STRICTLY TRANSVERSAL SLICES TO CONJUGACY CLASSES IN ALGEBRAIC GROUPS

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ABSTRACT. We show that for every conjugacy class $\mathcal{O}$ in a connected semisimple algebraic group G over an algebraically closed field of characteristic good for G one can find a special transversal slice Σ to the set of conjugacy classes in G such that $\mathcal{O}$ intersects Σ and $\dim \mathcal{O} = \operatorname{codim} \Sigma$.

To Victor Kac on the occasion of his 70th birthday.

1. INTRODUCTION

Let G be a connected finite–dimensional semisimple algebraic group over an algebraically closed field $\mathbf{k}$ of characteristic good for G, $\mathfrak{g}$ its Lie algebra. It is well known that if char $\mathbf{k} = 0$ or char $\mathbf{k} > 4h - 2$, where h is the Coxeter number of $\mathfrak{g}$, then for any nilpotent element $e \in \mathfrak{g}$ one can construct a special transversal slice $S(e)$, called the Slodowy slice, such that $S(e)$ is transversal to the set of adjoint orbits in $\mathfrak{g}$, the adjoint orbit $\mathcal{O}_e$ of e intersects $S(e)$, and $\dim \mathcal{O}_e = \operatorname{codim} S(e)$ (see [20], Section 7.4). In this paper transversal slices with similar properties are defined for arbitrary conjugacy classes in the algebraic group G.

A general scheme for constructing transversal slices to conjugacy classes in algebraic groups was suggested in [18]. The slices defined there are associated to conjugacy classes in the corresponding Weyl group W (see Section 3 for details). The definition given in [18] is a deep generalization of the Steinberg cross–section to the set of regular conjugacy classes in G (see [23]). In fact for each conjugacy class $\mathcal{O}_s \subset W$ of an element $s \in W$ the corresponding slice Σ_s depends also on the ordering of the eigenvalues of s which are not equal to 1 and have non–negative imaginary parts, in the complex reflection representation (note that s is of finite order, and hence all these eigenvalues are roots of unity). In this paper we show that for each element $g \in G$ one can find a conjugacy class $\mathcal{O}_s \subset W$ and an ordering of the eigenvalues of s with non–negative imaginary parts such that the conjugacy class $\mathcal{O}_g$ of g in G intersects Σ_s and $\dim \mathcal{O}_g = \operatorname{codim} \Sigma_s$.

In order to solve this problem we use a partition of G the strata of which $G_{\mathcal{O}_s}$, labeled by a special set $C(W)$ of conjugacy classes $\mathcal{O}_s$ in W, are unions of conjugacy classes of the same dimension,

$$G = \bigcup_{\mathcal{O}_s \in C(W)} G_{\mathcal{O}_s}.$$

This remarkable partition was recently introduced by Lusztig in [12]. We recall the definition of the Lusztig partition and the relevant combinatorics in Section 4.

It turns out that if the eigenvalues of $\epsilon_1, \ldots, \epsilon_k$ of s, which are not equal to 1 and have non–negative imaginary parts, are ordered so that

$$0 < \arg \epsilon_1 \leq \arg \epsilon_2 \leq \ldots \leq \arg \epsilon_k,$$

where arg stands for the principal argument of a complex number, then all conjugacy classes in the stratum $G_{\mathcal{O}_s}$ intersect the corresponding transversal slice Σ_s, and for each conjugacy class $\mathcal{O} \in G_{\mathcal{O}_s}$

$$(1.1) \qquad \dim \mathcal{O} = \operatorname{codim} \Sigma_s.$$

The first property is a consequence of inclusions of Bruhat cells in the Bruhat order. But condition (1.1) is checked by straightforward calculation in case of all simple algebraic groups and in case of all strata (see Theorem 5.2). In case of root systems of exceptional types a computer program is used for that purpose (see Appendix 2).

Note that in the case when s is elliptic, i.e. s acts without fixed points in the reflection representation of W, our results imply that $\Sigma_s = \dot{s} N_s$ and $\dim \Sigma_s = \underline{l}(s)$, where $\underline{l}(s)$ is the length of a representative $s \in \mathcal{O}_s$ of minimal possible length with respect to a system Γ of positive simple roots, $N_s = \{v \in N | \dot{s} v \dot{s}^{-1} \in \overline{N}\}$, N is the unipotent radical of G corresponding to Γ, $\overline{N}$ is the opposite unipotent radical and $\dot{s}$ is a representative of s in G (see the second part of the proof of Theorem 5.2). Therefore from Theorem 0.7 in [15] one immediately deduces that Σ_s coincides with the cross–section associated to $\mathcal{O}_s$ in [9].

Key words and phrases. Algebraic group, transversal slice.

The paper is organized as follows. After introducing the basic notation in Section 2 we recall the definition of the slices Σ_s and their properties. Then we review the definition and the properties of the Lusztig partition in Section 4. In the case of root systems of classical types the relevant combinatorics is explicitly described in terms of partitions. In Section 5 strict transversality condition (1.1) is checked in Theorem 5.2. Appendix 1 is not directly related to the main body of the text. It contains some auxiliary results which are important for the representation theory of quantum groups at roots of unity. The proofs of these results heavily rely on the numeric data contained in the tables in Appendix 2. These data are related to the transversal slices Σ_s. Appendix 3 contains the complete description of the root systems of exceptional types, including a numbering of all positive roots for each root system. These numberings are used in the tables in Appendix 2.

Acknowledgements

2. Notation

We fix the notation used throughout the text. Let G be a connected finite–dimensional semisimple algebraic group over an algebraically closed field $\mathbf{k}$ of characteristic good for G, $\mathfrak{g}$ the complex semisimple Lie algebra of the same type as G, G_p a connected finite–dimensional semisimple algebraic group of the same type as G over an algebraically closed field of characteristic exponent p. Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and let Δ be the set of roots of $(\mathfrak{g}, \mathfrak{h})$. Let α_i, $i = 1, \ldots l$, $l = \operatorname{rank} \mathfrak{g}$ be a system of simple roots, $\Delta_+ = \{\beta_1, \ldots, \beta_N\}$ the set of positive roots. Let $H_1, \ldots, H_l$ be the set of simple root generators of $\mathfrak{h}$.

Let a_{ij} be the corresponding Cartan matrix, and let $d_1, \ldots, d_l$, $i = 1, \ldots, l$ be coprime positive integers such that the matrix $b_{ij} = d_i a_{ij}$ is symmetric. There exists a unique non–degenerate invariant symmetric bilinear form $(\,,\,)$ on $\mathfrak{g}$ such that $(H_i, H_j) = d_j^{-1} a_{ij}$. It induces an isomorphism of vector spaces $\mathfrak{h} \simeq \mathfrak{h}^*$ under which $\alpha_i \in \mathfrak{h}^*$ corresponds to $d_i H_i \in \mathfrak{h}$. We denote by $\alpha^\vee$ the element of $\mathfrak{h}$ that corresponds to $\alpha \in \mathfrak{h}^*$ under this isomorphism. The induced bilinear form on $\mathfrak{h}^*$ is given by $(\alpha_i, \alpha_j) = b_{ij}$. We shall always identify $\mathfrak{h}$ and $\mathfrak{h}^*$ by means of the form $(\,,\,)$.

Let W be the Weyl group of the root system Δ. W is the subgroup of $GL(\mathfrak{h})$ generated by the fundamental reflections $s_1, \ldots, s_l$,

$$s_i(h) = h - \alpha_i(h) H_i, \ h \in \mathfrak{h}.$$

The action of W preserves the bilinear form $(\,,\,)$ on $\mathfrak{h}$. We denote a representative of $w \in W$ in G by $\dot{w}$. For $w \in W, g \in G$ we write $w(g) = \dot{w} g \dot{w}^{-1}$. For any root $\alpha \in \Delta$ we also denote by s_α the corresponding reflection. For every element $w \in W$ one can introduce the set $\Delta_w = \{\alpha \in \Delta_+ : w(\alpha) \in -\Delta_+\}$, and the number of the elements in the set Δ_w is equal to the length $\underline{l}(w)$ of the element w with respect to the system of simple roots in Δ_+.

Let $\mathfrak{b}_+$ be the Borel subalgebra associated to Δ_+ and $\mathfrak{b}_-$ the opposite Borel subalgebra; let $\mathfrak{n}_+ = [\mathfrak{b}_+, \mathfrak{b}_+]$ and $\mathfrak{n}_- = [\mathfrak{b}_-, \mathfrak{b}_-]$ be their nilradicals. Let B_+ be the Borel subgroup in G which corresponds to Δ_+, H the maximal torus in B_+ normalized by W, B_- the opposite Borel subgroup, and $N_\pm$ the unipotent radicals of $B_\pm$. Thus W is the Weyl group of the pair $(\mathfrak{g}, \mathfrak{h})$ and of the pair (G, H), $W = W(\mathfrak{g}, \mathfrak{h}) = W(G, H)$, and Δ is the root system of the pair (G, H), $\Delta = \Delta(G, H)$. This notation will also be used when G is reductive. Δ_+ can also be defined as the system of positive roots associated to $\mathfrak{b}_+$ or to B_+.

In this paper we denote by $\mathbb{N}$ the set of nonnegative integer numbers, $\mathbb{N} = \{0, 1, \ldots\}$.

3. Transversal slices to conjugacy classes in algebraic groups

In this section, following [18], we recall the general definition of transversal slices to conjugacy classes in algebraic groups. The slices are introduced with the help of systems of positive roots associated to Weyl group elements. We start with the definition of these systems of positive roots.

Let s be an element of the Weyl group W of the pair $(\mathfrak{g}, \mathfrak{h})$ and $\mathfrak{h}_\mathbb{R}$ the real form of $\mathfrak{h}$, the real linear span of simple coroots in $\mathfrak{h}$. The set of roots Δ is a subset of the dual space $\mathfrak{h}_\mathbb{R}^*$. The restriction of the canonical bilinear form to $\mathfrak{h}_\mathbb{R}$ induces a non–degenerate Weyl group invariant bilinear form on it. Let $\mathfrak{h}'_\mathbb{R}$ be the orthogonal complement in $\mathfrak{h}_\mathbb{R}$, with respect to the canonical bilinear form, to the subspace of $\mathfrak{h}_\mathbb{R}$ fixed by the natural action of s on $\mathfrak{h}_\mathbb{R}$, $\mathfrak{h}' \subset \mathfrak{h}$ the complexification of $\mathfrak{h}'_\mathbb{R}$. Let $\mathfrak{h}'^*$ be the image of $\mathfrak{h}'$ in $\mathfrak{h}^*$ under the identification $\mathfrak{h}^* \simeq \mathfrak{h}$ induced by the canonical bilinear form on $\mathfrak{g}$ and $P_{\mathfrak{h}'^*}$ the orthogonal projection operator onto $\mathfrak{h}'^*$ in $\mathfrak{h}^*$, with respect to canonical bilinear form. Now recall that by Theorem C in [3] s can be represented as a product of two involutions,

$$(3.1) \qquad\qquad\qquad\qquad\qquad s = s^1 s^2,$$

where $s^1 = s_{\gamma_1} \ldots s_{\gamma_n}$, $s^2 = s_{\gamma_{n+1}} \ldots s_{\gamma_{l'}}$, the roots in each of the sets $\gamma_1, \ldots \gamma_n$ and $\gamma_{n+1} \ldots \gamma_{l'}$ are positive and mutually orthogonal, and the roots $\gamma_1, \ldots \gamma_{l'}$ form a linear basis of $\mathfrak{h}'$.

The Weyl group element s naturally acts on $\mathfrak{h}_{\mathbb{R}}$ as an orthogonal transformation with respect to the scalar product induced by the canonical bilinear form on $\mathfrak{g}$. Now we recall some results of [4], Sect. 10.4 on the spectral decomposition for the action of s on $\mathfrak{h}_{\mathbb{R}}$.

Let $f_1, \ldots, f_{l'}$ be the vectors of unit length in the directions of $\gamma_1, \ldots \gamma_{l'}$, and $\widehat{f}_1, \ldots, \widehat{f}_{l'}$ the basis of $\mathfrak{h}'_{\mathbb{R}}$ dual to $f_1, \ldots, f_{l'}$. Let M be the $l' \times l'$ symmetric matrix with real entries $M_{ij} = (f_i, f_j)$. $I - M$ is also a symmetric real matrix, and hence it is diagonalizable and has real eigenvalues.

The following proposition gives a recipe for constructing a spectral decomposition for the action of the orthogonal transformation s on $\mathfrak{h}_{\mathbb{R}}$.

Proposition 3.1. *Let λ be a (real) eigenvalue of the symmetric matrix $I - M$, and $u \in \mathbb{R}^{l'}$ a corresponding non–zero real eigenvector with components u_i, $i = 1, \ldots, l'$. Let $a_u, b_u \in \mathfrak{h}_{\mathbb{R}}$ be defined by*

$$(3.2) \qquad a_u = \sum_{i=1}^{n} u_i \widehat{f}_i, \ b_u = \sum_{i=n+1}^{l'} u_i \widehat{f}_i.$$

(i) If $\lambda \neq 0$ then the angle θ between a_u and b_u satisfies $\cos \theta = \lambda$, the plane $\mathfrak{h}_\lambda \subset \mathfrak{h}_{\mathbb{R}}$ spanned by a_u and b_u is invariant with respect to the involutions $s^{1,2}$, s^1 acts on $\mathfrak{h}_\lambda$ as the reflection in the line spanned by b_u, and s^2 acts on $\mathfrak{h}_\lambda$ as the reflection in the line spanned by a_u. The orthogonal transformation $s = s^1 s^2$ acts on $\mathfrak{h}_\lambda$ as a rotation through the angle 2θ.

(ii) If $\lambda \neq 0, \pm 1$ is an eigenvalue of $I - M$ then $-\lambda$ is also an eigenvalue of $I - M$, and if $\lambda \neq \mu$ are two positive eigenvalues of $I - M$, $\lambda, \mu \neq 1$ then the planes $\mathfrak{h}_\lambda$ and $\mathfrak{h}_\mu$ are mutually orthogonal.

(iii) Let $\lambda \neq 0, \pm 1$ be an eigenvalue of $I - M$ of multiplicity greater than 1, and $u^k \in \mathbb{R}^{l'}$, $k = 1, \ldots,$ mult λ a basis of the eigenspace corresponding to λ. If the basis u^k is orthonormal with respect to the standard scalar product on $\mathbb{R}^{l'}$ then the corresponding planes $\mathfrak{h}_\lambda^k$ defined with the help of u^k, $k = 1, \ldots,$ mult λ are mutually orthogonal.

(iv) $\lambda = \pm 1$ are not eigenvalues of $I - M$.

(v) If $\lambda = 0$ is an eigenvalue of $I - M$, then there is a basis $u^k \in \mathbb{R}^{l'}$, $k = 1, \ldots,$ mult 0 of the eigenspace corresponding to 0 orthonormal with respect to the standard scalar product on $\mathbb{R}^{l'}$ and such that the corresponding non–zero elements a_{u^k}, b_{u^k} are all mutually orthogonal. Moreover, $s^1 a_{u^k} = -a_{u^k}$, $s^2 a_{u^k} = a_{u^k}$, $s^1 b_{u^k} = b_{u^k}$, $s^2 b_{u^k} = -b_{u^k}$ for non–zero elements a_{u^k}, b_{u^k}. In particular, for non–zero elements a_{u^k}, b_{u^k} we have $s a_{u^k} = -a_{u^k}$, $s b_{u^k} = -b_{u^k}$, and non–zero elements a_{u^k}, b_{u^k} is a basis of the subspace of $\mathfrak{h}_{\mathbb{R}}$ on which s acts by multiplication by -1.

Proof. All statements of this proposition, except for the last two parts, are proved by repeating the arguments given in the proofs of Lemma 10.4.2, Proposition 10.4.3 in [4] and using the spectral theory of orthogonal transformations.

For the last two statements one has to use some calculations from the proof of Lemma 10.4.3 in [4]. More precisely, by definition the matrix M can be written in a block form,

$$(3.3) \qquad M = \begin{pmatrix} I_n & A \\ A^\top & I_{l'-n} \end{pmatrix},$$

where A is an $n \times (l' - n)$ matrix, $A^\top$ is the transpose to A, I_n and $I_{l'-n}$ are the unit matrices of sizes n and $l' - n$. M^{-1} is also symmetric and has a similar block form,

$$(3.4) \qquad M^{-1} = \begin{pmatrix} B & C \\ C^\top & D \end{pmatrix}, \ B = B^\top, \ D = D^\top,$$

with the entries $M_{ij}^{-1} = (\widehat{f}_i, \widehat{f}_j)$.

For any vector $u \in \mathbb{R}^{l'}$ we introduce its $\mathbb{R}^n$ and $\mathbb{R}^{l'-n}$ components $\widetilde{u}$ and $\widetilde{\widetilde{u}}$ in a similar way,

$$(3.5) \qquad u = \begin{pmatrix} \widetilde{u} \\ \widetilde{\widetilde{u}} \end{pmatrix}.$$

We shall consider both $\widetilde{u}$ and $\widetilde{\widetilde{u}}$ as elements of $\mathbb{R}^{l'}$ using natural embeddings $\mathbb{R}^n, \mathbb{R}^{l'-n} \subset \mathbb{R}^{l'}$ associated to decomposition (3.5).

