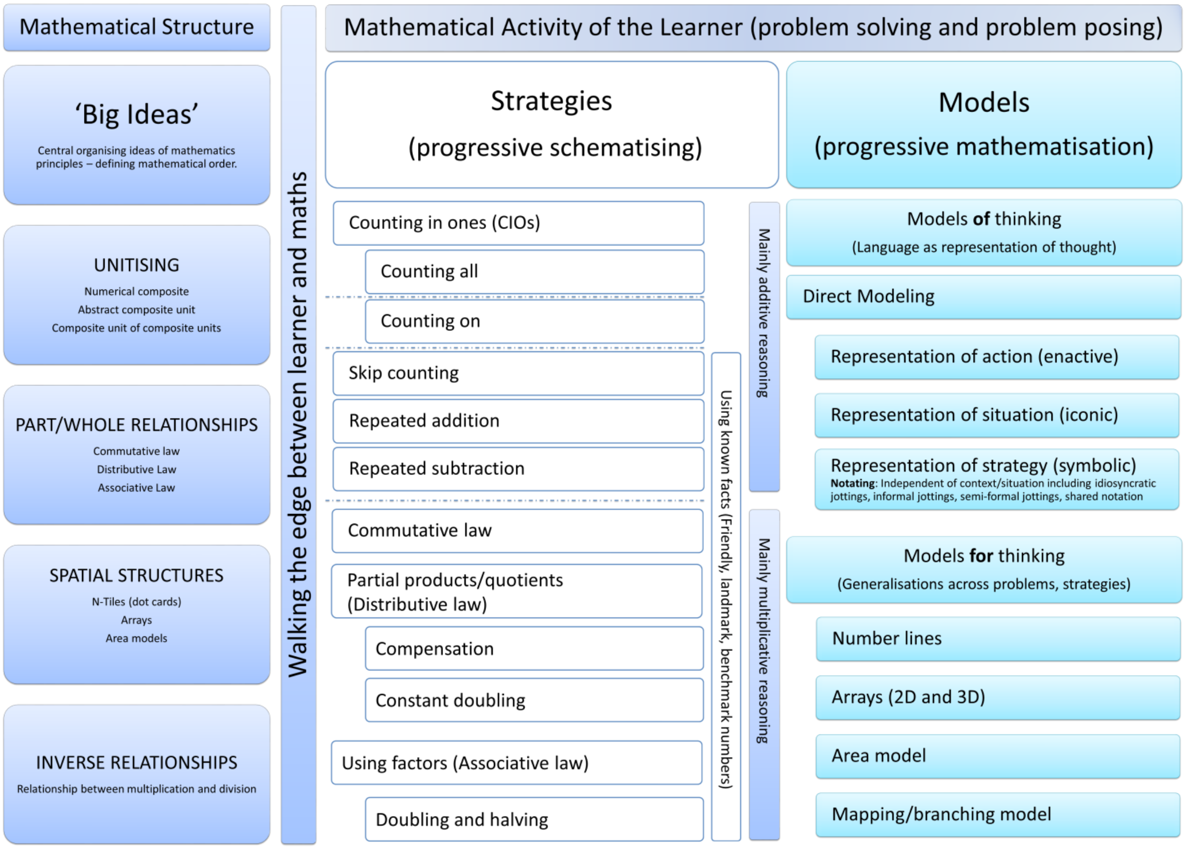

Figure 3: Example of landscape produced

Fosnot and Dolk also consider mini-lessons which are more teacher-directed, short focused approaches such as 'number strings.' The latter are a structured set of computation problems: written up one at a time; strategies used are discussed before the next problem is presented so that children can consider strategies as well as numbers – prompting them to think about relationships between problems.

