The existence of Aubry-Mather sets for the Fermi-Ulam model

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Abstract

We consider the Fermi-Ulam model, which can be described as a particle moving freely between two vertical rigid walls; the left one being fixed, whereas the right one moves according to a regular periodic function. The particle is elastically reflected when hitting the walls. We show that the dynamics of the model can be described by an area-preserving monotone twist map. Thus, the Aubry-Mather sets exist for every rotation number in the rotation interval. Consequently, this gives a description of global dynamics behavior, particularly periodic orbits and quasiperiodic orbits of a wide class for the model.

Keywords: Fermi-Ulam model, Aubry-Mather set, twist map

1. Introduction

A widely studied one-dimension impact system is the Fermi-Ulam model. It was originally proposed by Fermi in an attempt to study high-energy cosmic rays [1]. Ulam [2] developed a simplified model which is described as a particle confined in and bouncing between two rigid walls, one of which is fixed and the other moves in time. The main question about this model is whether particle will have unlimited energy? The answer is no when the motion of the wall is sufficiently smooth, since in such case there exist invariant curves in phase space. These curves act as barriers such that all motions have bounded velocity, see [3]. The model was later studied in different versions and using different approaches. Pustylnikov [4] considered the case when one wall is fixed and the other wall oscillates according to a periodic analytic function and proved that the velocities of the ball are always bounded for all initial values. In the general case when both walls move according to periodic analytic functions, Pustyl'nikov [5] proved the existence of invariant curves. Laederich and Levi [6] considered the case when both walls move according to a periodic C^5 function and the existence of invariant curves is proved by Moser's small twist theorem.

The Fermi-Ulam model in an external gravitational field and the possibility of unbounded motion is studied in [7]. Kunze and Ortega [8] considered the Fermi-Ulam model when one plate is fixed and the other one moves according to a given non-periodic function.

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Under some assumptions, the existence of infinitely many bounded orbits and unbounded orbits are established, and the unbounded orbits follow bounded orbits for long times. Some extensions to the quasi-periodic case with diophantine conditions can be found in [9]. Zharnitsky [10] found linearly escaping orbits in a piecewise linear Fermi-Ulam model. In a piecewise Fermi-Ulam model with one singularity, De Simoi and Dolgopyat [11] showed that bounded orbits co-exist with escaping ones.

In general, KAM theory gives the stability results of an integrable system under small perturbations [12, 13]. However, dynamical persistence under large perturbations is also important. This is because the behavior that persists under large perturbations is very fundamental to the system. Thus, a key question is what happens to the KAM invariant curves when the perturbation away from the integrable case is large. Aubry-Mather theory provides an answer to this question. Their study of the dynamics of an area-preserving monotone twist map of the annulus pointed out the existence of many invariant sets, which are obtained by means of variational methods [14, 15]. Actually, these sets are more plentiful than invariant curves, because they exist for every twist diffeomorphism and for every rotation number. On the other hand, the existence of invariant curves, proved by the KAM theory, requires further hypotheses.

In this paper, we study the existence of Aubry-Mather sets for the Fermi-Ulam model. We assume that the wall moves according to a periodic function of C^3 class. The remaining of this paper is organized in the following way. In Section 2, we present the Fermi-Ulam model in detail and establish its Poincaré map. In Section 3, we construct a generating function for the model and extend it to the whole plane. The existence of Aubry-Mather sets is proved in Section 4. The last section summarizes the results of the paper.

2. The model and its Poincaré map

Fermi-Ulam model which is described as a particle moving freely between two vertical perfectly reflecting walls, the left one being fixed at $x = 0$, whereas the right one moves according to a C^3 and 1−periodic function $p(t)$. Suppose that $0 < \inf p(t) = a$. At an instant τ of impact, the change of velocity is assumed to be elastic collision. See Fig. 2. So we consider the problem

$$\begin{cases} \ddot{x} = 0, \quad 0 \le x(t) \le p(t), \\ x(\tau) = 0 \Rightarrow x(\tau^+) = x(\tau^-), \\ x(\tau) = p(\tau) \Rightarrow \dot{x}(\tau^+) = -\dot{x}(\tau^-) + 2\dot{p}(\tau). \end{cases} \tag{1}$$