If u is a non–zero eigenvector of $I - M$ corresponding to an eigenvalue λ then the equation $(I - M)u = \lambda u$ gives

$$(3.6) \qquad -A\widetilde{\widetilde{u}} = \lambda \widetilde{u}, \ -A^\top \widetilde{u} = \lambda \widetilde{\widetilde{u}}.$$

Since $M^{-1}M = I$ one has

$$(3.7) \qquad BA + C = 0, \ C^\top + DA^\top = 0.$$

Multiplying the first and the second equations in (3.6) from the left by B and D, respectively, and using (3.7) we obtain that

$$(3.8) \qquad C\widetilde{\widetilde{u}} = \lambda B\widetilde{u}, \ C^{\top}\widetilde{u} = \lambda D\widetilde{\widetilde{u}}.$$

Now if $u^{1,2}$ are two non–zero eigenvectors of $I - M$ corresponding to an eigenvalue λ then by (3.4) we have

$$(3.9) \qquad (a_{u^1}, a_{u^2}) = \sum_{i,j=1}^{n} u_i^1 u_j^2 (\widehat{f}_i, \widehat{f}_j) = \sum_{i,j=1}^{n} u_i^1 u_j^2 B_{ij} = \widetilde{u}^1 \cdot B\widetilde{u}^2,$$

where $\cdot$ stands for the standard scalar product in $\mathbb{R}^{l'}$.

Similarly,

$$(3.10) \qquad (b_{u^1}, b_{u^2}) = D\widetilde{\widetilde{u}}^1 \cdot \widetilde{\widetilde{u}}^2, (a_{u^1}, b_{u^2}) = \widetilde{u}^1 \cdot C\widetilde{\widetilde{u}}^2$$

From (3.8), (3.9) and the first identity in (3.10) we also obtain that if $\lambda \neq 0$ then

$$(3.11) \qquad (a_{u^1}, a_{u^2}) = \widetilde{u}^1 \cdot B\widetilde{u}^2 = \frac{1}{\lambda}\widetilde{u}^1 \cdot C\widetilde{\widetilde{u}}^2 = \frac{1}{\lambda}C^{\top}\widetilde{u}^1 \cdot \widetilde{\widetilde{u}}^2 = D\widetilde{\widetilde{u}}^1 \cdot \widetilde{\widetilde{u}}^2 = (b_{u^1}, b_{u^2}).$$

Similarly, for any real eigenvalue λ we have

$$(3.12) \qquad (a_{u^1}, b_{u^2}) = \widetilde{u}^1 \cdot C\widetilde{\widetilde{u}}^2 = \lambda(a_{u^1}, a_{u^2}), \ (b_{u^1}, a_{u^2}) = \widetilde{\widetilde{u}}^1 \cdot C^{\top}\widetilde{u}^2 = \lambda(a_{u^1}, a_{u^2}).$$

Now if $\lambda \neq 0, \pm 1$ then (3.9), (3.11), (3.12) and the identity $M^{-1}u^2 = \frac{1}{1-\lambda}u^2$ yield

$$(a_{u^1} + b_{u^1}, a_{u^2} + b_{u^2}) = 2(a_{u^1}, a_{u^2})(\lambda + 1) = u^1 \cdot M^{-1}u^2 = \frac{1}{1-\lambda}u^1 \cdot u^2.$$

Thus if $\lambda \neq 0, \pm 1$ and $u^{1,2}$ are mutually orthogonal a_{u^1}, a_{u^2} are also mutually orthogonal, and from (3.11) and (3.12) we obtain that b_{u^1} and b_{u^2}, a_{u^1} and b_{u^2}, a_{u^2} and b_{u^1} are mutually orthogonal. Therefore the planes spanned by a_{u^1}, b_{u^1} and by a_{u^2}, b_{u^2} are mutually orthogonal.

$\lambda = 1$ is not an eigenvalue of $I - M$ since the matrix M is invertible. $\lambda = -1$ is not an eigenvalue of $I - M$ since otherwise the corresponding elements a_u, b_u would span a non–trivial fixed point subspace for the action of s in $\mathfrak{h}'_{\mathbb{R}}$ which is impossible as s acts on $\mathfrak{h}'_{\mathbb{R}}$ without non–trivial fixed points.

If $\lambda = 0$ is an eigenvalue of $I - M$ then $\widetilde{u}$ and $\widetilde{\widetilde{u}}$ are the components of an eigenvector u of $I - M$ with eigenvalue 0 if and only if $A\widetilde{u} = 0$ and $A^{\top}\widetilde{u} = 0$. Therefore using the usual orthogonalization procedure one can construct a basis $u^k \in \mathbb{R}^{l'}$, $k = 1, \ldots,$ mult 0 of the eigenspace corresponding to 0 orthonormal with respect to the standard scalar product on $\mathbb{R}^{l'}$ and such that the components $\widetilde{u}^k$ and $\widetilde{\widetilde{u}}^k$ $k = 1, \ldots,$ mult 0 are all mutually orthogonal.

Now using the definition of the eigenvector we deduce that for the basis u^k the following relations hold: $B\widetilde{u}^k = \widetilde{u}^k$, $D\widetilde{\widetilde{u}}^k = \widetilde{\widetilde{u}}^k$, $C\widetilde{\widetilde{u}}^k = 0$, $C^{\top}\widetilde{u}^k = 0$. Recalling also formulas (3.9), (3.10), and (3.12), and repeating the arguments given in the proofs of Lemma 10.4.2, Proposition 10.4.3 in [4] we immediately deduce the last statement of this proposition.

This completes the proof.

$\square$

Let $\mathfrak{h}_{-1}$ be the subspace of $\mathfrak{h}_{\mathbb{R}}$ on which s acts by multiplication by -1. According to the previous proposition one can choose one–dimensional $s^{1,2}$–invariant subspaces in $\mathfrak{h}_{-1}$ such that $\mathfrak{h}_{-1}$ is the orthogonal direct sum of those subspaces.

Using the previous proposition we can decompose $\mathfrak{h}_{\mathbb{R}}$ into a direct orthogonal sum of s–invariant subspaces,

$$(3.13) \qquad \mathfrak{h}_{\mathbb{R}} = \bigoplus_{i=0}^{K} \mathfrak{h}_i,$$

where each $\mathfrak{h}_i \subset \mathfrak{h}_{\mathbb{R}}$, $i = 0, \ldots, K$ is invariant with respect to both involutions $s^{1,2}$ in the decomposition $s = s^1 s^2$, and there are the following three possibilities for each $\mathfrak{h}_i$: $\mathfrak{h}_i$ is two–dimensional ($\mathfrak{h}_i = \mathfrak{h}_\lambda^k$ for an eigenvalue $0 < \lambda < 1$ of the matrix $I - M$, and $k = 1, \ldots,$ mult λ) and the Weyl group element s acts on it as rotation with angle θ_i, $0 < \theta_i < \pi$ or $\mathfrak{h}_i = \mathfrak{h}_\lambda^k$, $\lambda = 0$, $k = 1, \ldots,$ mult λ has dimension 1 and s acts on it by multiplication by -1 or $\mathfrak{h}_i$ coincides with the linear subspace of $\mathfrak{h}_{\mathbb{R}}$ fixed by the action of s. Note that since s has finite order $\theta_i = \frac{2\pi n_i}{m_i}$, $n_i, m_i \in \{1, 2, \ldots\}$.

Since the number of roots in the root system Δ is finite one can always choose elements $h_i \in \mathfrak{h}_i$, $i = 0, \ldots, K$, such that $h_i(\alpha) \neq 0$ for any root $\alpha \in \Delta$ which is not orthogonal to the s–invariant subspace $\mathfrak{h}_i$ with respect to the natural pairing between $\mathfrak{h}_{\mathbb{R}}$ and $\mathfrak{h}_{\mathbb{R}}^*$.

Now we consider certain s–invariant subsets of roots $\overline{\Delta}_i$, $i = 0, \ldots, K$, defined as follows

$$(3.14) \qquad \overline{\Delta}_i = \{\alpha \in \Delta : h_j(\alpha) = 0, j > i, \ h_i(\alpha) \neq 0\},$$

where we formally assume that $h_{K+1} = 0$. Note that for some indexes i the subsets $\overline{\Delta}_i$ are empty, and that the definition of these subsets depends on the order of terms in direct sum (3.13).

Now assume that

$$(3.15) \qquad |h_i(\alpha)| > |\sum_{p \leq j < i} h_j(\alpha)|, \text{ for any } \alpha \in \overline{\Delta}_i, \ i = 0, \ldots, K, \ p < i.$$

Condition (3.15) can be always fulfilled by suitable rescalings of the elements h_i. If $\overline{\Delta}_i = \emptyset$ for some i then the condition (3.15) is assumed to be void for this i.

Consider the element

$$(3.16) \qquad \bar{h} = \sum_{k=0}^{K} h_i \in \mathfrak{h}_{\mathbb{R}}.$$

From definition (3.14) of the sets $\overline{\Delta}_i$ we obtain that for $\alpha \in \overline{\Delta}_i \neq \emptyset$

$$(3.17) \qquad \bar{h}(\alpha) = \sum_{j \leq i} h_j(\alpha) = h_i(\alpha) + \sum_{j < i} h_j(\alpha)$$

Now condition (3.15), the previous identity and the inequality $|x + y| \geq ||x| - |y||$ imply that for $\alpha \in \overline{\Delta}_i \neq \emptyset$ we have

$$|\bar{h}(\alpha)| \geq ||h_i(\alpha)| - |\sum_{j < k} h_j(\alpha)|| > 0.$$

Observe that Δ is the disjoint union of the subsets $\overline{\Delta}_i$,

$$\Delta = \bigcup_{i=0}^{K} \overline{\Delta}_i,$$

and hence the last inequality ensures that $\bar{h}$ belongs to a Weyl chamber of the root system Δ and therefore one can define the subset of positive roots Δ_+ and the set of simple positive roots Γ with respect to that chamber. We call Δ_+ introduced in this way a system of positive roots associated to (the conjugacy class of) the Weyl group element s. From condition (3.15) and formula (3.17) we also obtain that a root $\alpha \in \overline{\Delta}_i \neq \emptyset$ is positive if and only if

$$(3.18) \qquad h_i(\alpha) > 0.$$

We denote by $(\overline{\Delta}_i)_+$ the set of positive roots contained in $\overline{\Delta}_i$, $(\overline{\Delta}_i)_+ = \Delta_+ \cap \overline{\Delta}_i$.

Now consider the nonempty s–invariant subsets of roots $\overline{\Delta}_{i_k}$, $k = 0, \ldots, T$. For convenience we assume that indexes i_k are labeled in such a way that $i_j < i_k$ if and only if $j < k$. The root system Δ is the disjoint union of the subsets $\overline{\Delta}_{i_k}$,

$$\Delta = \bigcup_{k=0}^{T} \overline{\Delta}_{i_k}.$$

We shall also need s–invariant subset of roots Δ_{i_k} defined as follows

$$(3.19) \qquad \Delta_{i_k} = \bigcup_{i_j \leq i_k} \overline{\Delta}_{i_j}.$$

We have a chain of strict inclusions

$$(3.20) \qquad \Delta_{i_T} \supset \Delta_{i_{T-1}} \supset \ldots \supset \Delta_{i_0},$$

such that $\Delta_{i_T} = \Delta$ and $\Delta_{i_k} \setminus \Delta_{i_{k-1}} = \overline{\Delta}_{i_k}$.

Note that by definition the sets of roots Δ_{i_k} are additively closed. Therefore each Δ_{i_k} is a root subsystem in Δ. The following property of the root systems Δ_{i_k} will be important in subsequent considerations.

Lemma 3.2. $\Gamma \cap \Delta_{i_k}$ *is a set of simple roots in* Δ_{i_k}.

Proof. Indeed, let $\alpha \in \Delta_{i_k} \cap \Delta_+$, $\alpha = \sum_{i=1}^{l} n_i \alpha_i$, where $n_i \in \mathbb{N}$ and $\Gamma = \{\alpha_1, \ldots, \alpha_l\}$. Assume that α does not belong to the linear span of roots from $\Gamma \cap \Delta_{i_k}$ and $t > i_k$ is maximal possible such that for some $\alpha_q \in \overline{\Delta}_t$ one has $n_q > 0$. Then by (3.14) and (3.18) $h_t(\alpha) = \sum_{i=1}^{l} n_i h_t(\alpha_i) = \sum_{\alpha_i \in \overline{\Delta}_t} n_i h_t(\alpha_i) > 0$, and by the choice of t $h_r(\alpha) = 0$ for $r > t$. Therefore $\alpha \in \overline{\Delta}_t$, and hence $\alpha \notin \Delta_{i_k}$. Thus we arrive at a contradiction.

$\square$

Now assume that $\mathfrak{h}_0$ is the linear subspace of $\mathfrak{h}_{\mathbb{R}}$ fixed by the action of s. In this case $\overline{\Delta}_0 = \{\alpha \in \Delta : s\alpha = \alpha\}$ is the set of roots fixed by the action of s. We shall need the parabolic subgroup P of G associated to the subset $\Gamma_0 = \Gamma \cap \overline{\Delta}_0 \subset \Gamma$ of simple roots. Note that by Lemma 3.2 Γ_0 is a set of simple roots in $\overline{\Delta}_0 = \Delta_0$.

Let N and L be the unipotent radical and the Levi factor of P, respectively. Note that we have natural inclusions $P \supset B_+ \supset N$, where B_+ is the Borel subgroup of G corresponding to the system Γ of simple roots, and by Lamma 3.2 $\overline{\Delta}_0$ is the root system of the reductive group L. We also denote by $\overline{N}$ the unipotent radical opposite to N.

Now we can define certain transversal slices to conjugacy classes in algebraic groups. Let Z be the subgroup of G generated by the semisimple part M of the Levi factor L and by the centralizer of $\dot{s}$ in H.

Proposition 3.3. ([18], **Propositions 2.1 and 2.2**) *Let* $N_s = \{v \in N | \dot{s}v\dot{s}^{-1} \in \overline{N}\}$. *Then the conjugation map*

$$(3.21) \qquad\qquad N \times \dot{s}ZN_s \to N\dot{s}ZN$$

is an isomorphism of varieties. Moreover, the variety $\Sigma_s = \dot{s}ZN_s$ *is a transversal slice to the set of conjugacy classes in* G.

Propositions 2.1 and 2.2 in [18] were proved when **k** is the field of complex numbers. But the proof can be repeated verbatim the case of an arbitrary algebraically closed field **k**.

4. THE LUSZTIG PARTITION

Recall that the purpose in this paper is show that for every conjugacy class $\mathcal{O}$ in G one can find a transversal slice Σ_s such that $\mathcal{O}$ intersects Σ_s and $\dim \mathcal{O} = \operatorname{codim} \Sigma_s$. It turns out that there is a remarkable partition of the group G the strata of which are unions of conjugacy classes of the same dimension (see [12]). For each stratum of this partition there is a Weyl group element s such that all conjugacy classes $\mathcal{O}$ from that stratum intersect a transversal slice Σ_s, and $\dim \mathcal{O} = \operatorname{codim} \Sigma_s$. In this section following [12] we describe this partition which is called the Lusztig partition.

For any Weyl group W let $\widehat{W}$ be the set of isomorphism classes of irreducible representations of W over $\mathbb{Q}$. For any $E \in \widehat{W}$ let b_E be the smallest nonnegative integer such that E appears with non–zero multiplicity in the b_E-th symmetric power of the reflection representation of W. If this multiplicity is equal to 1 then one says that E is good. If $W' \subset W$ are two Weyl groups, and $E \in \widehat{W'}$ is good then there is a unique $\widetilde{E} \in \widehat{W}$ such that $\widetilde{E}$ appears in the decomposition of the induced representation $\operatorname{Ind}_{W'}^{W} E$, $b_{\widetilde{E}} = b_E$, and $\widetilde{E}$ is good. The representation $\widetilde{E}$ is called j-induced from E, $\widetilde{E} = j_{W'}^{W} E$.

Let $g \in G_p$, and $g = g_s g_u$ its decomposition as a product of the semisimple part g_s and the unipotent part g_u. Let $C = Z_{G_p}(g_s)^0$ be the identity component of the centralizer of g_s in G_p. C is a reductive subgroup of G_p of the same rank as G_p. Let H_p be a maximal torus of C. H_p is also a maximal torus in G_p, and hence one has a natural imbedding

$$W' = N_C(H_p)/H_p \to N_{G_p}(H_p)/H_p = W,$$

where $N_C(H_p), N_{G_p}(H_p)$ stand for the normalizers of H_p in C and in G_p, respectively, W' is the Weyl group of C and W is the Weyl group of G_p.

Let E be the irreducible representation of W' associated with the help of the Springer correspondence to the conjugacy class of g_u and the trivial local system on it. Then E is good, and let $\widetilde{E}$ be the j-induced representation of W. This gives a well-defined map $\phi_{G_p} : G_p \to \widehat{W}$. The fibers of this map are called the strata of G_p. By definition the map ϕ_{G_p} is constant on each conjugacy class in G_p. Therefore the strata are unions of conjugacy classes.

Moreover, by 1.3 in [12] we have the following formula for the dimension of the centralizer $Z_{G_p}(g)$ of any element $g \in G_p$ in G_p:

$$(4.1) \qquad\qquad \dim Z_{G_p}(g) = \operatorname{rank} G_p + 2b_{\phi_{G_p}(g)},$$

where rank G_p is the rank of G_p.