Across both investigations and mini-lessons there is a detailed discussion about mathematical structuring (big ideas), schematizing (strategies) and modelling (mathematising) within the notion of landscapes (figure 3) and hypothetical learning trajectories. Within this, the approach highlights the idea of variation and multiple experiences that allows children to reflectively abstract and generalize (Watson and Mason 2005). For example, this can occur by exploring different contexts in which the same model is likely to be used or using the same numbers in different contexts to explore strategies or using a set of number tasks that have the same solution in order to bring a particular strategy to their attention.

In summary, the formal curriculum gives a skeletal frame focused on the development of number types but with very little detail on strategies, models or mathematical structures. In contrast Maths Recovery, with its strong emphasis on assessment, provides more detail and structure in terms of strategies and moving between word problems with and without manipulatives to bare number tasks. However, its conventional approach to frameworks such as Learning Framework in Number is a weakness in developing student teachers' pedagogical design flexibility. This weakness is compensated by the CGI work with its detailed discussion of task types and modelling where the complementary strength is in its focus on children's sense-making and teacher questioning. A weakness of CGI is its reliance on short contextual tasks which is compensated by RME and its development of sense-making in the design of investigations with a 'realisable' context that allow all children access to the problem. This approach also opens up the possibility of using ill-defined problems as a way to generate mathematical ideas rather than the application of previously learned procedures. The complementary strengths across all three different perspectives, cluster around the pupil strategies encouraging an appreciation of the diversity and the varying terminology used.

Having discussed the formal curriculum and three approaches, this paper now turns to develop a better understanding of how task design in a mathematics teacher education classroom impacts on task design in school classrooms.

Methods

This educational design research (Gravemeijer and Cobb 2006) takes an iterative approach to studying and refining task design within a teacher education classroom: to develop our understanding of:

--how does this embodied approach to task design within teacher education affect student teachers' pedagogical design capacity?

The design principles, valuing diversity, developing flexibility and making connections, are based on mathematics as a constructive activity (Watson and Mason 2005) with an emphasis on mathematical practices and a more participatory view of learning and teaching mathematics.

The project is based in the North-East of Scotland with year 3 undergraduate student teachers (pre-service teachers) and a mathematics education course. Each semester consists of six 3-hour sessions followed by five weeks in a primary school where the second semester course is focused on multiplicative reasoning. The task design in this latest cycle has been discussed earlier and the course assignment requires student teachers to report on their analysis of the children's multiplicative reasoning based on their video-taped task-based interviews with a small group of children. The data has been collected over the past four years throughout the iterative changes made to the course. This paper illustrates the latest cycle of the design research using multimodal analysis of data provided by the most recent cohort of 38 students during academic session 2017/18 working with children aged 8-12 years old: student teacher assignments and planning documentation (n=36), videos and transcripts (n=62) of task-based interviews with children and images of child-writing are deployed to better understand how the design of the course influences the tasks proposed by student teachers.

The cohort included 38 student teachers of which 36 completed the course on schedule and 2 completed later in the year after the data had been collated. In Scotland, primary education goes from primary 1 to 7 (P1 – P7) where children normally start school at age 4 or 5 years old. The students were placed in a variety of different year groups (table 2) and the ages of the children are given in the brackets. Nine of the student teachers were placed in a composite class with more than one year-group.

Table 2. Distribution of student teachers to primary year-groups

	P3 (7)	P4 (8)	P5 (9)	P6 (10)	P7 (11)
Number of student teachers	2	8	11	14	10

Analysis and Findings

All 36 assignments and 62 video files were analysed within NVivo using the following nodes for task types in line with table 3: mathematical operations (multiplication, division, partitive division, quotative division); mathematical type (discrete without remainders; discrete with remainders as more, less, remainder or fractional; continuous data as no context, rates or ratio; symmetric as arrays, areas or combinations).