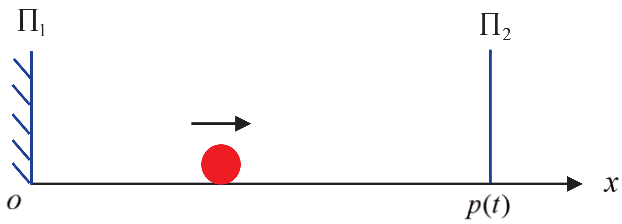

Figure 1: The Fermi-Ulam model

As in [16], by a solution of (1) we understand a function $x \in C(\mathbb{R})$ and a sequence $(\ldots, t_n, \ldots)$ of impact times such that

(i) $\inf_{n\in\mathbb{Z}}(t_{n+1}-t_n)>0$;

(ii) $x(t_n)=0$ for all $n\in\mathbb{Z}$ and $0<x(t)<p(t)$ for $t\in(t_n,t_{n+1})$;

(iii) the function x is of class C^2 and satisfies $\ddot{x}=0$ on every interval $[t_n,t_{n+1}]$;

(iv) $\dot{x}(t_{n+1}^+)=-\dot{x}(t_{n+1}^-)$ and $\dot{x}(\tilde{t}_n)=-\dot{x}(t_n)+2\dot{p}(\tilde{t}_n)$ hold for all $n\in\mathbb{Z}$, where $\tilde{t}_n\in(t_n,t_{n+1})$.

The problem (1) can be formulated in a discrete form. Let Π_1 and Π_2 denote the left and the right wall, respectively. Assume that the initial velocity of the particle is sufficiently large such that two consecutive collisions occur with different walls. Suppose that the particle collides with Π_1 at time t_n and attains a velocity $\dot{x}(t_n^+)=v_n>0$. Then it moves from Π_1 to Π_2 with the velocity v_n and at time $\tilde{t}_n$ the particle reaches Π_2. After collision with Π_2, the particle attains a velocity $\dot{x}(t_n^+)=\tilde{v}_n<0$. Continue this process, the particle moves from Π_2 to Π_1 with the velocity $\tilde{v}_n$ and at time t_{n+1} it reaches Π_1. After the particle collides with Π_1, it attains a velocity $\dot{x}(t_{n+1}^+)=v_{n+1}>0$, see Fig. 2.

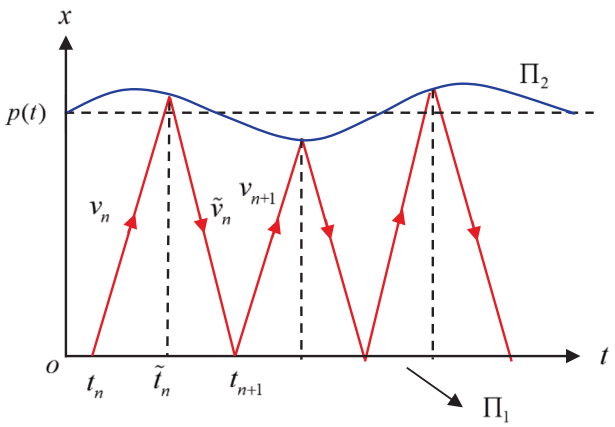

Figure 2: The Fermi-Ulam model

According to the above analysis, we have

$$p(\tilde{t}_n)=v_n(\tilde{t}_n-t_n),\quad \tilde{v}_n=-v_n+2\dot{p}(\tilde{t}_n),\quad -p(\tilde{t}_n)=\tilde{v}_n(t_{n+1}-\tilde{t}_n),\quad v_{n+1}=-\tilde{v}_n. \tag{2}$$

Thus, we obtain the map $(t_n,v_n)\mapsto(t_{n+1},v_{n+1})$ defined as

$$\begin{cases} t_{n+1}=t_n+\dfrac{p(\tilde{t}_n)}{v_n}+\dfrac{p(\tilde{t}_n)}{v_{n+1}},\\ v_{n+1}=v_n-2\dot{p}(\tilde{t}_n). \end{cases}$$

Without loss of generality, we will take $n=0$. Then we obtain the following map $(t_0,v_0)\mapsto(t_1,v_1)$

$$\begin{cases} t_1=t_0+\dfrac{p(\tilde{t}_0)}{v_0}+\dfrac{p(\tilde{t}_0)}{v_1},\\ v_1=v_0-2\dot{p}(\tilde{t}_0). \end{cases} \tag{3}$$

Lemma 2.1. *There exists a sufficiently large $v^*>0$ such that if $v_0>v^*$ and for every $t_0\in\mathbb{R}$ then the map $(t_0,v_0)\to(t_1,v_1)$ is well defined and is C^3.*