It turns out that the image $\mathcal{R}(W)$ of ϕ_{G_p} only depends on W. It can be described as follows. Let $\mathcal{N}(G_p)$ be the unipotent variety of G_p and $\underline{\mathcal{N}}(G_p)$ the set of unipotent classes in G_p. Let $\mathcal{X}^p(W)$ be the set of irreducible representations of W associated by the Springer correspondence to unipotent classes in $\underline{\mathcal{N}}(G_p)$ and the trivial local systems on them. We shall identify $\mathcal{X}^p(W)$ and $\underline{\mathcal{N}}(G_p)$. Let $f_p : \underline{\mathcal{N}}(G_p) \to \mathcal{X}^p(W)$ be the corresponding bijective map.

Proposition 4.1. ([12], **Proposition 1.4**) *We have*

$$\mathcal{R}(W) = \mathcal{X}^1(W) \bigcup_{r \text{ prime}} \mathcal{X}^r(W).$$

If G_p is of type A_n, $(n \geq 1)$ or E_6 then $\mathcal{R}(W) = \mathcal{X}^1(W)$.

If G_p is of type B_n $(n \geq 2)$, C_n $(n \geq 3)$, D_n $(n \geq 4)$, F_4 or E_7 then $\mathcal{R}(W) = \mathcal{X}^2(W)$.

If G_p is of type G_2 then $\mathcal{R}(W) = \mathcal{X}^3(W)$.

If G_p is of type E_8 then $\mathcal{R}(W) = \mathcal{X}^2(W) \bigcup \mathcal{X}^3(W)$, and $\mathcal{X}^2(W) \cap \mathcal{X}^3(W) = \mathcal{X}^1(W)$.

The above description of the set $\mathcal{R}(W)$ and the bijections $\underline{\mathcal{N}}(G_p) \to \mathcal{X}^p(W)$ yield certain maps between sets $\underline{\mathcal{N}}(G_p)$ which preserve dimensions of conjugacy classes by (4.1). For instance, one always has an inclusion $\mathcal{X}^1(W) \subset \mathcal{X}^r(W)$ for any $r \geq 2$. The corresponding inclusion $\underline{\mathcal{N}}(G_1) \subset \underline{\mathcal{N}}(G_p)$ coincides with the Spaltenstein map $\pi_p^G : \underline{\mathcal{N}}(G_1) \to \underline{\mathcal{N}}(G_p)$ (see [21], Théorème III.5.2).

Fix a system of positive roots in $\Delta = \Delta(G_p, H_p)$, and let B_p be the corresponding Borel subgroup in G_p, $H_p \subset B_p$ the maximal torus, and $\underline{l}$ the corresponding length function on W. Denote by $\underline{W}$ the set of conjugacy classes in W. For each $w \in W = N_{G_p}(H_p)/H_p$ one can pick up a representative $\dot{w} \in G_p$. If p is good for G_p, we simply write $G_p = G$, $B_p = B$, $\underline{\mathcal{N}}(G_p) = \underline{\mathcal{N}}(G)$.

Let $\mathcal{C}$ be a conjugacy class in W. Pick up a representative $w \in \mathcal{C}$ of minimal possible length with respect to $\underline{l}$. By Theorem 0.4 in [15] there is a unique conjugacy class $\mathcal{O} \in \underline{\mathcal{N}}(G)$ of minimal possible dimension which intersects the Bruhat cell $B\dot{w}B$ and does not depend on the choice of the minimal possible length representative w in $\mathcal{C}$. We denote this class by $\Phi_1^G(\mathcal{C})$.

As shown in Section 1.1 in [15], one can always find a representative $w \in \mathcal{C}$ of minimal possible length with respect to $\underline{l}$ which is elliptic in a parabolic Weyl subgroup $W' \subset W$, i.e. w acts without fixed points in the reflection representation of W'. Indeed, by Theorem 3.2.12 in [7] there is a parabolic subgroup $W' \subset W$ such that $\mathcal{C} \cap W'$ is a cuspidal conjugacy class in W', i.e. every element in it is elliptic in W'. By Lemma 3.1.14 in [7] if $w \in \mathcal{C} \cap W'$ is of minimal possible length with respect to the restriction of $\underline{l}$ to W' then it is also of minimal possible length with respect to $\underline{l}$.

Let $P' \subset G$ be the parabolic subgroup which corresponds to W', and M' the semi-simple part of the Levi factor of P', so that W' is the Weyl group of M'. Let $\Phi_p^G(\mathcal{C})$ be the unipotent class in G_p containing the class $\pi_p^{M'} \Phi_1^{M'}(\mathcal{C})$. This class only depends on the conjugacy class $\mathcal{C}$, and hence one has a map $\Phi_p^G : \underline{W} \to \underline{\mathcal{N}}(G_p)$ which is in fact surjective by 4.5(a) in [15].

Let $\mathcal{C} \in \underline{W}$, and $m_{\mathcal{C}}$ the dimension of the fixed point space for the action of any $w \in \mathcal{C}$ in the reflection representation. Then by Theorem 0.2 in [14] for any $\gamma \in \underline{\mathcal{N}}(G_p)$ there is a unique $\mathcal{C}_0 \in (\Phi_p^G)^{-1}(\gamma)$ such that the function $m_{\mathcal{C}} : (\Phi_p^G)^{-1}(\gamma) \to \mathbb{N}$ reaches its minimum at $\mathcal{C}_0$. We denote $\mathcal{C}_0$ by $\Psi_p^G(\gamma)$. Thus one obtains an injective map $\Psi_p^G : \underline{\mathcal{N}}(G_p) \to \underline{W}$.

Now recall that using identifications $f_p : \underline{\mathcal{N}}(G_p) \to \mathcal{X}^p(W)$ one can define a bijection

$$F : \widehat{\mathcal{N}}(G) = \underline{\mathcal{N}}(G_1) \bigcup_{r \text{ prime}} \underline{\mathcal{N}}(G_r) \to \mathcal{X}^1(W) \bigcup_{r \text{ prime}} \mathcal{X}^r(W) = \mathcal{R}(W).$$

Using maps Φ_p^G one can also define a surjective map $\Phi^W : \underline{W} \to \widehat{\mathcal{N}}(G)$ as follows. If $\Phi_r^G(\mathcal{C}) \in \underline{\mathcal{N}}(G_1)$ for all $r > 1$ then $\Phi_r^G(\mathcal{C})$ is independent of r, and one puts $\Phi^W(\mathcal{C}) = \Phi_r^G(\mathcal{C})$ for any $r > 1$. If $\Phi_r^G(\mathcal{C}) \notin \underline{\mathcal{N}}(G_1)$ for some $r > 1$ then r is unique, one defines $\Phi^W(\mathcal{C}) = \Phi_r^G(\mathcal{C})$.

By definition there is a right-sided injective inverse Ψ^W to Φ^W such that if $\gamma \in \underline{\mathcal{N}}(G_1)$ then $\Psi^W(\gamma) = \Psi_1^G(\gamma)$, and if $\gamma \notin \underline{\mathcal{N}}(G_1)$, and $\gamma \in \underline{\mathcal{N}}(G_r)$ then $\Psi^W(\gamma) = \Psi_r^G(\gamma)$.

Denote by $C(W)$ the image of $\widehat{\mathcal{N}}(G)$ in $\underline{W}$ under the map Ψ^W, $C(W) = \Psi^W(\widehat{\mathcal{N}}(G))$. We shall identify $C(W)$, $\widehat{\mathcal{N}}(G)$ and $\mathcal{R}(W)$.

Now assume that p is not a bad prime for G_p. In this case the strata of the Lusztig partition can be described geometrically as follows. Let $\mathcal{C} \in C(W)$. Pick up a representative $w \in \mathcal{C}$ of minimal possible length with respect to $\underline{l}$. Denote by $\underline{G}_p$ the set of conjugacy classes in G_p, and by $G'_{\mathcal{C}}$ the set of all conjugacy classes in G_p which intersect the Bruhat cell $B_p \dot{w} B_p$. This definition does not depend on the choice of the the minimal possible length representative w. Let

$$d_{\mathcal{C}} = \min_{\gamma \in G'_{\mathcal{C}}} \dim \gamma.$$

Then the stratum $G_{\mathcal{C}} = \phi_{G_p}^{-1}(F(\Phi^W(\mathcal{C})))$ can be described as follows (see Theorem 2.2, [12]),

$$G_{\mathcal{C}} = \bigcup_{\gamma \in G'_{\mathcal{C}}, \ \dim \gamma = d_{\mathcal{C}}} \gamma.$$

Thus we have a disjoint union

$$G_p = \bigcup_{\mathcal{C} \in C(W)} G_{\mathcal{C}}.$$

Note that by the definition of the stratum , if $\mathcal{C} \in \mathrm{Im}(\Psi_1^G)$ then $G_\mathcal{C}$ contains a unique unipotent class, and if $\mathcal{C} \notin \mathrm{Im}(\Psi_1^G)$ then $G_\mathcal{C}$ does not contain unipotent classes.

For good p the maps introduced above are summarized in the following diagram

$$
\begin{array}{ccccccc}
\mathcal{X}^1(W) & \xleftarrow{\; f_1 \;} & \underline{\mathcal{N}}(G) & & & & \\
\downarrow \iota & & \downarrow \pi^G & & & & \\
G \xrightarrow{\; \phi_G \;} \mathcal{R}(W) & \xleftarrow{\; F \;} & \widehat{\mathcal{N}}(G) & \overset{\Phi^W}{\underset{\Psi^W}{\rightleftarrows}} & \underline{W}, & &
\end{array}
$$

(4.2)

where ι is an inclusion, bijections f_1 and F are induced by the Springer correspondence with the trivial local data, and the inclusion π^G is induced by the Spaltenstein map.

For exceptional groups the maps f_1 and F can be described explicitly using tables in [22], the maps Φ^W and Ψ^W can be described using the tables in Section 2 in [14], and the maps ι and π^G can be described explicitly using the tables of unipotent classes in [11], Chapter 22 or [22] (note that the labeling for unipotent classes in bad characteristics in [11] differs from that in [22]). The dimensions of the conjugacy classes in the strata in G can be obtained using dimension tables of centralizers of unipotent elements in case when a stratum contains a unipotent class (see [5, 11]), the tables for dimensions of the centralizers of unipotent elements in bad characteristic when a stratum does not contain a unipotent class (see [11]) or formula (4.1) and the tables of the values of the b–invariant b_E for representations of Weyl groups (see [5, 7]). Note that formula (4.1) implies that if $\mathcal{O}$ is any conjugacy class in $G_\mathcal{C}$, $\mathcal{O} \in G_\mathcal{C}$ then

(4.3) $\dim \mathcal{O} = \dim \Phi^W(\mathcal{C})$.

In case of classical groups all those maps and dimensions are described in terms of partitions (see [5, 8, 11, 13, 14, 15, 21]). In case of classical matrix groups the strata can also be described explicitly (see [12]). We recall this description below. By (4.3) the dimensions of the conjugacy classes in every stratum of G are equal to the dimension of the corresponding conjugacy class in $\widehat{\mathcal{N}}(G)$. The dimensions of centralizers of unipotent elements in arbitrary characteristic can be found in [10, 11].

If $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m)$ is a partition we denote by $\lambda^* = (\lambda_1^* \geq \lambda_2^* \geq \ldots \geq \lambda_m^*)$ the corresponding dual partition. It is defined by the property that $\lambda_1^* = m$ and $\lambda_i^* - \lambda_{i+1}^* = l_i(\lambda)$, where $l_i(\lambda)$ is the number of times i appears in the partition λ. We also denote by $\tau(\lambda)$ the length of λ, $\tau(\lambda) = m$. If a partition μ is obtained from λ by adding a number of zeroes, we shall identify λ and μ.

$\mathbf{A_n}$. G is of type $\mathrm{SL}(V)$ where V is a vector space of dimension $n + 1 \geq 1$ over an algebraically closed field $\mathbf{k}$ of characteristic exponent $p \geq 1$. W is the group of permutations of $n + 1$ elements. All sets in (4.2), except for G, are isomorphic to the set of partitions of $n + 1$, and all maps, except for ϕ_G, are the identity maps.

To describe ϕ_G for $G = \mathrm{SL}(V)$ we choose a sufficiently large $m \in \mathbb{N}$. Let $g \in G$. For any $x \in \mathbf{k}^*$ let V_x be the generalized x–eigenspace of $g : V \to V$ and let $\lambda_1^x \geq \lambda_2^x \geq \ldots \geq \lambda_m^x$ be the sequence in $\mathbb{N}$ whose terms are the sizes of the Jordan blocks of $x^{-1}g : V_x \to V_x$. Then $\phi_G(g)$ is the partition $\lambda(g)_1 \geq \lambda(g)_2 \geq \ldots \geq \lambda(g)_m$ given by $\lambda(g)_j = \sum_{x \in \mathbf{k}^*} \lambda_j^x$.

If g is any element in the stratum G_λ corresponding to a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m)$, $\lambda_m \geq 1$, then

(4.4) $\dim Z_G(g) = n + 2 \sum_{i=1}^{m} (i - 1)\lambda_i$.

The element of $\underline{W}$ which corresponds to λ is the Coxeter class in the Weyl subgroup of the type

(4.5) $A_{\lambda_1 - 1} + A_{\lambda_2 - 1} + \ldots + A_{\lambda_m - 1}$.

The summands in the diagram above are called blocks. Blocks of type A_0 are called trivial.

$\mathbf{C_n}$. G is of type $\mathrm{Sp}(V)$ where V is a symplectic space of dimension $2n$, $n \geq 2$ over an algebraically closed field $\mathbf{k}$ of characteristic exponent $p \neq 2$. W is the group of permutations of the set $E = \{\varepsilon_1, \ldots, \varepsilon_n, -\varepsilon_1, \ldots, -\varepsilon_n\}$ which also commute with the involution $\varepsilon_i \mapsto -\varepsilon_i$. Each element $s \in W$ can be expressed as a product of disjoint cycles of the form

$$\varepsilon_{k_1} \to \pm\varepsilon_{k_2} \to \pm\varepsilon_{k_3} \to \ldots \to \pm\varepsilon_{k_r} \to \pm\varepsilon_{k_1}.$$

The cycle above is of length r; it is called positive if $s^r(\varepsilon_{k_1}) = \varepsilon_{k_1}$ and negative if $s^r(\varepsilon_{k_1}) = -\varepsilon_{k_1}$. The lengths of the cycles together with their signs give a set of positive or negative integers called the signed cycle-type of s. To each positive cycle of s of length r there corresponds a pair of positive orbits $X, -X$, $|X| = r$, for the action of the group $\langle s \rangle$ generated by s on the set $E = \{\varepsilon_1, \ldots, \varepsilon_n, -\varepsilon_1, \ldots, -\varepsilon_n\}$, and to each negative cycle of s of length r there corresponds a negative orbit X, $|X| = 2r$, for the action of $\langle s \rangle$ on E. A positive cycle of length 1 is called trivial. It corresponds to a pair of fixed points for the action of $\langle s \rangle$ on E.

Elements of $\underline{W}$ are parametrized by pairs of partitions (λ, μ), where the parts of λ are even (for any $w \in \mathcal{C} \in \underline{W}$ they are the numbers of elements in the negative orbits X, $X = -X$, in E for the action of the group $\langle w \rangle$ generated by w), μ consists of pairs of equal parts (they are the numbers of elements in the positive $\langle w \rangle$–orbits X in E; these orbits appear in pairs $X, -X, X \neq -X$), and $\sum \lambda_i + \sum \mu_j = 2n$. We denote this set of pairs of partitions by $\mathcal{A}_{2n}^1$. An element of $\underline{W}$ which corresponds to a pair (λ, μ), $\lambda = (\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_m)$ and $\mu = (\mu_1 = \mu_2 \leq \ldots \leq \mu_{2k-1} = \mu_{2k})$ is the Coxeter class in the Weyl subgroup of the type

$$(4.6) \qquad C_{\frac{\lambda_1}{2}} + C_{\frac{\lambda_2}{2}} + \ldots + C_{\frac{\lambda_m}{2}} + A_{\mu_1-1} + A_{\mu_3-1} + \ldots + A_{\mu_{2k-1}-1}.$$

Elements of $\underline{\mathcal{N}}(G)$ are parametrized by partitions λ of $2n$ for which $l_j(\lambda)$ is even for odd j. We denote this set of partitions by $\mathcal{T}_{2n}$. In case of $G = \mathrm{Sp}(V)$ the parts of λ are just the sizes of the Jordan blocks in V of the unipotent elements from the conjugacy class corresponding to λ.

In this case $\underline{\widehat{\mathcal{N}}}(G) = \underline{\mathcal{N}}(G_2)$, and G_2 is of type $\mathrm{Sp}(V_2)$ where V_2 is a symplectic space of dimension $2n$ over an algebraically closed field of characteristic 2. Elements of $\underline{\mathcal{N}}(G_2)$ are parametrized by pairs (λ, ε), where $\lambda = (\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_m) \in \mathcal{T}_{2n}$, and $\varepsilon : \{\lambda_1, \lambda_2, \ldots, \lambda_m\} \to \{0, 1, \omega\}$ is a function such that

$$(4.7) \qquad \varepsilon(k) = \begin{cases} \omega & \text{if } k \text{ is odd}; \\ 1 & \text{if } k = 0; \\ 1 & \text{if } k > 0 \text{ is even}, l_k(\lambda) \text{ is odd}; \\ 0 \text{ or } 1 & \text{if } k > 0 \text{ is even}, l_k(\lambda) \text{ is even}. \end{cases}$$

We denote the set of such pairs (λ, ε) by $\mathcal{T}_{2n}^2$.

Elements of $\widehat{W}$ are parametrized by pairs of partitions (α, β) written in non–decreasing order, $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_{\tau(\alpha)}$, $\beta_1 \leq \beta_2 \leq \ldots \leq \beta_{\tau(\beta)}$, and such that $\sum \alpha_i + \sum \beta_i = n$. By adding zeroes we can assume that the length $\tau(\alpha)$ of α is related to the length of β by $\tau(\alpha) = \tau(\beta) + 1$. The set of such pairs is denoted by $X_{n,1}$.