Mathematical Operation	Mathematical Type										
	Discrete without remainders	Discrete with Remainders				Continuous quantities			Symmetric		
		more	less	remainder	fractional	Continuous (no context)	Rates (time, cost)	Ratio (comparison)	Arrays (discrete)	Areas (continuous)	Combinations (discrete)
Multiplication	61					5	2		13	3	
Division (no context)	10				1	4					
Partitive division	23			2	1	1		1			
Quotative division	31	26	7	1		1	1				

Table 3. Mathematical type against operation

From the number of references across all sources (table 3), there is a predominant use of discrete quantities (countable objects) across both multiplication and division problems. Of note is the use of division tasks involving remainders across all year-groups where traditionally these would only occur in the later years. Where remainders are involved these are most commonly used in quotative problems where the context requires the solution to be rounded up (more). The extract below from ST18 is a typical example of these tasks.

> "I have a box of juice bottles; each box has twelve bottles. I have sixty-six customers, all wanting a bottle of juice. How many boxes will I need and will I have any juice bottles left?" (1718ST18: P4)

Where continuous quantities have been used these are primarily with no context i.e. bare number tasks. Where arrays and areas are used these are only with multiplication problems and there were no examples of the use of combination problems with this group of student teachers. The only noticeable difference between year-groups is the use of continuous quantities that only appears once with P5 children and the remaining tasks are with P6 and P7.

The pedagogical type was originally coded as bare number task, investigation or number talk. However, when analysing the videos with this latter group it became clear that the delineation between these nodes was not distinct enough. Many student teachers would write about using number talks but were focusing on the relational understanding between the numbers in one task rather than the relational understanding between a series of connected tasks. As a result, the coding used ‘number talks’ to indicate the discussion around strategies and models for a single bare number task and ‘number strings’ (Fosnot and Dolk 2001) was used to identify a connected series of bare number tasks. Similarly, many task-based interviews were recoded as ‘contextual tasks’ as used in the CGI approach rather than investigations where there was no evidence of a set of connected tasks where the patterns in the data led to children proposing conjectures (relationships) or explaining the connections they noticed.

In table 4 below, pedagogical approaches were identified as Bare tasks (B) including number talks, number strings and bare number tasks; Contextual tasks (C) and Investigations (I) where co-occurrence is noted by eg B & C indicating the use of both bare and contextual tasks. The mathematical operations are multiplication; partitive division and quotative division with a similar recognition for co-occurrences.

Table 4. Number of student teachers using pedagogical type against mathematical operation

Pedagogical Approaches	Mathematical Operations							
	Multiplication	Partitive Division	Quotative Division	Mult. & Partitive Division	Mult. & Quotative Division	Partitive & Quotative Division	Mult., Partitive & Quotative Division	Subtotal
Bare number tasks (B): incl. number talks and number strings	3						1	4
Contextual tasks (C)		2	2	2	2	2	2	12
Investigations (I)	1							1
B & C	2			2	1	1	4	10
B & I	1				3			4
B,C & I	1				2		2	5
Subtotal	8	2	2	4	8	3	9	36

In terms of mathematical operations 24 (67%) of the student teachers designed tasks that used more than one operation. This is important in terms of decision-making as traditionally in Scottish primary schools these operations are taught separately. Similarly, with pedagogical approaches there is evidence of decision-making where 20 (56%) student teachers used more than one approach and only 4 students used bare tasks alone. The lack of structural understanding of designing investigations leading to recoding meant that contextual tasks were far more common with 27 (75%) students. When looking across both operations and approaches there were 28 (78%) student teachers who worked with more than one operation and/or approach. This data was also considered with respect to year-groups however there were no noticeable differences.

The following discussion will focus on two student teachers, Grace (1718ST25: P7) and Lucy (1718ST27: P6), from the most recent cohort to illustrate the key findings of this analysis.

Discussion

Grace worked with a group of five P7 children in a large urban school with 497 pupils and 31 staff and Lucy worked with a group of seven P6 children in a very remote small primary school with 52 pupils and 5 staff on the west coast of Scotland. Both students designed tasks and analysed children’s mathematical thinking (figure 4) involving both multiplication and division with discrete quantities (with and without remainders) and symmetric problems with arrays. They both experimented with investigations and bare number tasks.