Proof. By (2), we have

$$\tilde{t}_0 = t_0 + \frac{p(\tilde{t}_0)}{v_0}. \tag{4}$$

Consider the function

$$F(\tilde{t}_0, t_0, v_0) = \tilde{t}_0 - t_0 - \frac{p(\tilde{t}_0)}{v_0}.$$

Then

$$\partial_{\tilde{t}_0} F(\tilde{t}_0, t_0, v_0) = 1 - \frac{\dot{p}(\tilde{t}_0)}{v_0} > 0$$

for v_0 sufficiently large. Now, for every (t_0, v_0) we get a unique $\tilde{t}_0 = \tilde{t}_0(t_0, v_0)$ such that the equation (4) is solved. Moreover, by applying the implicit function theorem we have, by uniqueness, that $\tilde{t}_0(t_0, v_0)$ is a C^3 function. Furthermore, it follows from (4) that

$$0 < \partial_{t_0} \tilde{t}_0 = \frac{1}{1 - \frac{\dot{p}(\tilde{t}_0)}{v_0}} < 1. \tag{5}$$

In view of (3), consider

$$\begin{aligned} G(t_1, t_0, v_0) &= t_1 - t_0 - \frac{p(\tilde{t}_0)}{v_0} - \frac{p(\tilde{t}_0)}{v_1} \\ &= t_1 - t_0 - \frac{p(\tilde{t}_0)}{v_0} - \frac{p(\tilde{t}_0)}{v_0 - 2\dot{p}(\tilde{t}_0)}. \end{aligned}$$

Then, in combination with (5), we have

$$\partial_{t_1} G(t_1, t_0, v_0) = 1 - \frac{\dot{p}(\tilde{t}_0)\partial_{t_0}\tilde{t}_0}{v_0} - \frac{\dot{p}(\tilde{t}_0)\partial_{t_0}\tilde{t}_0(v_0 - 2\dot{p}(\tilde{t}_0) + 2\ddot{p}(\tilde{t}_0)\partial_{t_0}\tilde{t}_0}{(v_0 - 2\dot{p}(\tilde{t}_0))^2} > 0$$

for v_0 sufficiently large. Now, for every (t_0, v_0) we get a unique $t_1 = t_1(t_0, v_0)$ such that the first equation of (3) is solved for v_0 sufficiently large. Moreover, applying the implicit function theorem we have, by uniqueness, that $t_1(t_0, v_0)$ is also a C^3 function. □

Since we intend to apply Aubry-Mather theory which concerns symplectic maps, the map $(t_0, v_0) \mapsto (t_1, v_1)$ from (3) is not symplectic, a short computation shows that $v_1 dv_1 \wedge dt_1 = v_0 dv_0 \wedge dt_0$, this fact suggests the use of new variables: time t and kinetic energy $E = \frac{1}{2}v^2$. Then we obtain from (3) a map S given by

$$\begin{cases} t_1 = t_0 + \dfrac{p(\tilde{t}_0)}{\sqrt{2E_0}} - \dfrac{p(\tilde{t}_0)}{\sqrt{2E_1}}, \\ E_1 = E_0 - 2\sqrt{2E_0}\dot{p}(\tilde{t}_0) + 2\dot{p}(\tilde{t}_0)^2. \end{cases}$$

3. The generating function

In order to calculate the generating function of the map S, we consider the Dirichlet problem

$$\begin{cases} \ddot{x} = 0, \quad x(t_0) = x(t_1) = 0, \\ x(\tilde{t}) = p(\tilde{t}), \quad \dot{x}(\tilde{t}^+) = -\dot{x}(\tilde{t}^-) + 2\dot{p}(\tilde{t}). \end{cases} \tag{6}$$

Let $w = -\dot{x}(\tilde{t}-)$. The solution of equation (6) with initial velocity w can be divided into two parts, which are given by

$$x(t) = w(t - t_0), \qquad t \in [t_0, \tilde{t}], \tag{7}$$

and

$$x(t) = (-w + 2\dot{p}(\tilde{t}))(t - \tilde{t}) + w(t - t_0), \quad t \in [\tilde{t}, t_1]. \tag{8}$$

It follows from (7) and the condition $x(\tilde{t}) = p(t)$ that

$$w = \frac{p(\tilde{t})}{\tilde{t} - t_0}. \tag{9}$$

By (8) and the condition $x(\tilde{t}) = 0$, we have

$$w = \frac{2\dot{p}(\tilde{t})(t_1 - \tilde{t})}{t_0 + t_1 - 2\tilde{t}}. \tag{10}$$