The maps f_1, F can be described as follows. Let $\lambda = (\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{2m+1}) \in \mathcal{T}_{2n}$, and assume that $\lambda_1 = 0$. If $f_1(\lambda) = ((c'_1, c'_3, \ldots, c'_{2m+1}), (c'_2, c'_4, \ldots, c'_{2m}))$ then the parts c'_i are defined by induction starting from $c'_1 = 0$,

$$\begin{aligned} c'_i &= \tfrac{\lambda_i}{2} && \text{if } \lambda_i \text{ is even and } c'_{i-1} \text{ is already defined}; \\ c'_i &= \tfrac{\lambda_i+1}{2} && \text{if } \lambda_i = \lambda_{i+1} \text{ is odd and } c'_{i-1} \text{ is already defined}; \\ c'_{i+1} &= \tfrac{\lambda_i-1}{2} && \text{if } \lambda_i = \lambda_{i+1} \text{ is odd and } c'_i \text{ is already defined}. \end{aligned}$$

The image of f_1 consists of all pairs $((c'_1, c'_3, \ldots, c'_{2m+1}), (c'_2, c'_4, \ldots, c'_{2m})) \in X_{n,1}$ such that $c'_i \leq c'_{i+1} + 1$ for all i. If $F(\lambda, \varepsilon) = ((c_1, c_3, \ldots, c_{2m+1}), (c_2, c_4, \ldots, c_{2m}))$ then the parts c_i are defined by induction starting from $c_1 = 0$,

$$\begin{aligned} c_i &= \tfrac{\lambda_i}{2} && \text{if } \lambda_i \text{ is even}, \varepsilon(\lambda_i) = 1 \text{ and } c_{i-1} \text{ is already defined}; \\ c_i &= \tfrac{\lambda_i+1}{2} && \text{if } \lambda_i = \lambda_{i+1} \text{ is odd and } c_{i-1} \text{ is already defined}; \\ c_{i+1} &= \tfrac{\lambda_i-1}{2} && \text{if } \lambda_i = \lambda_{i+1} \text{ is odd and } c_i \text{ is already defined}; \\ c_i &= \tfrac{\lambda_i+2}{2} && \text{if } \lambda_i = \lambda_{i+1} \text{ is even}, \varepsilon(\lambda_i) = \varepsilon(\lambda_{i+1}) = 0 \text{ and } c_{i-1} \text{ is already defined}; \\ c_{i+1} &= \tfrac{\lambda_i-2}{2} && \text{if } \lambda_i = \lambda_{i+1} \text{ is even}, \varepsilon(\lambda_i) = \varepsilon(\lambda_{i+1}) = 0 \text{ and } c_i \text{ is already defined}. \end{aligned}$$

The image $\mathcal{R}(W)$ of F consists of all pairs $((c_1, c_3, \ldots, c_{2m+1}), (c_2, c_4, \ldots, c_{2m})) \in X_{n,1}$ such that $c_i \leq c_{i+1} + 2$ for all i.

The map Φ^W is defined by $\Phi^W(\lambda, \mu) = (\nu, \varepsilon)$, where the set of parts of ν is just the union of the sets of parts of λ and μ, and

$$\varepsilon(k) = \begin{cases} 1 & \text{if } k \in 2\mathbb{N} \text{ is a part of } \lambda; \\ 0 & \text{if } k \in 2\mathbb{N} \text{ is not a part of } \lambda; \\ \omega & \text{if } k \text{ is odd}. \end{cases}$$

The map Ψ^W associates to each pair (ν, ε) a unique point (λ, μ) in the preimage $(\Phi^W)^{-1}(\nu, \varepsilon)$ such that the number of parts of μ is minimal possible. This point is defined by the conditions

$$l_k(\lambda) = \begin{cases} 0 & \text{if } k \text{ is odd or } k \text{ is even}, l_k(\nu) \geq 2 \text{ is even and } \varepsilon(k) = 0; \\ l_k(\nu) & \text{otherwise}, \end{cases}$$

$$l_k(\mu) = \begin{cases} l_k(\nu) & \text{if } k \text{ is odd or } k \text{ is even}, l_k(\nu) \geq 2 \text{ is even and } \varepsilon(k) = 0; \\ 0 & \text{otherwise}. \end{cases}$$

The map π^G is given by $\pi^G(\lambda) = (\lambda, \varepsilon')$, where

$$\varepsilon'(k) = \begin{cases} \omega & \text{if } k \text{ is odd}; \\ 1 & \text{if } k \text{ is even}. \end{cases}$$

The map π^G is injective and its image consists of pairs $(\lambda, \varepsilon) \in \mathcal{T}_{2n}^2$, where ε satisfies the conditions above.

To describe ϕ_G for $G = \mathrm{Sp}(V)$ we choose a sufficiently large $m \in \mathbb{N}$. Let $g \in G$. For any $x \in \mathbf{k}^*$ let V_x be the generalized x–eigenspace of $g : V \to V$. For any $x \in \mathbf{k}^*$ such that $x^2 \neq 1$ let $\lambda_1^x \geq \lambda_2^x \geq \ldots \geq \lambda_{2m+1}^x$ be the sequence in $\mathbb{N}$ whose terms are the sizes of the Jordan blocks of $x^{-1}g : V_x \to V_x$.

For any $x \in \mathbf{k}^*$ with $x^2 = 1$ let $\lambda_1^x \geq \lambda_2^x \geq \ldots \geq \lambda_{2m+1}^x$ be the sequence in $\mathbb{N}$, where $((\lambda_1^x \geq \lambda_3^x \geq \ldots \geq \lambda_{2m+1}^x), (\lambda_2^x \geq \lambda_4^x \geq \ldots \geq \lambda_{2m}^x))$ is the pair of partitions such that the corresponding irreducible representation of the Weyl group of type $B_{\dim V_x/2}$ is the Springer representation attached to the unipotent element $x^{-1}g \in \mathrm{Sp}(V_x)$ and to the trivial local data.

Let $\lambda(g)$ be the partition $\lambda(g)_1 \geq \lambda(g)_2 \geq \ldots \geq \lambda(g)_{2m+1}$ given by $\lambda(g)_j = \sum_x \lambda_j^x$, where x runs over a set of representatives for the orbits of the involution $a \mapsto a^{-1}$ of $\mathbf{k}^*$. Now $\phi_G(g)$ is the pair of partitions $((\lambda(g)_1 \geq \lambda(g)_3 \geq \ldots \geq \lambda(g)_{2m+1}), (\lambda(g)_2 \geq \lambda(g)_4 \geq \ldots \geq \lambda(g)_{2m}))$.

If g is any element in the stratum $G_{(\lambda, \varepsilon)}$ corresponding to a pair $(\lambda, \varepsilon) \in \mathcal{T}_{2n}^2$, $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m)$ then

$$(4.8) \qquad \dim Z_G(g) = n + \sum_{i=1}^m (i-1)\lambda_i + \frac{1}{2}|\{i : \lambda_i \text{ is odd}\}| + |\{i : \lambda_i \text{ is even and } \varepsilon(\lambda_i) = 0\}|.$$

B_n. G is of type $\mathrm{SO}(V)$ where V is a vector space of dimension $2n+1$, $n \geq 2$ over an algebraically closed field $\mathbf{k}$ of characteristic exponent $p \neq 2$ equipped with a non–degenerate symmetric bilinear form. W is the same as in case of C_n.

An element of $\underline{W}$ which corresponds to a pair (λ, μ), $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m)$ and $\mu = (\mu_1 = \mu_2 \geq \ldots \geq \mu_{2k-1} = \mu_{2k})$ is the class represented by the sum of the blocks in the following diagram (we use the notation of [3], Section 7)

$$(4.9) \qquad \begin{aligned} & A_{\mu_1-1} + A_{\mu_3-1} + \ldots + A_{\mu_{2k-1}-1} + \\ & + D_{\frac{\lambda_1+\lambda_2}{2}}\left(a_{\frac{\lambda_2}{2}-1}\right) + D_{\frac{\lambda_3+\lambda_4}{2}}\left(a_{\frac{\lambda_4}{2}-1}\right) + \ldots + D_{\frac{\lambda_{m-2}+\lambda_{m-1}}{2}}\left(a_{\frac{\lambda_{m-1}}{2}-1}\right) + B_{\frac{\lambda_m}{2}} \ (m \text{ is odd}), \\ & A_{\mu_1-1} + A_{\mu_3-1} + \ldots + A_{\mu_{2k-1}-1} + \\ & + D_{\frac{\lambda_1+\lambda_2}{2}}\left(a_{\frac{\lambda_2}{2}-1}\right) + D_{\frac{\lambda_3+\lambda_4}{2}}\left(a_{\frac{\lambda_4}{2}-1}\right) + \ldots + D_{\frac{\lambda_{m-1}+\lambda_m}{2}}\left(a_{\frac{\lambda_m}{2}-1}\right) \ (m \text{ is even}), \end{aligned}$$

where it is assumed that $D_k(a_0) = D_k$.

The elements of $\underline{\mathcal{N}}(G)$ are parametrized by partitions λ of $2n+1$ for which $l_j(\lambda)$ is even for even j. We denote this set of partitions by $\mathcal{Q}_{2n+1}$. In case of $G = \mathrm{SO}(V)$ the parts of λ are just the sizes of the Jordan blocks in V of the unipotent elements from the conjugacy class corresponding to λ.

In this case $\widehat{\mathcal{N}}(G) = \underline{\mathcal{N}}(G_2)$, and G_2 is of type $\mathrm{SO}(V_2)$ where V_2 is a vector space of dimension $2n+1$ over an algebraically closed field of characteristic 2 equipped with a bilinear form $(\cdot, \cdot)$ and a non–zero quadratic form Q such that

$$(x, y) = Q(x+y) - Q(x) - Q(y), \ x, y \in V_2,$$

and the restriction of Q to the null space $V_2^\perp = \{x \in V_2 : (x, y) = 0 \ \forall \ y \in V_2\}$ of $(\cdot, \cdot)$ has zero kernel. In fact G_2 is isomorphic to a group of type $\mathrm{Sp}(V_2)$, $\dim V_2 = 2n$, and hence $\underline{\mathcal{N}}(G_2) \simeq \mathcal{T}_{2n}^2$.

We also have $\widehat{W} \simeq X_{n,1}$, and the map F is the same as in case of C_n.

The map f_1 can be described as follows. Let $\lambda = (\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{2m+1}) \in \mathcal{Q}_{2n+1}$. If

$$f_1(\lambda) = ((c_1', c_3', \ldots, c_{2m+1}'), (c_2', c_4', \ldots, c_{2m}'))$$

then the parts c_i' are defined by induction starting from c_1',

$$\begin{aligned} c_i' &= \frac{\lambda_i - 1}{2} + i - 1 - 2\left[\frac{i-1}{2}\right] && \text{if } \lambda_i \text{ is odd and } c_{i-1}' \text{ is already defined;} \\ c_i' &= \frac{\lambda_i}{2} && \text{if } \lambda_i = \lambda_{i+1} \text{ is even and } c_{i-1}' \text{ is already defined;} \\ c_{i+1}' &= \frac{\lambda_i}{2} && \text{if } \lambda_i = \lambda_{i+1} \text{ is even and } c_i' \text{ is already defined.} \end{aligned}$$

The image of f_1 consists of all pairs $((c_1', c_3', \ldots, c_{2m+1}'), (c_2', c_4', \ldots, c_{2m}')) \in X_{n,1}$ such that $c_i' \leq c_{i+1}'$ for all odd i and $c_i' \leq c_{i+1}' + 2$ for all even i.

The image $\mathcal{R}(W)$ of F consists of all pairs $((c_1, c_3, \ldots, c_{2m+1}), (c_2, c_4, \ldots, c_{2m})) \in X_{n,1}$ such that $c_i \leq c_{i+1} + 2$ for all i.

The maps Φ^W and Ψ^W are the same as in case of C_n.

The map π^G is given by $\pi^G(\lambda) = (\nu, \varepsilon')$, $\lambda = (\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{2m+1}) \in \mathcal{Q}_{2n+1}$, where

$$\nu_i = \begin{cases} \lambda_i - 1 & \text{if } \lambda_i \text{ and } i \text{ are odd and } \lambda_{i-1} < \lambda_i; \\ \lambda_i + 1 & \text{if } \lambda_i \text{ is odd, } i \text{ is even and } \lambda_i < \lambda_{i+1}; \\ \lambda_i & \text{otherwise,} \end{cases}$$

and

$$\varepsilon'(k) = \begin{cases} \omega & \text{if } k \text{ is odd;} \\ 0 & \text{if } k \text{ is even, there exists even } \lambda_i = k \text{ with even } i \text{ such that } \lambda_{i-1} < \lambda_i; \\ 1 & \text{otherwise.} \end{cases}$$

The map π^G is injective and its image consists of pairs $(\nu, \varepsilon) \in \mathcal{T}^2_{2n}$ such that $\varepsilon(k) \neq 0$ if ν^*_k is odd and for each even i such that ν^*_i is even we have $\nu^*_{i-1} = \nu^*_i$, i.e. $i-1$ does not appear in the partition ν. Here $\nu^*_1 \geq \nu^*_2 \geq \ldots \geq \nu^*_m$ is the partition dual to ν.

To describe ϕ_G for $G = \mathrm{SO}(V)$ we choose a sufficiently large $m \in \mathbb{N}$. Let $g \in G$. For any $x \in \mathbf{k}^*$ let V_x be the generalized x–eigenspace of $g : V \to V$. For any $x \in \mathbf{k}^*$ such that $x^2 \neq 1$ let $\lambda^x_1 \geq \lambda^x_2 \geq \ldots \geq \lambda^x_{2m+1}$ be the sequence in $\mathbb{N}$ whose terms are the sizes of the Jordan blocks of $x^{-1}g : V_x \to V_x$.

For any $x \in \mathbf{k}^*$ with $x^2 = 1$ let $\lambda^x_1 \geq \lambda^x_2 \geq \ldots \geq \lambda^x_{2m+1}$ be the sequence in $\mathbb{N}$, where $((\lambda^x_1 \geq \lambda^x_3 \geq \ldots \geq \lambda^x_{2m+1}), (\lambda^x_2 \geq \lambda^x_4 \geq \ldots \geq \lambda^x_{2m}))$ is the pair of partitions such that the corresponding irreducible representation of the Weyl group of type $B_{(\dim V_x-1)/2}$ (if $x \neq -1$) or $D_{\dim V_x/2}$ (if $x = -1$) is the Springer representation attached to the unipotent element $x^{-1}g \in \mathrm{SO}(V_x)$ and to the trivial local data.

Let $\lambda(g)$ be the partition $\lambda(g)_1 \geq \lambda(g)_2 \geq \ldots \geq \lambda(g)_{2m+1}$ given by $\lambda(g)_j = \sum_x \lambda^x_j$, where x runs over a set of representatives for the orbits of the involution $a \mapsto a^{-1}$ of $\mathbf{k}^*$. Now $\phi_G(g)$ is the pair of partitions $((\lambda(g)_1 \geq \lambda(g)_3 \geq \ldots \geq \lambda(g)_{2m+1}), (\lambda(g)_2 \geq \lambda(g)_4 \geq \ldots \geq \lambda(g)_{2m}))$.

If g is any element in the stratum $G_{(\lambda,\varepsilon)}$ corresponding to a pair $(\lambda, \varepsilon) \in \mathcal{T}^2_{2n}$, $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m)$ then the dimension of the centralizer of g in G is given by formula (4.8),

$$(4.10) \qquad \dim Z_G(g) = n + \sum_{i=1}^m (i-1)\lambda_i + \frac{1}{2}|\{i : \lambda_i \text{ is odd}\}| + |\{i : \lambda_i \text{ is even and } \varepsilon(\lambda_i) = 0\}|.$$

$\mathbf{D_n.}$ G is of type $\mathrm{SO}(V)$ where V is a vector space of dimension $2n$, $n \geq 3$ over an algebraically closed field $\mathbf{k}$ of characteristic exponent $p \neq 2$ equipped with a non–degenerate symmetric bilinear form. W is the group of even permutations of the set $E = \{\varepsilon_1, \ldots, \varepsilon_n, -\varepsilon_1, \ldots, -\varepsilon_n\}$ which also commute with the involution $\varepsilon_i \mapsto -\varepsilon_i$. W can be regarded as a subgroup in the Weyl group W' of type C_n.

Let $\widetilde{W}$ be the set of W'–conjugacy classes in W. Elements of $\widetilde{W}$ are parametrized by pairs of partitions (λ, μ), where the parts of λ are even (for any $w \in \mathcal{C} \in \widetilde{W}$ they are the numbers of elements in the negative orbits X, $X = -X$, in E for the action of the group $\langle w \rangle$ generated by w), the number of parts of λ is even, μ consists of pairs of equal parts (they are the numbers of elements in the positive $\langle w \rangle$–orbits X in E; these orbits appear in pairs $X, -X, X \neq -X$), and $\sum \lambda_i + \sum \mu_j = 2n$. We denote this set of pairs of partitions by $\mathcal{A}^0_{2n}$. To each pair $(-, \mu)$, where all parts of μ are even, there correspond two conjugacy classes in W. To all other elements of $\mathcal{A}^0_{2n}$ there corresponds a unique conjugacy class in W.