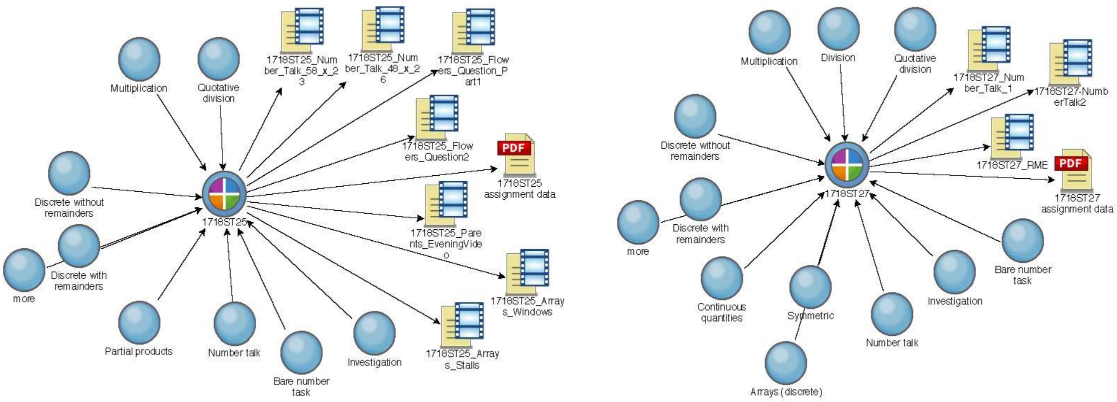

Figure 4 NVivo coding for Grace (ST25) and Lucy (ST27)

One of the more obvious influences of the course has been a refocusing on solution strategies and models (figure 3) used by children and the promotion of

There are 78 parents attending parent's evening next Wednesday. The school has decided to order in refreshments. The company that the school will be buying from sell crates

The school has also decided to buy in squeezable coffee pods. Each coffee pod can make 15 cups of coffee. The school has ordered a huge bulk batch of 264 pods to keep for big events

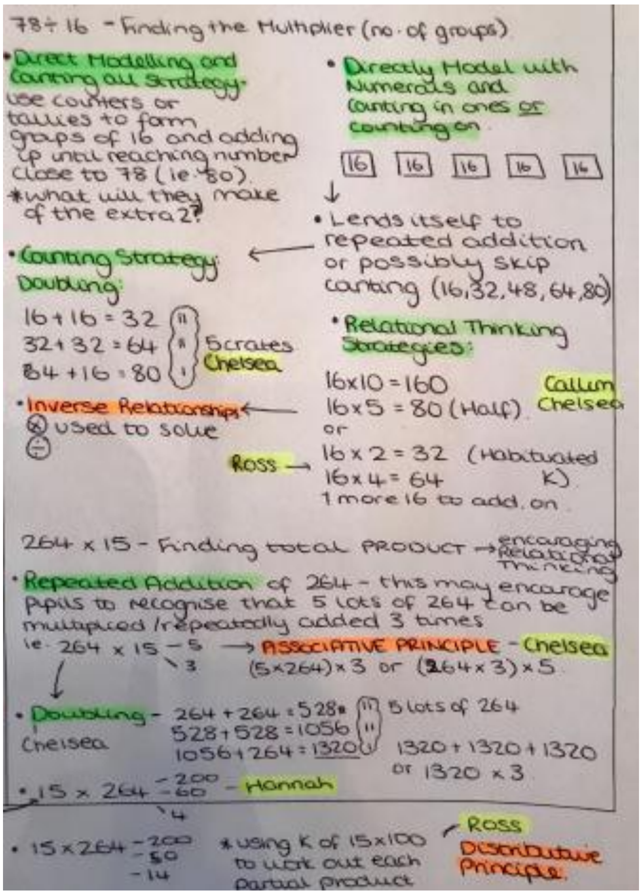

Figure 5: Planning example from Grace

discussion based on these that was evident across all student teachers. Examples of planning, using an annotated structure, from Grace (figure 5) and Lucy (figure 6) illustrate some of the tasks designed alongside possible strategies and models that children might use in solving the problems.