From (9) and (10), i.e., the continuity of x at $t = \tilde{t}$, we get

$$2\dot{p}(\tilde{t})(t_1 - \tilde{t})(\tilde{t} - t_0) = p(\tilde{t})(t_0 + t_1 - 2\tilde{t}). \tag{11}$$

Consider the action

$$h(t_0, t_1) = \int_{t_0}^{t_1} \frac{1}{2}\dot{x}(t; t_0, t_1)^2 dt.$$

The relations (9) and (11) imply that

$$\begin{aligned} h(t_0, t_1) &= \frac{w^2}{2}(\tilde{t} - t_0) + \frac{1}{2}(-w + 2\dot{p}(\tilde{t}))^2(t_1 - \tilde{t}) \\ &= \frac{1}{2}\frac{p(\tilde{t})}{\tilde{t} - t_0} + \frac{1}{2}\Big(-\frac{p(\tilde{t})}{\tilde{t} - t_0} + 2\dot{p}(\tilde{t})\Big)^2(t_1 - \tilde{t}) \\ &= \frac{p(\tilde{t})^2}{2(\tilde{t} - t_0)} + \frac{p(\tilde{t})^2}{2(t_1 - \tilde{t})}. \end{aligned}$$

Lemma 3.1. *If $t_1 - t_0 > 0$ is sufficiently small, then the relation in (11) has a unique solution $\tilde{t} = \tilde{t}(t_0, t_1) \in (t_0, t_1)$. Furthermore, $\tilde{t}$ is a C^2 and $1-$periodic function.*

Proof. Write $\tilde{t} = \frac{t_1+t_0}{2} + \theta$ for $|\theta| < \eta = \frac{t_1-t_0}{2}$. Then (11) is equivalent to

$$\psi(\theta) = \theta p\Big(\frac{t_1 + t_0}{2} + \theta\Big) + (\eta^2 - \theta^2)\dot{p}\Big(\frac{t_1 + t_0}{2} + \theta\Big) = 0. \tag{12}$$

Note that for $t_1 - t_0$ small enough, we have

$$\dot{\psi}(\theta) = (\eta^2 - \theta^2)\ddot{p}\Big(\frac{t_1 + t_0}{2} + \theta\Big) - \theta\dot{p}\Big(\frac{t_1 + t_0}{2} + \theta\Big) + p\Big(\frac{t_1 + t_0}{2} + \theta\Big) \geq \frac{a}{2} > 0, \ |\theta| < \eta. \tag{13}$$

Furthermore,

$$\begin{aligned} \psi(\eta) &= \psi(0) + \textstyle\int_0^{\eta} \dot{\psi}(\theta)d\theta \geq \eta^2\dot{p}(\frac{t_1+t_0}{2}) + \frac{a}{2}\eta > 0, \\ \psi(-\eta) &= \psi(0) + \textstyle\int_0^{-\eta} \dot{\psi}(\theta)d\theta \leq \eta^2\dot{p}(\frac{t_1+t_0}{2}) - \frac{a}{2}\eta < 0. \end{aligned} \tag{14}$$

By (14), there exists a unique θ_0 in $(-\eta, \eta)$ such that $\psi(\theta_0) = 0$. Therefore, the unique solution to (12) is $\tilde{t} = \frac{t_1+t_0}{2} + \theta_0$, and $\tilde{t} \in C^2$ follows from (13) and the implicit function theorem. Moreover, by (11) we obtain

$$\tilde{t}(t_0, t_1) = \tilde{t}(t_0 + 1, t_1 + 1).$$

This complete the proof of the lemma. □

From a directly but long calculation (see Appendix), we have

$$\partial_{t_0} h(t_0, t_1) = \frac{p(\tilde{t})^2}{2(\tilde{t} - t_0)^2}, \quad \partial_{t_1} h(t_0, t_1) = -\frac{p(\tilde{t})^2}{2(t_1 - \tilde{t})^2}. \tag{15}$$

Thus, h is a generating function for the map S.

Proposition 3.1. *There exists a positive number ϵ sufficiently small such that the generating function $h(t_0, t_1)$ satisfies*
(P1) $h : W \to \mathbb{R} \in C^2(W)$;
(P2) $h(t_0 + 1, t_1 + 1) = h(t_0, t_1)$;
(P3) $\partial_{t_0 t_1} h \leq \kappa < 0$ for some κ,
where $W := \{(t_0, t_1) \,|\, 0 < t_1 - t_0 < \epsilon\}$.