An element of $\widetilde{W}$ which corresponds to a pair (λ, μ), $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m)$ and $\mu = (\mu_1 = \mu_2 \geq \ldots \geq \mu_{2k-1} = \mu_{2k})$ is the class represented by the sum of the blocks in the following diagram (we use the notation of [3], Section 7)

$$(4.11) \qquad A_{\mu_1-1} + A_{\mu_3-1} + \ldots + A_{\mu_{2k-1}-1} + D_{\frac{\lambda_1+\lambda_2}{2}}(a_{\frac{\lambda_2}{2}-1}) + D_{\frac{\lambda_3+\lambda_4}{2}}(a_{\frac{\lambda_4}{2}-1}) + \ldots + D_{\frac{\lambda_{m-1}+\lambda_m}{2}}(a_{\frac{\lambda_m}{2}-1}).$$

Let G' be the extension of G by the Dynkin graph automorphism of order 2. Then G' is of type $\mathrm{O}(V)$. Denote by $\widetilde{\mathcal{N}}(G)$ the set of unipotent classes of G'. Note that they are all contained in G. The elements of $\widetilde{\mathcal{N}}(G)$ are parametrized by partitions λ of $2n$ for which $l_j(\lambda)$ is even for even j. Note that the number of parts of such partitions is even. We denote this set of partitions by $\mathcal{Q}_{2n}$. In case when $G = \mathrm{SO}(V)$ the parts of λ are just the sizes of the Jordan blocks in V of the unipotent elements from the conjugacy class corresponding to λ. If λ has only even parts then λ corresponds to two unipotent classes in G of the same dimension. In all other cases there is a unique unipotent class in G which corresponds to λ.

Let V_2 be a vector space of dimension $2n$ over an algebraically closed field of characteristic 2 equipped with a non–degenerate bilinear form $(\cdot, \cdot)$ and a non–zero quadratic form Q such that

$$(x, y) = Q(x + y) - Q(x) - Q(y), \ x, y \in V_2.$$

We remind that $\mathrm{SO}(V_2)$ is the connected component containing the identity of the group of linear automorphisms of V_2 preserving the quadratic, and hence the bilinear, form.

One has $\widehat{\mathcal{N}}(G) = \mathcal{N}(G_2)$, and G_2 is of type $\mathrm{SO}(V_2)$.

Let G'_2 be the extension of G_2 by the Dynkin graph automorphism of order 2. Then G'_2 is of type $\mathrm{O}(V_2)$. Denote by $\widetilde{\mathcal{N}}(G_2)$ the set of unipotent classes of G'_2 contained in G_2. Since the bilinear form $(\cdot, \cdot)$ is also alternating in characteristic 2 there is a natural injective homomorphism from $\mathrm{O}(V_2)$ to $\mathrm{Sp}(V_2)$, $\dim V_2 = 2n$, and hence $\widetilde{\mathcal{N}}(G_2) \simeq \widetilde{\mathcal{T}}^2_{2n}$, where $\widetilde{\mathcal{T}}^2_{2n}$ is the set of elements $(\lambda, \varepsilon) \in \mathcal{T}^2_{2n}$ such that λ has an even number of parts.

Let $\widehat{\widetilde{W}}$ be the set of orbits of irreducible characters of W under the action of W'. Elements of $\widehat{\widetilde{W}}$ are parametrized by unordered pairs of partitions (α, β) written in non–decreasing order, $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_{\tau(\alpha)}$, $\beta_1 \leq \beta_2 \leq \ldots \leq \beta_{\tau(\beta)}$, and such that $\sum \alpha_i + \sum \beta_i = n$. By adding zeroes we can assume that the length of α is equal to the length of β. The set of such pairs is denoted by $Y_{n,0}$.

Instead of the maps in (4.2) we shall describe the following maps

(4.12)
$$
\begin{array}{ccc}
\widetilde{\mathcal{X}}^1(W) & \xleftarrow{\widetilde{f}_1} & \widetilde{\underline{\mathcal{N}}}(G) \\
\downarrow \iota & & \downarrow \widetilde{\pi}^G \\
G \xrightarrow{\widetilde{\phi}_G} \widetilde{\mathcal{R}}(W) & \xleftarrow{\widetilde{F}} \widetilde{\underline{\mathcal{N}}}(G_2) & \underset{\widetilde{\Psi}^W}{\overset{\widetilde{\Phi}^W}{\rightrightarrows}} \widetilde{W},
\end{array}
$$

where $\widetilde{f}_1$ and $\widetilde{F}$ are induced by the restrictions of the maps f_1 and F for G', G'_2 to $\widetilde{\underline{\mathcal{N}}}(G)$, $\widetilde{\underline{\mathcal{N}}}(G_2)$, respectively, $\widetilde{\mathcal{X}}^1(W)$ and $\widetilde{\mathcal{R}}(W)$ are their images, $\widetilde{\phi}_G, \widetilde{\Psi}^W, \widetilde{\Phi}^W$ and $\widetilde{\pi}^G$ are also induced by the corresponding maps for G', G'_2 and W'.

The map $\widetilde{f}_1$ is defined as in case of B_n. The image of $\widetilde{f}_1$ consists of all pairs $((c'_1, c'_3, \ldots, c'_{2m+1}), (c'_2, c'_4, \ldots, c'_{2m})) \in Y_{n,0}$ such that $c'_i \leq c'_{i+1}$ for all odd i and $c'_i \leq c'_{i+1} + 2$ for all even i.

If $(\lambda, \varepsilon) \in \widetilde{\mathcal{T}}^2_{2n}$, $\lambda = (\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{2m})$ and $\widetilde{F}(\lambda, \varepsilon) = ((c_1, c_3, \ldots, c_{2m-1}), (c_2, c_4, \ldots, c_{2m}))$ then the parts c_i are defined by induction starting from c_1,

$$
\begin{array}{ll}
c_i = \frac{\lambda_i - 2}{2} + 2(i-1) - 4\left[\frac{i-1}{2}\right] & \text{if } \lambda_i \text{ is even, } \varepsilon(\lambda_i) = 1 \text{ and } c_{i-1} \text{ is already defined;} \\
c_i = \frac{\lambda_i - 1}{2} + 2(i-1) - 4\left[\frac{i-1}{2}\right] & \text{if } \lambda_i = \lambda_{i+1} \text{ is odd and } c_{i-1} \text{ is already defined;} \\
c_{i+1} = \frac{\lambda_i - 3}{2} + 2i - 4\left[\frac{i}{2}\right] & \text{if } \lambda_i = \lambda_{i+1} \text{ is odd and } c_i \text{ is already defined;} \\
c_i = \frac{\lambda_i}{2} + 2(i-1) - 4\left[\frac{i-1}{2}\right] & \text{if } \lambda_i = \lambda_{i+1} \text{ is even, } \varepsilon(\lambda_i) = 0 \text{ and } c_{i-1} \text{ is already defined;} \\
c_{i+1} = \frac{\lambda_i}{2} + 2(i-1) - 4\left[\frac{i-1}{2}\right] & \text{if } \lambda_i = \lambda_{i+1} \text{ is even, } \varepsilon(\lambda_i) = 0 \text{ and } c_i \text{ is already defined.}
\end{array}
$$

The image $\widetilde{\mathcal{R}}(W)$ of $\widetilde{F}$ consists of all pairs $((c_1, c_3, \ldots, c_{2m+1}), (c_2, c_4, \ldots, c_{2m})) \in Y_{n,0}$ such that $c_i \leq c_{i+1}$ for all odd i and $c_i \leq c_{i+1} + 4$ for all even i.

The maps $\widetilde{\Phi}^W$ and $\widetilde{\Psi}^W$ are defined by the same formulas as in case of C_n.

The map $\widetilde{\pi}^G$ is given by $\widetilde{\pi}^G(\lambda) = (\nu, \varepsilon')$, $\lambda = (\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_{2m}) \in \mathcal{Q}_{2n}$, where

$$
\nu_i = \begin{cases}
\lambda_i - 1 & \text{if } \lambda_i \text{ is odd, } i \text{ is even and } \lambda_{i-1} < \lambda_i; \\
\lambda_i + 1 & \text{if } \lambda_i \text{ and } i \text{ are odd, and } \lambda_i < \lambda_{i+1}; \\
\lambda_i & \text{otherwise,}
\end{cases}
$$

and

$$
\varepsilon'(k) = \begin{cases}
\omega & \text{if } k \text{ is odd;} \\
0 & \text{if } k \text{ is even, there exists even } \lambda_i = k \text{ with odd } i \text{ such that } \lambda_{i-1} < \lambda_i; \\
1 & \text{otherwise.}
\end{cases}
$$

The map $\widetilde{\pi}^G$ is injective and its image consists of pairs $(\nu, \varepsilon) \in \widetilde{\mathcal{T}}^2_{2n}$ such that $\varepsilon(k) \neq 0$ if ν^*_k is odd and for each even i such that ν^*_i is even we have $\nu^*_{i-1} = \nu^*_i$, i.e. $i-1$ does not appear in the partition ν. Here $\nu^*_1 \geq \nu^*_2 \geq \ldots \geq \nu^*_m$ is the partition dual to ν.

To describe $\widetilde{\phi}_G$ for $G = \mathrm{SO}(V)$ we choose a sufficiently large $m \in \mathbb{N}$. Let $g \in G$. For any $x \in \mathbf{k}^*$ let V_x be the generalized x–eigenspace of $g : V \to V$. For any $x \in \mathbf{k}^*$ such that $x^2 \neq 1$ let $\lambda^x_1 \geq \lambda^x_2 \geq \ldots \geq \lambda^x_{2m}$ be the sequence in $\mathbb{N}$ whose terms are the sizes of the Jordan blocks of $x^{-1}g : V_x \to V_x$.

For any $x \in \mathbf{k}^*$ with $x^2 = 1$ let $\lambda^x_1 \geq \lambda^x_2 \geq \ldots \geq \lambda^x_{2m}$ be the sequence in $\mathbb{N}$, where $((\lambda^x_1 \geq \lambda^x_3 \geq \ldots \geq \lambda^x_{2m-1}), (\lambda^x_2 \geq \lambda^x_4 \geq \ldots \geq \lambda^x_{2m}))$ is the pair of partitions such that the corresponding irreducible representation of the Weyl group of type $D_{\dim V_x/2}$ is the Springer representation attached to the unipotent element $x^{-1}g \in \mathrm{SO}(V_x)$ and to the trivial local data.

Let $\lambda(g)$ be the partition $\lambda(g)_1 \geq \lambda(g)_2 \geq \ldots \geq \lambda(g)_{2m+1}$ given by $\lambda(g)_j = \sum_x \lambda^x_j$, where x runs over a set of representatives for the orbits of the involution $a \mapsto a^{-1}$ of $\mathbf{k}^*$. Now $\widetilde{\phi}_G(g)$ is the pair of partitions $((\lambda(g)_1 \geq \lambda(g)_3 \geq \ldots \geq \lambda(g)_{2m-1}), (\lambda(g)_2 \geq \lambda(g)_4 \geq \ldots \geq \lambda(g)_{2m}))$.

The preimage $\widetilde{\phi}_G^{-1}(\lambda, \mu)$ is a stratum in G in all cases except for the one when the pair (λ, μ) is of the form $((\lambda_1 \geq \lambda_3 \geq \ldots \geq \lambda_{2m-1}), (\lambda_1 \geq \lambda_3 \geq \ldots \geq \lambda_{2m-1}))$. In that case $\widetilde{\phi}_G^{-1}(\lambda, \mu)$ is a union of two strata, and the conjugacy classes in each of them have the same dimension.

If g is any element in the stratum $G_{(\lambda,\varepsilon)}$ corresponding to a pair $(\lambda,\varepsilon) \in \widetilde{\mathcal{T}}_{2n}^2$, $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m)$ then the dimension of the centralizer of g in G is given by the following formula

$$(4.13) \qquad \dim Z_G(g) = n + \sum_{i=1}^{m}(i-1)\lambda_i - \frac{1}{2}|\{i : \lambda_i \text{ is odd}\}| - |\{i : \lambda_i \text{ is even and } \varepsilon(\lambda_i) = 1\}|.$$

5. The strict transversality condition

In this section for every conjugacy class $\mathcal{C} \in C(W)$ we define a transversal slice Σ_s, $s \in \mathcal{C}$ such that every conjugacy class $\mathcal{O} \in G_{\mathcal{C}}$ intersects Σ_s and

$$(5.1) \qquad\qquad\qquad\qquad\qquad \dim \mathcal{O} = \operatorname{codim} \Sigma_s.$$

Recall that the definition of Σ_s is given in terms of a system of positive roots Δ_+ associated to s, and Δ_+ depends on the choice of ordering of terms in decomposition (3.13). It turns out that in order to fulfill condition (5.1) the subspaces $\mathfrak{h}_i$ in (3.13) should be ordered in such a way that $\mathfrak{h}_0$ is the linear subspace of $\mathfrak{h}_{\mathbb{R}}$ fixed by the action of s and if $\mathfrak{h}_i = \mathfrak{h}_\lambda^k$, $\mathfrak{h}_j = \mathfrak{h}_\mu^l$ and $0 \leq \lambda < \mu < 1$ then $i < j$, where λ and μ are eigenvalues of the corresponding matrix $I - M$ for s. In case of exceptional root systems this is verified using a computer program, and in case of classical root systems this is confirmed by explicit computation based on a technical lemma. In order to formulate this lemma we recall realizations of classical irreducible root systems.

Let V be a real Euclidean n–dimensional vector space with an orthonormal basis $\varepsilon_1, \ldots, \varepsilon_n$. The root systems of types A_{n-1}, B_n, C_n and D_n can be realized in V as follows.

A_n. The roots are $\varepsilon_i - \varepsilon_j$, $1 \leq i,j \leq n$, $i \neq j$, $\mathfrak{h}_{\mathbb{R}}$ is the hyperplane in V consisting of the points the sum of whose coordinates is zero.

B_n. The roots are $\pm\varepsilon_i \pm \varepsilon_j$, $1 \leq i < j \leq n$, $\pm\varepsilon_i$, $1 \leq i \leq n$, $\mathfrak{h}_{\mathbb{R}} = V$.

C_n. The roots are $\pm\varepsilon_i \pm \varepsilon_j$, $1 \leq i < j \leq n$, $\pm 2\varepsilon_i$, $1 \leq i \leq n$, $\mathfrak{h}_{\mathbb{R}} = V$.

D_n. The roots are $\pm\varepsilon_i \pm \varepsilon_j$, $1 \leq i < j \leq n$, $\mathfrak{h}_{\mathbb{R}} = V$.

In all cases listed above the corresponding Weyl group W is a subgroup of the Weyl group of type C_n acting on the elements of the basis $\varepsilon_1, \ldots, \varepsilon_n$ by permuting the basis vectors and changing the sign of an arbitrary subset of them.

Now we formulate the main lemma.

Lemma 5.1. *Let s be an element of the Weyl group of type C_n operating on the set $E = \{\varepsilon_1, \ldots, \varepsilon_n, -\varepsilon_1, \ldots, -\varepsilon_n\}$ as indicated in Section 4, where $\varepsilon_1, \ldots, \varepsilon_n$ is the basis of V introduced above. Assume that s has either only one nontrivial cycle of length $k/2$ (k is even), which is negative, or only one nontrivial cycle of length k, which is positive, $1 < k \leq n$. Let Δ be a root system of type A_{n-1}, B_n, C_n or D_n realized in V as above.*

(i) If s has only one nontrivial cycle of length $k/2$, which is negative, then k is even, the spectrum of s in the complexification $V_{\mathbb{C}}$ of V is $\epsilon_r = \exp(\frac{2\pi i(k-2r+1)}{k})$, $r = 1, \ldots, k/2$, and possibly $\epsilon_0 = 1$, all eigenvalues are simple except for possibly 1.

(ii) If s only has one nontrivial cycle of length k, which is positive, then the spectrum of s in the complexification of V is $\epsilon_r = \exp(\frac{2\pi i(k-r)}{k})$, $r = 1, \ldots, k-1$, and $\epsilon_0 = 1$, all eigenvalues are simple except for possibly 1.

In both cases we denote by V_r the invariant subspace in V which corresponds to $\epsilon_r = \exp(\frac{2\pi i([k/2]+1-r)}{k})$, $r = 1, \ldots, \left[\frac{k}{2}\right]$ or $\epsilon_0 = 1$ in case of a positive nontrivial cycle and to $\epsilon_r = \exp(\frac{2\pi i(2\left[\frac{k/2+1}{2}\right]+1-2r)}{k})$, $r = 1, \ldots, \left[\frac{k/2+1}{2}\right]$ or $\epsilon_0 = 1$ in case of a negative cycle. For $r \neq 0$ the space V_r is spanned by the real and the imaginary parts of a nonzero eigenvector of s in $V_{\mathbb{C}}$ corresponding to ϵ_r, and V_0 is the subspace of fixed points of s in V.

V_r is two–dimensional if $\epsilon_r \neq \pm 1$, one–dimensional if $\epsilon_r = -1$ or may have arbitrary dimension if $\epsilon_r = 1$.

Let Δ_+ be a system of positive roots associated to s and defined as in Section 3, where we use the decomposition

$$(5.2) \qquad\qquad\qquad\qquad\qquad V = \bigoplus_i V_i$$

as (3.13) in the definition of Δ_+. Denote by $\overline{\Delta}_i \subset \Delta$ the corresponding subsets of roots defined as in (3.14).

Let Δ_0^s be the root subsystem fixed by the action of s and $l(s)$ the number of positive roots which become negative under the action of s.