In the planning example above, Grace has added the names of pupils, Chelsea, Callum, Ross and Hannah, who have used some of the strategies. Grace uses selected transcripts of the children solving problems in her assignment to evidence her understanding of strategies, models and the impact of her task design. The extract below from the first problem demonstrates a participatory view of doing mathematics exemplified in the way that Grace responds to Callum, 'how could we use that to help us work out…', where she then steps back to leave the rest of the discussion, questioning and explaining, between two of the pupils.

Callum: Sixteen times ten…that would not work because it's way over. It's one hundred and sixty.

T: One hundred and sixty.

Callum: But we only need seventy-eight bottles.

T: So, you know that sixteen times ten is one hundred and sixty, how could we use that to help us work out…

Chelsea: Oh…You can half it, which is eighty…So you can then, so you then times it by five and then you've got two spare bottles.

T: You get that Callum?

Callum: Wait, what do you times by five?

Chelsea: But no, so you half so you did sixteen times ten which is one hundred and sixty and half of one hundred and sixty is eighty, and then so and half of ten is five so you do sixteen times five which is eighty. So we can't get exactly seventy eight but we can still get eighty.

Chelsea: Oh…could you do, well could you work out what sixteen times two is?

T: yeah

Chelsea: Which is thirty-two, and then you double it, which is sixty-four

T: yeah...

Chelsea: And then you see how many six, and then how many more sixteen's you need to get from sixty four so six add the four is seventy and then the seventy add the ten is eighty, so then that's another sixteen. So that's five.

The tasks themselves are similar to ones from the texts, based on a parent's evening and visits, and this minimal 'tinkering' (Gravemeijer 1994) is quite common. It also demonstrates two limitations; the first is that their annotations are fairly similar to the example used during the course. By using one example of planning it is possible that we have restricted the student teacher's capacity to go much beyond what is given. The second limitation is the nature of the programme as a whole in that we are not normally in contact with student teachers once they are out in school given the range of requirements so there are missed opportunities to work together on their task design.

In describing her tasks, Grace moves between quotative and measurement as equivalent labels used by different authors. She notices that including tasks with remainders within a realisable context means that children can deal sensibly with these eg where Chelsea says ‘you’ve got two spare bottles’. This meant that Grace was comfortable designing the second task with numbers beyond the children’s experience and the expectation of the formal curriculum (264 × 15), ‘I was surprised by the diverse ways in which this problem was tackled.’

Within the assignment, Grace recognizes and highlights the children’s understanding of mathematical structures (figure 3) including the inverse relationship between division and multiplication, the distributive law and associative law in the second problem. Alongside this level of understanding, she also illustrates balanced reflections on the children’s learning and her own. The extract below is from Grace’s reflections about a bare number task, 58 × 23, where her question was ‘how many ways can you work out this problem?’ and demonstrates how deep seated a teacher’s own experience of solving a problem can be:

> “I got quite confused as the pupils began to work out 60 x 23. I was convinced that factoring 60 into 6 and 10 and then working out 23 x 6 x 10 would not work as I was thinking of splitting 60 into two addends instead of two factors. This was a huge learning curve for me personally. My confusion really got the pupils thinking and questioning, which meant my confusion benefited both myself as well as the pupils although I did not give the children the opportunity to talk to each other much about why I was incorrect. Next Steps: Try not to talk too much and instead of telling the children that something doesn’t work, ask them to explain it so that they have the chance to recognize this for themselves. In this case, if I had asked questions like ‘Why have you done 26 x 6 then times by 10’ to Chelsea, I would have made sense of this using Chelsea’s explanation instead of confusing myself and the group.”