Proof. $(P1)$ and $(P2)$ follow from Lemma 3.1 and the expression of h. It remains to verify the property $(P3)$. Differentiating (15) once more yields

$$\begin{aligned}\partial_{t_1 t_0} h(t_0, t_1) &= -\frac{1}{2}\Big(\frac{2p(\tilde{t})\dot{p}(\tilde{t})\partial_{t_0}\tilde{t}(t_1 - \tilde{t}) + \partial_{t_0}\tilde{t}p(\tilde{t})^2}{t_1 - \tilde{t}}\Big) \\ &= -\frac{1}{2}\partial_{t_0}\tilde{t}\Big(\frac{2p(\tilde{t})\dot{p}(\tilde{t})(t_1 - \tilde{t}) + p(\tilde{t})^2}{t_1 - \tilde{t}}\Big).\end{aligned}$$

Moreover, by (11) we obtain

$$\partial_{t_0}\tilde{t} = \frac{p(\tilde{t}) + 2\dot{p}(\tilde{t})(t_1 - \tilde{t})}{2(t_1 - \tilde{t})(\tilde{t} - t_0)\ddot{p}(\tilde{t}) + (t_0 + t_1 - 2\tilde{t})\dot{p}(\tilde{t}) + 2p(\tilde{t})} > 0.$$

Thus, there exists a positive number $\epsilon > t_1 - t_0$ sufficiently small such that

$$\partial_{t_1 t_0} h(t_0, t_1) \leq \kappa < 0$$

for some negative number κ. □

To apply the Aubry-Mather theory [17], we need to look for a modified generating function $\tilde{h} : \mathbb{R}^2 \to \mathbb{R}$ which satisfies the following properties
(M1) $\tilde{h} \in C^2$;
(M2) $\tilde{h}(t_0 + 1, t_1 + 1) = \tilde{h}(t_0, t_1)$;
(M3) $\partial_{t_0 t_1}\tilde{h} \leq \kappa < 0$,
and coincide with h when $t_1 - t_0$ is sufficiently small.

The idea of how to do this is presented in [18], see also [19, 20].

Lemma 3.2. *Take a small positive number $\delta \ll \epsilon$. Let $\beta_-(t_0) = t_0 - \delta$, $\beta_+(t_0) = t_0 + \epsilon - \delta$, and let $W_\delta = \{(t_0, t_1) \,|\, \beta_-(t_0) \leq t_1 \leq \beta_+(t_0)\}$. If $h : W_\delta \to \mathbb{R}$ satisfies the properties $(P1)-(P3)$, then it can be extended to a C^2 function $\tilde{h}(t_0, t_1)$ defined on all of $\mathbb{R}^2$ satisfying $(M1) - (M3)$.*

Proof. W_δ is invariant under the translation $(t_0, t_1) \mapsto (t_0 + 1, t_1 + 1)$. By Proposition 3.1, there exists a number $\kappa < 0$ such that $\partial_{t_0 t_1} h(t_0, t_1) \le \kappa$. According to the Whitney extension theorem [21], there exists a continuous function on $\mathbb{R}^2$ such that $\rho(t_0+1, t_1+1) = \rho(t_0, t_1), \rho \le \kappa$ and $\rho|W_\delta = \partial_{t_0 t_1} h$. In addition, we may suppose that $\rho = \kappa$ outside of the set $\{(t_0, t_1) \,|\, \beta_-(t_0) - \delta \le t_1 \le \beta_+(t_0) + \delta\}$ (see Fig. 3). Then there is a unique extension $\tilde{h}$ of h to all of $\mathbb{R}^2$ satisfying $\partial_{t_0 t_1} \tilde{h} = \rho$. Now, $\tilde{h}$ satisfying the properties $(M1) - (M3)$ follows from the procedure for determining it. □

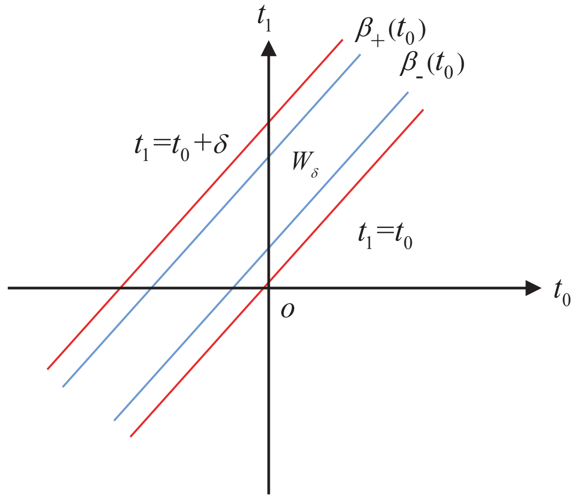

Figure 3: Domain W_δ

4. Existence of Aubry-Mather sets

We first provide a brief overview of the Aubry-Mather theory [17, 18].