(iii) If s has only one nontrivial cycle of length k, which is positive, we have

    (1) if $\Delta = A_{n-1}$ then $\Delta_0^s = A_{n-k-1}$, $l(s) = 2n - k - 1$;

    (2) if $\Delta = B_n(C_n)$ then $\Delta_0^s = B_{n-k}(C_{n-k})$, $l(s) = 4n - 2k$ for odd k and $l(s) = 4n - 2k + 1$ for even k;

    (3) if $\Delta = D_n$ then $\Delta_0^s = D_{n-k}$, $l(s) = 4n - 2k - 2$ for odd k and $l(s) = 4n - 2k - 1$ for even k.

(iv) If s has only one nontrivial cycle of length $\frac{k}{2}$, which is negative, we have

    (1) if $\Delta = B_n(C_n)$ then $\Delta_0^s = B_{n-k/2}(C_{n-k/2})$, $l(s) = 2n - k/2$;

    (2) if $\Delta = D_n$ then $\Delta_0^s = D_{n-k/2}$, $l(s) = 2n - k/2 - 1$.

(v) If s has only one nontrivial cycle of length k, which is positive, Δ is of type B_n, C_n or D_n, and k is even then $\Delta = \overline{\Delta}_{k/2} \bigcup \overline{\Delta}_{k/2-1} \bigcup \Delta_0^s$ (disjoint union), and all roots in $\overline{\Delta}_{k/2-1}$ are orthogonal to the fixed point subspace for the action of s on V.

(vi) In all other cases $\Delta = \overline{\Delta}_{i_{\max}} \bigcup \Delta_0^s$ (disjoint union), where $i_{\max}$ is the maximal possible index i which appears in decomposition (5.2).

Proof. The proof is similar in all cases. We only give details in the most complicated case when s has only one nontrivial cycle, which is positive, Δ is of type $B_n(C_n)$, and k is even. Without loss of generality one can assume that s corresponds to the cycle of the form

$$\varepsilon_1 \to \varepsilon_2 \to \varepsilon_4 \to \varepsilon_6 \to \cdots \to \varepsilon_{k-2} \to \varepsilon_k \to \varepsilon_{k-1} \to \varepsilon_{k-3} \to \cdots \to \varepsilon_3 \to \varepsilon_1 \ (k > 2), \ \varepsilon_1 \to \varepsilon_2 \to \varepsilon_1 \ (k = 2).$$

From this definition one easily sees that $\Delta_0^s = B_{n-k}(C_{n-k}) = \Delta \cap V'$, where $V' \subset V$ is the subspace generated by $\varepsilon_{k+1}, \ldots, \varepsilon_n$. Computing the eigenvalues of s in $V_{\mathbb{C}}$ is a standard exercise in linear algebra. The eigenvalues are expressed in terms of the exponents of the root system of type A_{k-1} (see [4], Ch. 10).

The invariant subspace V_r is spanned by the real and the imaginary parts of a nonzero eigenvector of s in $V_{\mathbb{C}}$ corresponding to the eigenvalue ϵ_r. If $\epsilon_r \neq \pm 1$ then V_r is two–dimensional, and for $\epsilon_r = -1$ V_r is one–dimensional. In the former case V_r will be regarded as the real form of a complex plane with the orthonormal basis $1, i$. Under this convention the orthogonal projection operator onto V_r acts on the basic vectors ε_j as follows

$$(5.3) \qquad \varepsilon_{2j+1} \mapsto c\epsilon_r^j, \ j = 0, \ldots, \frac{k}{2} - 1, \varepsilon_{2j} \mapsto c\epsilon_r^{-j}, \ j = 1, \ldots, \frac{k}{2},$$

where $c = \sqrt{\frac{2}{k}}$. Consider the case when $k > 2$; the case $k = 2$ can be analyzed in a similar way.

To compute $l(s)$ using the definition of Δ_+ given in Section 3 one should first look at all roots which have nonzero projections onto $V_{k/2}$ on which s acts by rotation with the angle $\frac{2\pi}{k}$.

From (5.3) we deduce that the roots which are not fixed by s and have zero orthogonal projections onto $V_{k/2}$ are $\pm(\varepsilon_j + \varepsilon_{k-j+1})$, $j = 1, \ldots \frac{k}{2}$. The number of those roots is equal to k, and they all have nonzero orthogonal projections onto $V_{k/2-1}$. From (5.3) we also obtain that all the other roots which are not fixed by s have nonzero orthogonal projections onto $V_{k/2}$, hence $|\overline{\Delta}_{k/2-1}| = k$. The number of roots fixed by s is $2(n-k)^2$ since it is equal to the number of roots in $\overline{\Delta}_0 = \Delta_0^s = B_{n-k}(C_{n-k})$. Hence $\Delta = \overline{\Delta}_{k/2} \bigcup \overline{\Delta}_{k/2-1} \bigcup \overline{\Delta}_0$ (disjoint union), the number of roots in $\overline{\Delta}_{k/2}$ is $|\Delta| - |\overline{\Delta}_0| - |\overline{\Delta}_{k/2-1}| = 2n^2 - 2(n-k)^2 - k = 4nk - 2k^2 - k$, $|\overline{\Delta}_{k/2}| = 4nk - 2k^2 - k$.

Now using the symmetry of the root system Δ as a subset of V and the fact that s acts as rotation by the angles $\frac{2\pi}{k}$ and $\frac{4\pi}{k}$ in $V_{k/2}$ and $V_{k/2-1}$, respectively, we deduce that the number of positive roots in $\overline{\Delta}_{k/2}$ ($\overline{\Delta}_{k/2-1}$) which become negative under the action of s is equal to the number of roots in $\overline{\Delta}_{k/2}$ ($\overline{\Delta}_{k/2-1}$) divided by the order of s in $V_{k/2}(V_{k/2-1})$. Therefore

$$l(s) = \frac{|\overline{\Delta}_{k/2}|}{k} + \frac{|\overline{\Delta}_{k/2-1}|}{k/2} = \frac{4nk - 2k^2 - k}{k} + \frac{k}{k/2} = 4n - 2k + 1.$$

This completes the proof in the considered case. $\qquad\square$

Now we are in a position to prove the main statement of this paper.

Theorem 5.2. *Let G be a connected semisimple algebraic group over an algebraically closed filed $\mathbf{k}$ of characteristic good for G, and $\mathcal{O} \in \widehat{\mathcal{N}}(G)$. Let H be a maximal torus of G, W the Weyl group of the pair (G, H), and $s \in W$ an element from the conjugacy class $\Psi^W(\mathcal{O})$. Let Δ be the root system of the pair (G, H) and Δ_+ a system of positive roots in Δ associated to $\Psi^W(\mathcal{O})$ and defined in Section 3 with the help of decomposition (3.13), where the subspaces $\mathfrak{h}_i$ are ordered in such a way that $\mathfrak{h}_0$ is the linear subspace of $\mathfrak{h}_{\mathbb{R}}$ fixed by the action of s and if $\mathfrak{h}_i = \mathfrak{h}_\lambda^k$, $\mathfrak{h}_j = \mathfrak{h}_\mu^l$ and $0 \leq \lambda < \mu < 1$ then $i < j$. In case of exceptional root systems we assume, in addition, that Δ_+ is chosen as in the tables in Appendix 2, so that $s = s^1 s^2$ is defined by the data from columns three and four in the tables in Appendix 2. Then all conjugacy classes in the stratum $G_{\mathcal{O}} = \phi_G^{-1}(F(\mathcal{O}))$ intersect the corresponding transversal slice Σ_s at some points of the subvariety $\dot{s} H_0 N_s$, where $H_0 \subset H$ is the centralizer of $\dot{s}$ in H. Moreover, if $\mathcal{O} \in \underline{\mathcal{N}}(G_p) \subset \widehat{\mathcal{N}}(G)$ for some p, then for any $g \in G_{\mathcal{O}}$*

$$(5.4) \qquad \dim Z_G(g) = \dim \Sigma_s = \mathrm{codim}_{G_p} \mathcal{O}.$$

Proof. We shall divide the proof into several lemmas. First we compute the dimension of the slice Σ_s, $s \in \Psi^W(\mathcal{O})$ and justify that for any $g \in G_{\mathcal{O}}$ equality (5.4) holds.

Lemma 5.3. *Assume that the conditions of Theorem 5.2 are satisfied. Then for any $g \in G_{\mathcal{O}}$, where $\mathcal{O} \in \underline{\mathcal{N}}(G_p) \subset \widehat{\underline{\mathcal{N}}}(G)$ for some p, equality (5.4) holds, i.e.*

$$\dim Z_G(g) = \dim \Sigma_s = \operatorname{codim}_{G_p} \mathcal{O}.$$

Proof. Observe that by the definition of the slice Σ_s

$$\dim \Sigma_s = \underline{l}(s) + |\overline{\Delta}_0| + \dim \mathfrak{h}_0,$$

where $\underline{l}(s)$ is the length of s with respect to the system of simple roots in Δ_+. Hence to compute $\dim \Sigma_s$ we have to find all numbers in the right hand side of the last equality.

Consider the case of classical Lie algebras when each Weyl group element is a product of cycles in a permutation group. In this case the identity (5.4) is proved by a straightforward calculation using Lemma 5.1.

If G is of type A_n let s be a representative in the conjugacy class of the Weyl group which corresponds to a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_m)$. The particular ordering of the invariant subspaces $\mathfrak{h}_i$ in the formulation of this theorem implies that the length $\underline{l}(s)$ should be computed by successive application of Lemma 5.1 to the cycles s_i of s, which correspond to λ_i placed in a non–increasing order.

We should first apply Lemma 5.1 to the cycle s_1 of s which corresponds to the maximal part λ_1. In this case $l(s_1) = 2n - \lambda_1 + 1$ and $\Delta_0^{s_1} = A_{n-\lambda_1} = \Delta \setminus \overline{\Delta}_{i_T}$ in the notation of Section 3. The remaining cycles $s_2, \ldots, s_m$ of s corresponding to $\lambda_2 \geq \ldots \geq \lambda_m$ act on $\Delta_0^{s_1}$, and we can apply Lemma 5.1 to s_2 acting on $\Delta_0^{s_1}$ to get $l(s_2) = 2(n - \lambda_1) - \lambda_2 + 1$ and $\Delta_0^{s_2} = A_{n-\lambda_1-\lambda_2} = \Delta \setminus \left(\overline{\Delta}_{i_T} \bigcup \overline{\Delta}_{i_{T-1}}\right)$. Iterating this procedure and observing that $\underline{l}(s)$ is equal to the number of positive roots which become negative under the action of s we obtain

$$(5.5) \qquad \underline{l}(s) = \sum l(s_k), \ l(s_k) = 2(n - \sum_{i=1}^{k-1} \lambda_i) - \lambda_k + 1,$$

where the first sum in (5.5) is taken over k for which $\lambda_k > 1$.

The number of roots fixed by s can be represented in a similar form,

$$(5.6) \qquad |\overline{\Delta}_0| = \sum l(s_k), \ l(s_k) = 2(n - \sum_{i=1}^{k-1} \lambda_i) - \lambda_k + 1,$$

where the sum in (5.6) is taken over k for which $\lambda_k = 1$.

Finally the dimension of the fixed point space $\mathfrak{h}_0$ of s in $\mathfrak{h}$ is $m - 1$, $\dim \mathfrak{h}_0 = m - 1$.

Observe now that

$$(5.7) \qquad \dim \Sigma_s = \underline{l}(s) + |\overline{\Delta}_0| + \dim \mathfrak{h}_0,$$

and hence

$$\dim \Sigma_s = \sum_{k=1}^{m} l(s_k) + m - 1 = \sum_{k=1}^{m} \left(2(n - \sum_{i=1}^{k-1} \lambda_i) - \lambda_k + 1\right) + m - 1.$$

Exchanging the order of summation and simplifying this expression we obtain that

$$\dim \Sigma_s = n + 2 \sum_{i=1}^{m} (i - 1)\lambda_i$$

which coincides with (4.4).

The computations of $\dim \Sigma_s$ in case of B_n and of C_n are the same. If $(\nu, \varepsilon) \in \mathcal{T}_{2n}^2$, $\nu = (\nu_1 \geq \nu_2 \geq \ldots \geq \nu_m)$, corresponds to $\mathcal{O} \in \widehat{\underline{\mathcal{N}}}(G) = \underline{\mathcal{N}}(G_2)$ then $\Psi^W(\nu, \varepsilon) = (\lambda, \mu) \in \mathcal{A}_{2n}^1 \simeq \underline{W}$ is defined in Section 4, part $\mathbf{C_n}$. λ consists of even parts ν_i of ν for which $\varepsilon(\nu_i) = 1$, and μ consists of all odd parts of ν and of even parts ν_i of ν for which $\varepsilon(\nu_i) = 0$, the last two types of parts appear in pairs of equal parts. Let s be a representative in the conjugacy class $\Psi^W(\nu, \varepsilon)$. Then each part λ_i corresponds to a negative cycle of s of length $\frac{\lambda_i}{2}$, and each pair $\mu_i = \mu_{i+1}$ of equal parts of μ corresponds to a positive cycle of s of length μ_i. We order the cycles s_k of s associated to the (pairs of equal) parts of the partition ν in a way compatible with a non–increasing ordering of the parts of the partition $\nu = (\nu_1 \geq \nu_2 \geq \ldots \geq \nu_m)$, i.e. if we denote by s_k the cycle that corresponds to an even part ν_k of ν for which $\varepsilon(\nu_k) = 1$ or to a pair $\nu_k = \nu_{k+1}$ of odd parts of ν or of even parts of ν for which $\varepsilon(\nu_k) = 0$ then $s_k \geq s_l$ if $\nu_k \geq \nu_l$.

Similarly to the case of A_n, by the definition of Δ_+ and by Lemma 5.1 applied iteratively to the cycles s_k in the order defined above, the length $\underline{l}(s)$ of s is the sum of the following terms $l(s_k)$.

To each even part ν_k of ν for which $\varepsilon(\nu_k) = 1$ we associate the term

$$l(s_k) = 2(n - \sum_{i=1}^{k-1} \frac{\nu_i}{2}) - \frac{\nu_k}{2};$$

to each pair of odd parts $\nu_k = \nu_{k+1} > 1$ we associate the term

$$l(s_k) = 4(n - \sum_{i=1}^{k-1}\frac{\nu_i}{2}) - 2\nu_k = \left(2(n - \sum_{i=1}^{k-1}\frac{\nu_i}{2}) - \frac{\nu_k}{2}\right) + \left(2(n - \sum_{i=1}^{k}\frac{\nu_i}{2}) - \frac{\nu_{k+1}}{2}\right);$$

note that the sum of these terms over all pairs $\nu_k = \nu_{k+1} = 1$ gives the number $|\overline{\Delta}_0|$ of the roots fixed by s;

to each pair of even parts $\nu_k = \nu_{k+1}$ for which $\varepsilon(\nu_k) = 0$ we associate the term

$$l(s_k) = 4(n - \sum_{i=1}^{k-1}\frac{\nu_i}{2}) - 2\nu_k + 1 = \left(2(n - \sum_{i=1}^{k-1}\frac{\nu_i}{2}) - \frac{\nu_k}{2} + \frac{1}{2}\right) + \left(2(n - \sum_{i=1}^{k}\frac{\nu_i}{2}) - \frac{\nu_{k+1}}{2} + \frac{1}{2}\right).$$

The dimension of the fixed point space $\mathfrak{h}_0$ of s in $\mathfrak{h}$ is equal to a half of the sum of the number of all even parts ν_k for which $\varepsilon(\nu_k) = 0$ and of the number of all odd parts ν_k,

(5.8) $$\dim \mathfrak{h}_0 = \frac{1}{2}|\{i : \nu_i \text{ is odd}\}| + \frac{1}{2}|\{i : \nu_i \text{ is even and } \varepsilon(\nu_i) = 0\}|.$$

Finally substituting all the computed contributions into formula (5.7) we obtain

$$\dim \Sigma_s = \sum_{k=1}^{m}\left(2(n - \sum_{i=1}^{k-1}\frac{\nu_i}{2}) - \frac{\nu_k}{2}\right) + \frac{1}{2}|\{i : \nu_i \text{ is even and } \varepsilon(\nu_i) = 0\}| +$$

$$+ \frac{1}{2}|\{i : \nu_i \text{ is odd}\}| + \frac{1}{2}|\{i : \nu_i \text{ is even and } \varepsilon(\nu_i) = 0\}|.$$

Exchanging the order of summation and simplifying this expression we obtain that

(5.9) $$\dim \Sigma_s = n + \sum_{i=1}^{m}(i-1)\nu_i + \frac{1}{2}|\{i : \nu_i \text{ is odd}\}| + |\{i : \nu_i \text{ is even and } \varepsilon(\nu_i) = 0\}|$$

which coincides with (4.8) or (4.10).

In case of D_n the number $\dim \Sigma_s$ can be easily obtained if we observe that the map $\widetilde{\Psi}^W$ is defined by the same formula as Ψ^W in case of C_n. In case when $\widetilde{\Psi}^W(\nu, \varepsilon) = (-, \mu)$, where all parts of μ are even, there are two conjugacy classes in W which correspond to $\widetilde{\Psi}^W(\nu, \varepsilon)$. However, the numbers $\underline{l}(s)$, $|\overline{\Delta}_0|$ and $\dim \mathfrak{h}_0$ are the same in both cases. They only depend on $\widetilde{\Psi}^W(\nu, \varepsilon)$ in all cases. Let $s \in W$ be a representative from the conjugacy class $\widetilde{\Psi}^W(\nu, \varepsilon)$, $\nu = (\nu_1 \geq \nu_2 \geq \ldots \geq \nu_m)$.