Lucy did a lot of work with a number talks approach working with decimal numbers at the request of the class teacher. She realised that these types of tasks had become detached from reality and the children's explanations became rule-based for example, when solving 26.4 ÷ 8 Sarah offered 'You could times it by 10, then divide that by 8 and put the decimal point back in.' Lucy later adapted the idea of local context to develop a set of problems for the children. The first of these investigations is given in figure 6.

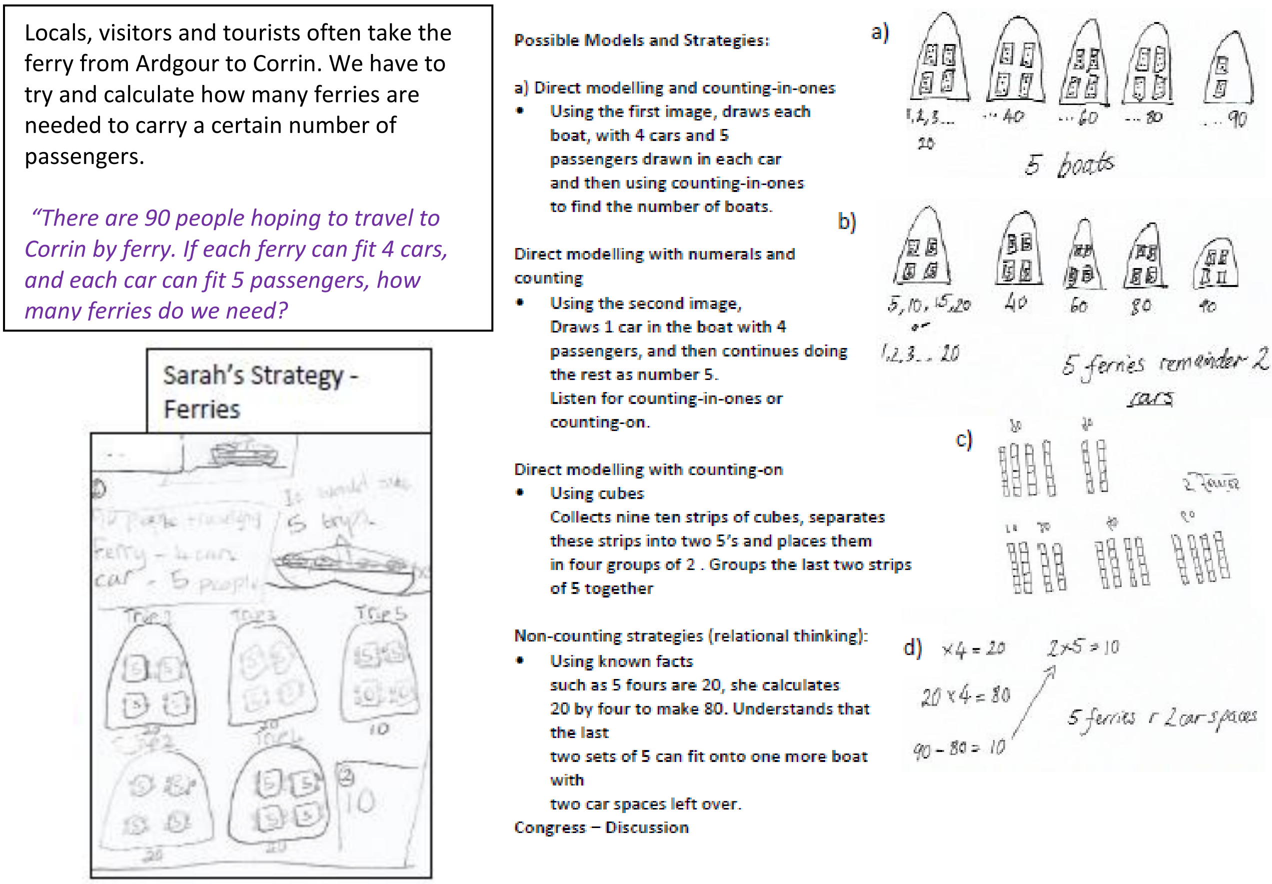

Figure 6: Examples from Lucy

Here we see her capacity to design using her own context and based on the children's experiences with an example drawn from the local ferries to the islands off the west coast of Scotland. Lucy selected the excerpt below from the quotative task above to illustrate Sarah's strategy, modelling (figure 6) and reactions to remainders.