Let $\tilde{h}$ be as in Section 3. By a configuration, we mean a bi-infinite sequence $t = (\dots, t_n, \dots)$ of real numbers. A configuration $(\dots, t_n, \dots)$ is periodic of type (p, q) if $t_{n+q} = t_n + p$. If a configuration $(\dots, t_n, \dots)$ satisfies

$$\partial_{t_n} \tilde{h}(t_{n-1}, t_n) + \partial_{t_n} \tilde{h}(t_n, t_{n+1}) = 0, \text{ for every } n \in \mathbb{Z},$$

then it is called a stationary configuration of $\tilde{h}$.

Given a segment $(t_j, \cdots, t_k)$, we can associate its action

$$\tilde{h}(t_j, \cdots, t_k) = \sum_{n=j}^{k-1} \tilde{h}(t_n, t_{n+1}).$$

The configuration $(\dots, t_n, \dots)$ is said to be $\tilde{h}$-minimal if, for every segment $(\xi_j, \cdots, \xi_k)$ such that $t_j = \xi_j$ and $t_k = \xi_k$ (but can be arbitrary for $j < n < k$), we have

$$\tilde{h}(t_j, \cdots, t_k) \le \tilde{h}(\xi_j, \cdots, \xi_k).$$

Every $\tilde{h}$-minimal configuration is stationary with respect to $\tilde{h}$. The set of minimal configuration of $\tilde{h}$ is denoted by $\mathcal{M}^{\tilde{h}}$.

Let $\tilde{G}$ be the set of lifts of orientation-preserving homeomprphisms of the circle $\mathbb{R}/\mathbb{Z}$. For every $t \in \mathcal{M}^{\tilde{h}}$ there exists $\varphi \in \tilde{G}$ such that $t_{n+1} = \varphi(t_n)$ for all $n \in \mathbb{Z}$. Therefore, for every $t \in \mathcal{M}^{\tilde{h}}$, the rotation number $\alpha(t)$ of t is well defined by

$$\alpha(t) = \lim_{|n| \to +\infty} \sum_{i=0}^{n-1} \frac{1}{n}(t_{i+1} - t_i). \tag{16}$$

For every $\alpha \in \mathbb{R}$, there exists a minimal configuration $t \in \mathcal{M}^{\tilde{h}}$ with rotation number α.

Let $\mathcal{M}_\alpha^{\tilde{h}}$ denote the set of all minimal configurations with rotation number α. A configuration $t \in \mathcal{M}^{\tilde{h}}$ is called recurrent, if there exist $(j_m, k_m) \in \mathbb{Z}^2$, such that $t_{i+j_m} - k_m \to t_i$ for $m \to \infty$. The set of the recurrent configurations is denoted by $\mathcal{M}^{\tilde{h},rec}$. The elements of $\mathcal{M}_\alpha^{\tilde{h},rec} = \mathcal{M}^{\tilde{h},rec} \cap \mathcal{M}_\alpha^{\tilde{h}}$ are called Aubry-Mather sets.

- For $\alpha \in \mathbb{Q}$, say $\alpha = p/q$ with p and q relatively prime, $\mathcal{M}_\alpha^{\tilde{h},rec}$ contains a configuration with period of type (p, q);
- For $\alpha \in \mathbb{R} - \mathbb{Q}$, either $p_0(\mathcal{M}_\alpha^{\tilde{h},rec}) = \mathbb{R}$ or $p_0(\mathcal{M}_\alpha^{\tilde{h},rec})$ is a Cantor set in $\mathbb{R}$, where $p_0 : \mathbb{R}^Z \to \mathbb{R}$ is the projection $t \mapsto t_0$.

Getting back to the physical problem of the Fermi-Ulam model in Section 2, we proof the following.

Theorem 4.1. *There exists α^* such that for every $\alpha \in (0, \alpha^*)$ there exists orbits $(\dots, (t_n, E_n), \dots)$ with rotation number α. Moreover,*
(i) if α is rational, say $\alpha = p/q$ with p and q relatively prime, then there exists a $p-$periodic bouncing orbit with q bounces in each period.
(ii) if α is irrational then there exists an orbit such that the corresponding sequence of impact times has dense projection to the circle $\mathbb{R}/\mathbb{Z}$ or there exists an orbit such that the closure of the projection of the corresponding impact time sequence to $\mathbb{R}/\mathbb{Z}$ is a minimal Cantor set (called Denjoy type set).