From Lemma 5.1 we deduce that in case of D_n the contributions of the cycles s_k of s to the formula for $\dim \Sigma_s$ can be obtained from the corresponding contributions in case of C_n in the following way: for each pair of odd parts $\nu_k = \nu_{k+1}$ and for each pair of even parts $\nu_k = \nu_{k+1}$ with $\varepsilon(\nu_k) = 0$ the corresponding contribution $l(s_k)$ to $\underline{l}(s)$ should be reduced by 2 and for each even part ν_k of ν with $\varepsilon(\nu_k) = 1$ the corresponding contribution $l(s_k)$ to $\underline{l}(s)$ should be reduced by 1. This observation and formula (5.9) yield

$$\dim \Sigma_s = n + \sum_{i=1}^{m}(i-1)\nu_i + \frac{1}{2}|\{i : \nu_i \text{ is odd}\}| + |\{i : \nu_i \text{ is even and } \varepsilon(\nu_i) = 0\}| -$$

$$- |\{i : \nu_i \text{ is odd}\}| - |\{i : \nu_i \text{ is even}\}| =$$

$$= n + \sum_{i=1}^{m}(i-1)\nu_i - \frac{1}{2}|\{i : \nu_i \text{ is odd}\}| - |\{i : \nu_i \text{ is even and } \varepsilon(\nu_i) = 1\}|$$

which coincides with (4.13).

In case of root systems of exceptional types $\dim \Sigma_s$ can be found in the tables in Appendix 2. According to those tables equality (5.4) holds in all cases.

$\square$

Now we show that all conjugacy classes in the stratum $G_{\mathcal{O}} = \phi_G^{-1}(F(\mathcal{O}))$ intersect the corresponding transversal slice Σ_s, $s \in \Psi^W(\mathcal{O})$. Let Δ_+ be the system of the positive roots introduced in the statement of Theorem 5.2,

$$\mathfrak{h}_{\mathbb{R}} = \bigoplus_{i=0}^{K} \mathfrak{h}_i$$

the corresponding decomposition of $\mathfrak{h}_{\mathbb{R}}$ and $h_i \in \mathfrak{h}_i$ the corresponding elements of the subspaces $\mathfrak{h}_i$.

Let ${\mathfrak{h}_{\mathbb{R}}'}^{\perp}$ be the subspace of $\mathfrak{h}_{\mathbb{R}}$ fixed by the action of s. If ${\mathfrak{h}_{\mathbb{R}}'}^{\perp} = 0$, let $\mathfrak{h}_i' = \mathfrak{h}_i$ and $h_i' = h_i$, $i = 0, \ldots, K$. Otherwise let $\mathfrak{h}_K' = {\mathfrak{h}_{\mathbb{R}}'}^{\perp} = \mathfrak{h}_0$, $\mathfrak{h}_i' = \mathfrak{h}_{i+1}$, $h_i' = h_{i+1}$, $i = 0, \ldots, K-1$ and choose an element $h_K' \in \mathfrak{h}_K'$ such that

$h'_K(\alpha) \neq 0$ for any root $\alpha \in \Delta$ which is not orthogonal to the s–invariant subspace $\mathfrak{h}'_K$ with respect to the natural pairing between $\mathfrak{h}_\mathbb{R}$ and $\mathfrak{h}^*_\mathbb{R}$.

By a suitable rescaling of h'_K we can assume that conditions (3.15) are satisfied for the elements h'_i and roots α from sets $\overline{\Delta}'_i$ defined as in (3.19) with h_j, h_i replaced by h'_j, h'_i. Indeed, observe that

$$\overline{\Delta}'_i = \{\alpha \in \Delta : h'_j(\alpha) = 0, j > i, \ h'_i(\alpha) \neq 0\} \subset \{\alpha \in \Delta : h_j(\alpha) = 0, j > i+1, \ h_{i+1}(\alpha) \neq 0\} = \overline{\Delta}_{i+1}, i = 0, \ldots, K-1$$

by the definition of the elements h'_i. Thus, since (3.15) is satisfied for h_i, $i = 0, \ldots, K$, it is also satisfied for $h'_i = h_i$ if $i < K$. By a suitable rescaling of h'_K we can assume that (3.15) is satisfied for h'_K as well.

Let Δ^1_+ be a system of positive roots in $\Delta = \Delta(G, H)$ which corresponds to the Weyl chamber containing the element $\bar{h}' = \sum_{i=0}^K h'_i$. By Lemma 3.2 the set of roots $\Delta(L_1, H)$ with zero orthogonal projections onto $\mathfrak{h}'^\perp$ is the root system of a standard Levi subgroup $L_1 \subset G$ with respect to the system of simple roots in Δ^1_+.

Using formula (3.1) and recalling that the roots $\gamma_1, \ldots \gamma_{l'}$ form a linear basis of $\mathfrak{h}'$, i.e. for $i = 1, \ldots, l'$ γ_i has zero orthogonal projection onto $\mathfrak{h}'^\perp$, we deduce that for $i = 1, \ldots, l'$ $\gamma_i \in \Delta(L_1, H)$, and hence s belongs to the Weyl group $W_1 \subset W$ of the root system $\Delta(L_1, H)$. Note that, as L_1 is a standard Levi subgroup in G, W_1 is a parabolic subgroup in W with respect to the system of simple roots in Δ^1_+. Since $\gamma_1, \ldots \gamma_{l'}$ form a linear basis of $\mathfrak{h}'$, the linear span of roots from $\Delta(L_1, H)$ coincides with $\mathfrak{h}'$, and hence the element s is elliptic in W_1 as s acts without fixed points on $\mathfrak{h}'$.

Let w be a minimal length representative in the conjugacy class of s in W_1 with respect to the system of simple roots in $\Delta(L_1, H)_+ = \Delta^1_+ \cap \Delta(L_1, H)$. By Lemma 3.1.14 in [7] if $w \in \Psi^W(\mathcal{O}) \cap W_1$ is of minimal possible length with respect to the system of simple reflections in W_1 then it is also of minimal possible length with respect to the system of simple reflections in W, where in both cases the simple reflections are the reflections with respect to the simple roots in Δ^1_+. Note that w is elliptic in W_1 as well.

Let B_1 be the Borel subgroup in G corresponding to Δ^1_+, $P_1 \supset B_1$ the parabolic subgroup of G corresponding to W_1. Thus L_1 is the Levi factor of P_1.

Denote by $B_2 = B_1 \cap L_1$ the Borel subgroup in L_1. One can always find a representative $\dot{w} \in L_1$ of w.

Lemma 5.4. *Any conjugacy class in $G_\mathcal{O}$ intersects $B_2 \dot{w} B_2 \subset B_1 \dot{w} B_1$.*

Proof. By the definition the stratum $G_\mathcal{O}$ consists of all conjugacy classes of minimal possible dimension which intersect the Bruhat cell $B_1 \dot{w} B_1$. Denote by U_1 the unipotent radical of P_1. Then by the definition of parabolic subgroups one can always find a one parameter subgroup $\rho : \mathbf{k}^* \to Z_G(L_1)$ such that

$$(5.10) \qquad \lim_{t \to 0} \rho(t) n \rho(t^{-1}) = 1$$

for any $n \in U_1$.

Let $\gamma \in G_\mathcal{O}$ be a conjugacy class which intersects $B_1 \dot{w} B_1$ at point $b \dot{w} b'$, $b, b' \in B_1$ such that $b \dot{w} b' \notin B_2 \dot{w} B_2$. Since by definitions of B_2 and U_1 we have $B_1 = B_2 U_1$ there are unique factorizations $b = un$, $b' = u'n'$, $u, u' \in B_2$, $n, n' \in U_1$. By (5.10) we have

$$\lim_{t \to 0} \rho(t) b \dot{w} b' \rho(t^{-1}) = \lim_{t \to 0} u \rho(t) n \rho(t^{-1}) \dot{w} u' \rho(t) n' \rho(t^{-1}) = u \dot{w} u' \in B_2 \dot{w} B_2,$$

and hence the closure of γ contains a conjugacy class γ' which intersects $B_1 \dot{w} B_1$ at some point of $B_2 \dot{w} B_2 \subset B_1 \dot{w} B_1$. In particular, $\dim \gamma > \dim \gamma'$. This is impossible by the definition of $G_\mathcal{O}$, and hence γ intersects $B_1 \dot{w} B_1$ at some point of $B_2 \dot{w} B_2 \subset B_1 \dot{w} B_1$. $\square$

Lemma 5.5. *Let G be a connected semisimple algebraic group over an algebraically closed field $\mathbf{k}$ of characteristic good for G. Let H be a maximal torus of G, W the Weyl group of the pair (G, H), and $s \in W$ an elliptic element. Denote by $\mathcal{O}_s$ the conjugacy class of s in W. Then $\Phi^W(\mathcal{O}_s) \subset \underline{\mathcal{N}}(G)$.*

Proof. The statement of this lemma is a consequence of the fact that s is elliptic. Indeed, it suffices to consider the case when G is simple. In case when G is of type A_n this is obvious since $\widehat{\mathcal{N}}(G)$ contains only unipotent classes. In fact in this case $\mathcal{O}_s$ is the Coxeter class, and $\Phi^W(\mathcal{O}_s)$ is the class of regular unipotent elements. If G is of type B_n, C_n or D_n, formula (5.8) implies that if $\Phi^W(\mathcal{O}_s)$ corresponds to $(\nu, \varepsilon) \in \mathcal{T}^2_{2n}(\widetilde{\mathcal{T}}^2_{2n})$ then ν has no odd parts and no even parts ν_i with $\varepsilon(\nu_i) = 0$. According to the description given in the previous section the map $\pi^G(\widetilde{\pi}^G)$ is injective and its image consists of pairs $(\nu, \varepsilon) \in \mathcal{T}^2_{2n}(\widetilde{\mathcal{T}}^2_{2n})$ such that $\varepsilon(k) \neq 0$ if ν^*_k is odd and for each even i such that ν^*_i is even we have $\nu^*_{i-1} = \nu^*_i$, i.e. $i-1$ does not appear in the partition ν. We deduce that $\Phi^W(\mathcal{O}_s)$ is contained in the image of $\pi^G(\widetilde{\pi}^G)$, i.e. $\Phi^W(\mathcal{O}_s) \in \underline{\mathcal{N}}(G)$ is a unipotent class in G. In case when G is of exceptional type this can be checked by examining the tables in Appendix 2. $\square$

Now we show that in fact one can always take $w = s$.

Lemma 5.6. *The element s is of minimal length representative in its conjugacy class in W_1 with respect to the system of simple roots in $\Delta(L_1, H)_+ = \Delta_+^1 \cap \Delta(L_1, H)$.*

Proof. Let M_1 be the semisimple part of L_1 and $\mathcal{O}_n = \Phi^{W_1}(\mathcal{O}_w) \subset M_1$, where $\mathcal{O}_w$ is the conjugacy class of w in the Weyl group $W_1 = W(L_1, H)$. By the previous lemma applied to the group M_1 and the elliptic element $w \in W_1$ we have $\mathcal{O}_n \in \underline{\mathcal{N}}(M_1)$.

Therefore $\mathcal{O}_n$ is the unipotent class of minimal possible dimension which intersects $B_2 \dot{w} B_2$. By Theorem 0.7 in [15] the codimension of $\mathcal{O}_n$ in M_1 is equal to $\underline{l}_1(w)$, where $\underline{l}_1$ is the length function in W_1 with respect to the system of simple roots in $\Delta(L_1, H)_+ = \Delta_+^1 \cap \Delta(L_1, H)$,

$$(5.11) \qquad\qquad\qquad \mathrm{codim}_{M_1} \, \mathcal{O}_n = \underline{l}_1(w).$$

Now we show that s has minimal length in the Weyl group W_1 with respect to the system of simple roots in the set of positive roots $\Delta(L_1, H)_+$.

Indeed, let Σ'_s be the transversal slice in M_1 associated to $s \in W_1$ in Section 3, where we use $\Delta(L_1, H)_+$ as the system of positive roots in the definition of Σ'_s.

Formula (5.4) confirmed in Lemma 5.3 is applicable to the slice Σ'_s and yields

$$\mathrm{codim}_{M_1} \, \mathcal{O}_n = \dim \Sigma'_s.$$

Formula (5.7) and the fact that s is elliptic in W_1 imply that

$$\dim \Sigma'_s = \underline{l}_1(s),$$

From the last two formulas we infer

$$\mathrm{codim}_{M_1} \, \mathcal{O}_n = \dim \Sigma'_s = \underline{l}_1(s).$$

The last formula and (5.11) yield $\underline{l}_1(w) = \underline{l}_1(s)$, and hence s has minimal possible length in its conjugacy class in W_1 with respect to the system of simple roots in $\Delta(L_1, H)_+$.   □

Now we can assume that $s = w$.

Lemma 5.7. *Any conjugacy class $\gamma \in G_{\mathcal{O}}$ intersects Σ_s at some point of $\dot{s} H_0 N_s$.*

Proof. Let $\alpha \in \Delta \cap \mathfrak{h}'_{\mathbb{R}}$. Then $\alpha \in \overline{\Delta}_i \cap \mathfrak{h}'_{\mathbb{R}}$ for some $i > 0$. Observe that by the definition of the subspaces $\mathfrak{h}'_k$ and by the choice of the elements h_k, $k = 0, \dots, K$

$$\overline{\Delta}_i \cap \mathfrak{h}'_{\mathbb{R}} = \{\beta \in \Delta : h_j(\beta) = 0, j > i, h_i(\alpha) \neq 0, h_0(\alpha) = 0\} = \{\beta \in \Delta : h'_j(\beta) = 0, j > i - 1, h'_{i-1}(\alpha) \neq 0, \} = \overline{\Delta}'_{i-1},$$

and hence by (3.18) $\alpha \in (\overline{\Delta}_i)_+ \cap \mathfrak{h}'_{\mathbb{R}}$ if and only if $h'_{i-1}(\alpha) = h_i(\alpha) > 0$, i.e. $\alpha \in (\overline{\Delta}_i)_+ \cap \mathfrak{h}'_{\mathbb{R}}$ if and only if $\alpha \in \Delta_+^1 \cap \overline{\Delta}'_i$. Therefore if we denote $B_3 = B \cap L_1$, where B is the Borel subgroup corresponding to Δ_+, then $B_3 = B_1 \cap L_1 = B_2$.

By Lemma 5.4 applied to the element $s \in W$ any conjugacy class $\gamma \in G_{\mathcal{O}}$ intersects $B_2 \dot{s} B_2$, and hence it also intersects $B_3 \dot{s} B_3 \subset B \dot{s} B$ as $B_3 = B_2$. But by the definition of L_1 s acts on the root system of the pair (L_1, H) without fixed points. Since $B_3 = B \cap L_1$ and s fixes all the roots of the pair (ZH, H) we have an inclusion $B_3 \subset HN$, where N is the unipotent radical of the parabolic subgroup P attached to s. Hence $B_3 \dot{s} B_3 \subset N \dot{s} HN$.

Let H_0 be the centralizer of $\dot{s}$ in H. Let $\mathfrak{h}_{\mathbf{k}}$ be the Lie algebra of H and $\mathfrak{h}_{\mathbf{k}}^0$ the Lie algebra of H_0. Let $\mathfrak{h}'_{\mathbf{k}}$ be the s–invariant complementary subspace to $\mathfrak{h}_{\mathbf{k}}^0$ in $\mathfrak{h}_{\mathbf{k}}$. Since $\mathfrak{h}_{\mathbf{k}}$ is abelian $\mathfrak{h}'_{\mathbf{k}} \subset \mathfrak{h}_{\mathbf{k}}$ is a Lie algebra. Let $H' \subset H$ be the subgroup which corresponds to $\mathfrak{h}'_{\mathbf{k}}$ in H. If $h, h' \in H'$, $h = e^x, h' = e^y$, $x, y \in \mathfrak{h}'_{\mathbf{k}}$ then

$$h' \dot{s} h (h')^{-1} = \dot{s} e^{(s^{-1} - 1)x + y},$$

and for any y one can find a unique $x = \frac{1}{1 - s^{-1}} y$ such that $h' \dot{s} h (h')^{-1} = \dot{s}$. Note also that H normalizes N. Therefore the factorization $H = H_0 H'$ implies that any element of $N \dot{s} HN$ can be conjugated by an element of H' to an element from $N \dot{s} H_0 N$.

Finally observe that $N \dot{s} H_0 N \subset N \dot{s} ZN$, and hence any conjugacy class $\gamma \in G_{\mathcal{O}}$ intersects $N \dot{s} H_0 N \subset N \dot{s} ZN$. By Proposition 3.3 γ also intersects Σ_s. The proof of Proposition 3.3 in [18] implies that the Z–component of any element from $N \dot{s} ZN$ is equal to the Z–component in $\Sigma_s = \dot{s} ZN_s$ of its image under the isomorphism $N \dot{s} ZN \simeq N \times \dot{s} ZN_s$. Therefore any conjugacy class $\gamma \in G_{\mathcal{O}}$ intersects Σ_s at some point of $\dot{s} H_0 N_s$. This completes the proof.

                                                                          □

By the previous lemma the statement of this theorem holds.

                                                                          □

APPENDIX 1. SOME AUXILIARY RESULTS.

In this section we obtain two auxiliary numeric results which are required for the study of representations of quantum groups at roots of unity.