Sarah: I drew out… you wouldn't need as many as 18 ferries you would only need four and a half.

T: Okay, why would you need… half a ferry, would you need half a ferry?

Steven: You can't get half a ferry, it would have to be 5.

T: So, can we all listen to (Sarah's) strategy? So what did you do?

Sarah: Well, I drew them all out… I did trip one, I drew a kind of oval shape, like that one, trip one. Points to my exemplar drawing.

Sarah: And then I did… I counted up how many people, there would be 20.

T: 20 here.

Sarah: 20 in trip one. And then trip two there would also be 20, and I drew little cars.

T: So you had another 20 there, did you draw your cars? What did you put in the car?

Sarah: Just the number 5. And then in trip three I did the same and wrote 20 people more.

[…]

Sarah: There was 80, and then I thought like, I would only need half a ferry next. I drew out all the car spaces and I just put the number 0 instead of 5.

T: So you basically had, you had the ferry, and then you had two cars with 5 people in them.

Sarah: And then I did two other cars, because I didn't really draw them as cars I drew them as car spaces, rather than the cars. So I did two with 0's.

T: So how many free spaces do you have on the fifth boat?

Steven: 2. Yeah, and you needed 5 ferries, I was right!

Both Grace and Lucy demonstrate that it is possible for student teachers to take informed risks, to try different approaches and analyse the different responses of the children. I will leave the last word to Lucy:

> It was clear to see that in both of these relatable activities, children were generally much more motivated and intrigued to find the solution. Thus, my hypothetical next steps with this group, as well as with future classes, would be to continue to use the

RME approach, gradually making problems more complex yet still related to the individual's real-life activities.

This multimodal approach, using video, transcripts and images, to analyse has been a key benefit for student teachers and teaching colleagues. Student teachers have become more comfortable using many of the labels and noticing more (Mason and Johnston-Wilder 2004) in a way that is grounded in their reality of teaching. Those marking the assignments benefit from seeing a student's understanding from both what they write and what they do in the classroom.

Conclusion

The aim of this paper is to develop a better understanding of how this embodied approach to task design within teacher education affects student teachers' pedagogical design capacity. The construction of synthetic landscapes of multiplicative reasoning and the use of this as an analytic framework are tasks designed around 'problematic situations that are experientially real for [student teachers]' (Watson and Ohtani 2015, 41). As a result of designing the multimodal assessment, we can see how student teachers interact with learners, the questions they ask, the tasks they design aligned with mathematics as a constructive activity (Watson and Mason 2005)

The findings indicate that all student teachers engaged in a participatory view of learning mathematics, discussing a range of strategies and models used by the children. This focus on children's sense-making (Carpenter et al 2015) values diversity and develops both children's and teacher's flexibility. Many student teachers experimented with different pedagogical approaches although very few understood the underlying relational structures of number strings or context-based investigations (Fosnot and Dolk 2001).

This embodied approach to task design within teacher education has enabled student teachers to build their awareness of themselves as mathematicians, researchers and teachers grounded in their reality of working with children. They have gone beyond 'sheer imitation' (Dewey 1910) to develop their pedagogical design capacity (Brown 2009) with an interest in people-in-action (Sfard 1998). Through the construction of their own landscapes of multiplicative reasoning they discern more closely and sensitively the children's reasoning (Mason & Johnston-Wilder 2004) and will hopefully sustain this ethical decision-making approach in their own professional lives.

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