Proof. Taking a large positive integer m, if if $|n| > m$ then

$$|t_n - t_0 - n\alpha(t)| < \epsilon_0$$

for some $\epsilon_0 > 0$ sufficiently small. Besides, since v_0 is sufficiently large and the map is a diffeomorphism, if $|n| \leq m$ then there exists a positive number ϵ_1 sufficiently small such that

$$|t_n - t_0 - n\alpha(t)| = |(t_n - t_{n-1}) + (t_{n-1} - t_{n-2}) + \cdots + (t_1 - t_0) - n\alpha(t)| < \epsilon_1.$$

Let $\tilde{\epsilon} = \max\{\epsilon_0, \epsilon_1\}$. Then

$$|t_n - t_0 - n\alpha(t)| < \tilde{\epsilon}, \qquad n \in \mathbb{Z}.$$

We have

$$t_0 + n\alpha(t) - \tilde{\epsilon} < t_n < t_0 + n\alpha(t) + \tilde{\epsilon} \tag{17}$$

and

$$t_0 + n\alpha(t) + \alpha(t) - \tilde{\epsilon} < t_{n+1} < t_0 + n\alpha(t) + \alpha(t) + \tilde{\epsilon}. \tag{18}$$

Then, by (17) and (18) we have

$$\alpha(t) - 2\tilde{\epsilon} < t_{n+1} - t_n < \alpha(t) + 2\tilde{\epsilon}.$$

Therefore, there exists α^* such that if $\alpha \in (0, \alpha^*)$, then

$$t_{n+1} - t_n < \epsilon^* \quad \text{for every } n \in \mathbb{Z}, \tag{19}$$

where $\epsilon^* \leq \epsilon$ is a sufficiently small constant.

By Lemma 3.2, the generating function $\tilde{h}(t_0, t_1)$ satisfies the properties $(M1) - (M3)$, then for every $\alpha \in \mathbb{R}$ there exists an Aubry-Mather set whose orbits have rotation number α. In view of (19), if $\alpha \in (0, \alpha^*)$ then $h|W_\delta = \tilde{h}|W_\delta$ for some δ. The thesis follows directly from the general Aubry-Mather theory, remembering that if $(\dots, t_n, \dots)$ is a stationary configuration of h then by (15) we have

$$E_n = -\partial_{t_n} h(t_{n-1}, t_n) = \partial_{t_n} h(t_n, t_{n+1}) \quad \text{for every } n \in \mathbb{Z},$$

i.e. $(\dots, (t_n, E_n), \dots)$ is an orbit of the map S. □

5. Conclusion

In this paper, we study the dynamical properties of the Fermi-Ulam model using the Aubry-Mather theory. It is shown that Aubry-Mather sets exist for every rotation number in the rotation interval. Therefore, such sets are more plentiful than invariant curves. For the Aubry-Mather sets with rational number p/q, the corresponding orbits are periodic of type (p, q), while for the ones with irrational numbers, the corresponding orbits are quasiperiodic (in a generalized sense). Moreover, compared to the assumptions in [4] for using KAM theory, the smoothness condition of $p(t)$ is reduced to C^3 and there is no additional restrictions on the amplitude of $p(t)$.

Acknowledgements

This work is supported by the National Natural Science Foundations of China (11732014).

Appendix

Differentiating (12) yields

$$\begin{aligned}
\partial_{t_0} h(t_0, t_1) &= \frac{2p(\tilde{t})\dot{p}(\tilde{t})\partial_{t_0}\tilde{t}(\tilde{t} - t_0) - (\partial_{t_0}\tilde{t} - 1)p(\tilde{t})^2}{2(\tilde{t} - t_0)^2} + \frac{2p(\tilde{t})\dot{p}(\tilde{t})\partial_{t_0}\tilde{t}(t_1 - \tilde{t}) + \partial_{t_0}\tilde{t}p(\tilde{t})^2}{2(t_1 - \tilde{t})^2} \\
&= p(\tilde{t})\dot{p}(\tilde{t})\partial_{t_0}\tilde{t}\Big(\frac{1}{\tilde{t} - t_0} + \frac{1}{t_1 - \tilde{t}}\Big) + \frac{1}{2}p(\tilde{t})^2\Big((1 - \partial_{t_0}\tilde{t})\frac{1}{(\tilde{t} - t_0)^2} + \frac{1}{(t_1 - \tilde{t})^2}\partial_{t_0}\tilde{t}\Big) \\
&= p(\tilde{t})\partial_{t_0}\tilde{t}\Big(\dot{p}(\tilde{t})\Big(\frac{1}{\tilde{t} - t_0} + \frac{1}{t_1 - \tilde{t}}\Big) - \frac{1}{2}p(\tilde{t})\Big(\frac{1}{(\tilde{t} - t_0)^2} + \frac{1}{(t_1 - \tilde{t})^2}\Big)\Big) + \frac{p(\tilde{t})^2}{2(\tilde{t} - t_0)^2} \\
&= p(\tilde{t})\partial_{t_0}\tilde{t}\Big(\frac{\dot{p}(\tilde{t})(t_1 - t_0)}{(\tilde{t} - t_0)(t_1 - \tilde{t})} - \frac{1}{2}p(\tilde{t})\Big(\frac{(t_1 + t_0 - 2\tilde{t})(t_1 - t_0)}{(\tilde{t} - t_0)^2(t_1 - \tilde{t})^2}\Big)\Big) + \frac{p(\tilde{t})^2}{2(\tilde{t} - t_0)^2} \\
&= \frac{p(\tilde{t})\partial_{t_0}\tilde{t}(t_1 - t_0)}{2(\tilde{t} - t_0)^2(t_1 - \tilde{t})^2}\Big(2\dot{p}(\tilde{t})(\tilde{t} - t_0)(t_1 - \tilde{t}) - (t_0 + t_1 - 2\tilde{t})p(\tilde{t})\Big) + \frac{p(\tilde{t})^2}{2(\tilde{t} - t_0)^2}.
\end{aligned}$$

and

$$\partial_{t_1} h(t_0,t_1) = \frac{2p(\tilde t)\dot p(\tilde t)\partial_{t_1}\tilde t(\tilde t - t_0) - \partial_{t_1}\tilde t p(\tilde t)^2}{2(\tilde t - t_0)^2} + \frac{2p(\tilde t)\dot p(\tilde t)\partial_{t_1}\tilde t(t_1 - \tilde t) - (1-\partial_{t_1}\tilde t)p(\tilde t)^2}{2(t_1 - \tilde t)^2}$$

$$= p(\tilde t)\dot p(\tilde t)\partial_{t_1}\tilde t\Big(\frac{1}{\tilde t - t_0} + \frac{1}{t_1 - \tilde t}\Big) + \frac{1}{2}p(\tilde t)^2\Big((-\partial_{t_1}\tilde t)\frac{1}{(\tilde t - t_0)^2} + \frac{1}{(t_1 - \tilde t)^2}\partial_{t_1}\tilde t\Big)$$

$$= p(\tilde t)\partial_{t_1}\tilde t\Big(\dot p(\tilde t)\Big(\frac{1}{t_1 - \tilde t} + \frac{1}{t_1 - \tilde t}\Big) + \frac{1}{2}p(\tilde t)\Big(-\frac{1}{(\tilde t - t_0)^2} + \frac{1}{(t_1 - \tilde t)^2}\Big)\Big) - \frac{p(\tilde t)^2}{2(t_1 - \tilde t)^2}$$

$$= p(\tilde t)\partial_{t_1}\tilde t\Big(\frac{\dot p(\tilde t)(t_1 - t_0)}{(\tilde t - t_0)(t_1 - \tilde t)} - \frac{1}{2}p(\tilde t)\Big(\frac{(t_1 + t_0 - 2\tilde t)(t_1 - t_0)}{(\tilde t - t_0)^2(t_1 - \tilde t)^2}\Big)\Big) - \frac{p(\tilde t)^2}{2(t_1 - \tilde t)^2}$$

$$= \frac{p(\tilde t)\partial_{t_1}\tilde t(t_1 - t_0)}{2(\tilde t - t_0)^2(t_1 - \tilde t)^2}\Big(2\dot p(\tilde t)(\tilde t - t_0)(t_1 - \tilde t) - (t_0 + t_1 - 2\tilde t)p(\tilde t)\Big) - \frac{p(\tilde t)^2}{2(t_1 - \tilde t)^2}.$$

Then, by (11) we get

$$\partial_{t_0} h(t_0,t_1) = \frac{p(\tilde t)^2}{2(\tilde t - t_0)^2}, \quad \partial_{t_1} h(t_0,t_1) = -\frac{p(\tilde t)^2}{2(t_1 - \tilde t)^2}.$$

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