Proposition A1.1. *Let Δ be an irreducible root system, Δ_+ the system of positive roots associated to a conjugacy class of a Weyl group element $s \in W$ in Theorem 5.2, $s = s_{\gamma_1} \ldots s_{\gamma_{l'}}$, representation (3.1) for s. Let q be a positive integer defined by the tables in Appendix 2 for some irreducible exceptional root systems. Assume that q is not defined for the other exceptional root systems and for irreducible root systems of classical types. Let m be an odd positive integer which is not divisible by q if q is defined. Then for any integers $p_j \in \mathbb{Z}$, $j = 1, \ldots, l$ the system of equations $Y_j(\sum_{i=1}^{l'} m_i \gamma_i) = m p_j$, $j = 1, \ldots, l$ for unknowns $m_i \in \{-m+1, \ldots, m-1\}$, $i = 1, \ldots, l'$ has no nontrivial solutions; here Y_j are defined by $Y_j(\alpha_i) = d_i \delta_{ij}$, and α_i are the simple roots of Δ_+.*

Proof. If Δ is of exceptional type the statement of the theorem is verified with the help of a computer program (see Appendix 2).

If Δ is of classical type, we rewrite the system of equations in question in the form

$$(A1.1) \qquad \sum_{i=1}^{l'} x_i \gamma_i = \sum_{j=1}^{l} \frac{p_j}{d_j} \alpha_j, \ x_i = \frac{m_i}{m}.$$

We have to show that for any $p_j \in \mathbb{Z}$ this system has no nontrivial rational solutions $-1 < x_i < 1$, $i = 1, \ldots, l'$ such that m is divisible by the lowest common multiple of the denominators of x_i, $i = 1, \ldots, l'$.

Let Δ'_+ be a system of positive roots in Δ, $\alpha'_1, \ldots \alpha'_l$ the system of simple roots in Δ'_+. Fix Δ'_+ in such a way that s is elliptic in a parabolic Weyl subgroup $W' \subset W$ generated by the simple reflections corresponding to roots from a subset of $\alpha'_1, \ldots \alpha'_l$ (for instance one can take $\Delta'_+ = \Delta^1_+$ from the proof of Theorem 5.2). We claim that the set $\gamma_1, \ldots, \gamma_{l'}$ can be completed to a basis Υ of $\mathfrak{h}^*$ by elements from the set $\alpha'_1, \ldots \alpha'_l$ so that every element of the set $\alpha'_1, \ldots \alpha'_l$ can be expressed as a linear combination of elements from Υ with half integer coefficients.

Assuming that the claim is true and $\Upsilon = \{\gamma_1, \ldots, \gamma_{l'}, \alpha'_{i_{l'+1}}, \ldots \alpha'_{i_l}\}$ every root α_j can be expressed as a linear combination of elements of Υ with half integer coefficients, $\alpha_j = \sum_{k=1}^{l'} c_j^k \gamma_k + \sum_{k=l'+1}^{l} c_j^k \alpha'_{i_k}$, and system (A1.1) takes the form

$$(A1.2) \qquad \sum_{i=1}^{l'} x_i \gamma_i = \sum_{j=1}^{l} \frac{p_j}{d_j} \left(\sum_{k=1}^{l'} c_j^k \gamma_k + \sum_{k=l'+1}^{l} c_j^k \alpha'_{i_k} \right).$$

Equations (A1.2) are equivalent to

$$(A1.3) \qquad x_i = \sum_{j=1}^{l} \frac{p_j}{d_j} c_j^i, \ i = 1, \ldots, l',$$

and we also have compatibility conditions $\sum_{j=1}^{l} c_j^i \frac{p_j}{d_j} = 0$, $i = l'+1, \ldots, l$. Since the numbers d_j are equal to 1 or 2 and the coefficients c_j^i are half integer, nontrivial solutions (A1.3), $-1 < x_i < 1$ may only have denominators equal to 2 or 4, and m is not divisible by 2 and 4.

Thus to complete the proof we have to construct the basis Υ. First observe that the conjugacy class of every element s corresponds to the sum of a number of blocks of type A_n, C_n, B_n or $D_{i+j}(a_{j-1})$ as in (4.5), (4.9), (4.6) or (4.11) (see [3], Section 7). The root system Δ'_0 fixed by the cycles of s corresponding to a block is of type A_n, C_n, B_n or D_n, and the remaining blocks are contained in Δ'_0. Therefore it suffices to find a basis Υ containing the set $\gamma_1, \ldots, \gamma_{l'}$ and simple roots of Δ'_0 in case when the conjugacy class of s corresponds to a diagram with a single nontrivial block different from blocks of type A_0 (which correspond to trivial cycles of s).

If the nontrivial block is of type $A_{l'}$ then one can choose Δ'_+ in such a way that $\{\gamma_1, \ldots, \gamma_{l'}\} = \{\alpha'_1, \ldots \alpha'_{l'}\}$, so $\Upsilon = \{\alpha'_1, \ldots \alpha'_l\}$.

If the nontrivial block is of type $C_{l'}$ then Δ is of type C_l, and one can choose Δ'_+ in such a way that $\Upsilon = \{\alpha'_1, \ldots, \alpha'_l\}$, $\{\gamma_1, \ldots, \gamma_{l'}\} = \{\alpha'_{l-l'+1}, \ldots, \alpha'_l\}$, the basis of simple roots of Δ'_0 is $\alpha'_1, \ldots, \alpha'_{l-l'-1}$.

If the nontrivial block is of type $B_{l'}$ then Δ is of type B_l. One can choose Δ'_+ in such a way that $\Upsilon = \{\alpha'_1, \ldots, \alpha'_l\}$, $\{\gamma_1, \ldots, \gamma_{l'}\} = \{\alpha'_{l-l'+1}, \ldots, \alpha'_l\}$, the basis of simple roots of Δ'_0 is $\alpha'_1, \ldots, \alpha'_{l-l'-1}$.

If the nontrivial block is of type $D_{i+j}(a_{j-1})$ and Δ is of type B_l then one can choose Δ'_+ in such a way that $\{\gamma_1, \ldots, \gamma_{i+j-1}\} = \{\alpha'_1, \ldots \alpha'_{i+j-1}\}$, $\gamma_{i+j} = \gamma_i + 2\Sigma_{k=i+1}^{l} \alpha'_k$, the basis of simple roots of Δ'_0 is $\alpha'_{i+j+1}, \ldots \alpha'_l$. So $\Upsilon = \{\gamma_1, \ldots \gamma_{i+j}, \alpha'_{i+j+1}, \ldots \alpha'_l\}$, and $\alpha'_{i+j} = \frac{\gamma_{i+j}}{2} - \frac{\gamma_i}{2} - \Sigma_{k=i+1}^{i+j-1} \gamma_k - \Sigma_{k=i+j+1}^{l} \alpha'_k$.

If the nontrivial block is of type $D_{i+j}(a_{j-1})$ and Δ is of type D_l then one can choose Δ'_+ in such a way that $\{\gamma_1, \ldots, \gamma_{i+j-1}\} = \{\alpha'_1, \ldots \alpha'_{i+j-1}\}$, $\gamma_{i+j} = \gamma_i + 2\Sigma_{k=i+1}^{l-2} \alpha'_k + \alpha'_{l-1} + \alpha'_l$ $(i < l-1)$ or $\gamma_{i+j} = \alpha'_l$ $(i = l-1, j = 1)$, the

basis of simple roots of Δ'_0 is $\alpha'_{i+j+1}, \ldots \alpha'_l$. So $\Upsilon = \{\gamma_1, \ldots \gamma_{i+j}, \alpha'_{i+j+1}, \ldots \alpha'_l\}$, and $\alpha'_{i+j} = \frac{\gamma_{i+j}}{2} - \frac{\gamma_i}{2} - \Sigma^{i+j-1}_{k=i+1}\gamma_k - \Sigma^{l-2}_{k=i+j+1}\alpha'_k - \frac{\alpha'_{l-1}}{2} - \frac{\alpha'_l}{2}$ $(i < l - 1)$.

Thus in all cases every element of the set $\alpha'_1, \ldots \alpha'_l$ can be expressed as a linear combination of elements from Υ with half integer coefficients. This completes the proof.

$\square$

Proposition A1.2. *Let Δ be an irreducible root system, Δ_+ the system of positive roots associated to a conjugacy class of a Weyl group element $s \in W$ in Theorem 5.2, $s = s_{\gamma_1} \ldots s_{\gamma_{l'}}$ representation (3.1) for s, $\alpha_1, \ldots, \alpha_l$ the system of simple roots in Δ_+. Then*

(i) if Δ is of exceptional type the lowest common multiple d of the denominators of the numbers $\frac{1}{d_j}\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha_i, \alpha_j\right)$, where $i, j = 1, \ldots, l$ and $i < j$ (or $i > j$) is given in the tables in Appendix 2; in the non simply–laced cases two numbers are given: the first one is for $i < j$ and the second one is for $i > j$;

(ii) if Δ is of classical type then the conjugacy class of s corresponds to the sum of a number of blocks as in (4.5), (4.6), (4.9) or (4.11). To each block of type X we associate an integer $d_{ij}(X)$, $i, j = 1, \ldots, l$ as follows:

if Δ is not of type A_l, D_l, an orbit with the smallest number of elements for the action of the group $\langle s \rangle$ on E corresponds to a block of type A_n and s does not fix any root from Δ then

for $\Delta = B_l$

$$(A1.4) \qquad d_{ij}(A_n) = \begin{cases} 2p+1 & \text{if } n = 2p \text{ is even;} \\ p+1 & \text{if } n = 2p+1, n \neq 4p-1 \text{ is odd;} \\ p & \text{if } n = 4p-1 \text{ is odd and } i < j; \\ 2p & \text{if } n = 4p-1 \text{ is odd and } i > j; \end{cases}$$

for $\Delta = C_l$

$$(A1.5) \qquad d_{ij}(A_n) = \begin{cases} 2p+1 & \text{if } n = 2p \text{ is even;} \\ p+1 & \text{if } n = 2p+1, n \neq 4p-1 \text{ is odd;} \\ 2p & \text{if } n = 4p-1 \text{ is odd and } i < j; \\ p & \text{if } n = 4p-1 \text{ is odd and } i > j; \end{cases}$$

for $\Delta = D_l$ if $A_{l-1} \subset D_l$ is the only nontrivial block of the conjugacy class of s then

$$(A1.6) \qquad d_{ij}(A_{l-1}) = \begin{cases} 2p+1 & \text{if } l = 2p+1 \text{ is odd;} \\ p+1 & \text{if } l = 2p+2, l \neq 4p \text{ is even;} \\ p & \text{if } l = 4p \text{ is even;} \end{cases}$$

for $\Delta = A_l$ if s is a representative in the Coxeter conjugacy class, i.e. the conjugacy class of s corresponds to the block of type A_l, then

$$(A1.7) \qquad\qquad\qquad\qquad d_{ij}(A_l) = 1;$$

in all other cases

$$(A1.8) \qquad\qquad d_{ij}(A_k) = \begin{cases} k+1 & \text{if } k \text{ is even;} \\ \frac{k-1}{2}+1 & \text{if } k \text{ is odd;} \end{cases}$$

in all cases

$$d_{ij}(C_n) = d_{ij}(B_n) = d_{ij}(D_{v+w}(a_{w-1})) = 1,$$

where, as before, we use the notation of [3], Section 7 for (blocks of) Weyl group conjugacy classes.

Then a common multiple d of the denominators of the numbers $\frac{1}{d_j}\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha_i, \alpha_j\right)$, where $i, j = 1, \ldots, l$ and $i < j$ $(i > j)$ is the lowest common multiple of the numbers $d_{ij}(X)$ for all blocks X of the conjugacy class of s and $i < j$ $(i > j$, respectively).

If $\alpha'_1, \ldots \alpha'_l$ is another system of simple roots then a common multiple of the denominators of the numbers $\frac{1}{d_j}\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha_i, \alpha_j\right)$ will be also a common multiple of the denominators of the numbers $\frac{1}{d_j}\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha'_i, \alpha'_j\right)$ and vice versa.

Proof. First observe that if Δ'_+ is another system of positive roots with the simple roots $\alpha'_1, \ldots, \alpha'_l$ then $\alpha_i = \sum_{k=1}^l c_i^k \alpha'_k$, $\alpha_j^\vee = \sum_{k=1}^l b_i^k \alpha'^\vee_k$, where c_i^k, b_i^k are integer coefficients. Hence

$$\frac{1}{d_j}\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha_i, \alpha_j\right) = \left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha_i, \alpha_j^\vee\right) = \sum_{k,p=1}^l c_i^k b_j^p \left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha'_k, \alpha'^\vee_p\right) = \sum_{k,p=1}^l c_i^k b_j^p \frac{1}{d_p}\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha'_k, \alpha'_p\right),$$

and a common multiple of the denominators of the numbers $\frac{1}{d_j}\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha'_i,\alpha'_j\right)$ will be also a common multiple of the denominators of the numbers $\frac{1}{d_j}\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha_i,\alpha_j\right)$ and vice versa.

In case of classical irreducible root systems we shall compute a common multiple d of the denominators of the numbers $\frac{1}{d_j}\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha'_i,\alpha'_j\right)$, where Δ'_+ is chosen in such a way that s is elliptic in a parabolic Weyl subgroup $W' \subset W$ generated by the simple reflections corresponding to roots from a subset of $\alpha'_1,\ldots,\alpha'_l$ (for instance, one can take $\Delta'_+ = \Delta^1_+$ from proof of Theorem 5.2).

Since different blocks of the conjugacy class of s correspond to different disjoint mutually orthogonal subsets of simple roots in $\alpha'_1,\ldots,\alpha'_l$ it suffices to consider the case when the conjugacy class of s corresponds to a diagram with a single nontrivial block. We shall compute d in case when this block is of type A_k, $k > 1$. Other cases can be considered in a similar way. Assume that the root system Δ is realized as in Section 5, where V is a real Euclidean n–dimensional vector space equipped with the standard scalar product, with an orthonormal basis $\varepsilon_1,\ldots,\varepsilon_n$. In that case simple roots are

A_n. $\alpha'_i = \varepsilon_i - \varepsilon_{i+1}$, $1 \le i \le n$;

B_n. $\alpha'_i = \varepsilon_i - \varepsilon_{i+1}$, $1 \le i < n$, $\alpha'_n = \varepsilon_n$;

C_n. $\alpha'_i = \varepsilon_i - \varepsilon_{i+1}$, $1 \le i < n$, $\alpha'_n = 2\varepsilon_n$;

D_n. $\alpha'_i = \varepsilon_i - \varepsilon_{i+1}$, $1 \le i < n$, $\alpha'_n = \varepsilon_{n-1} + \varepsilon_n$;

Then s is of the form
$$s = s^1 s^2, \quad s^1 = s_{\alpha'_{p+1}} s_{\alpha'_{p+3}} \ldots, \quad s^2 = s_{\alpha'_{p+2}} s_{\alpha'_{p+4}} \ldots,$$

where in the formulas for $s^{1,2}$ the products are taken over mutually orthogonal simple roots labeled by indexes of the same parity; the last simple root which appears in those products is $\alpha'_{p+k} = \varepsilon_{p+k} - \varepsilon_{p+k+1}$, so $\gamma_1,\ldots,\gamma_k = \alpha'_{p+1},\alpha'_{p+3},\ldots,\alpha'_{p+2},\alpha'_{p+4},\ldots$.

We have to compute the numbers $\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha'_i,\alpha'^\vee_j\right)$. We consider the case when $i < j$. The case when $i > j$ can be obtained from it by observing that

(A1.9)
$$\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha'_i,\alpha'^\vee_j\right) = -\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha'_j,\alpha'^\vee_i\right)\frac{(\alpha'_i,\alpha'_i)}{(\alpha'_j,\alpha'_j)}.$$

First observe that by Lemma 6.2 in [19]

(A1.10)
$$\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\gamma_i,\gamma_j\right) = \varepsilon_{ij}(\gamma_i,\gamma_j),$$

where
$$\varepsilon_{ij} = \begin{cases} -1 & i < j \\ 0 & i = j \\ 1 & i > j \end{cases}.$$

Let ω'_t be the fundamental weights of the root subsystem $A_k \subset \Delta$ with respect to the basis of simple roots α'_i, $i = p+1,\ldots,p+k$,
$$\omega'_t = \varepsilon_{p+1} + \ldots + \varepsilon_{p+t} - \frac{t}{k+1}\sum_{j=1}^{k+1}\varepsilon_{p+j}, t = 1,\ldots,k.$$

Since α'_{p+t}, $t = 1,\ldots,k$ form a linear basis of $\mathfrak{h}'^*$, and ω'_t, $t = 1,\ldots,k$ form the dual basis we have
$$\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha'_i,\alpha'^\vee_j\right) = \sum_{t,u=1}^k (\omega'^\vee_t,\alpha'_i)\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha'_{p+t},\alpha'^\vee_{p+u}\right)(\omega'_u,\alpha'^\vee_j).$$

Since the scalar product in V is normalized in such a way that $\alpha'^\vee_{p+u} = \alpha'_{p+u}$, $u = 1,\ldots,k$ we obtain using (A1.10)

(A1.11)
$$\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha'_i,\alpha'^\vee_j\right) = \sum_{t,u=1}^k (\omega'^\vee_t,\alpha'_i)\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha'_{p+t},\alpha'_{p+u}\right)(\omega'_u,\alpha'^\vee_j) =$$
$$= \sum_{t=1}^k (-1)^t(\omega'^\vee_t,\alpha'_i)(\omega'_{t-1} + \omega'_{t+1},\alpha'^\vee_j),$$

where we assume that $\omega'_0 = \omega'_{k+1} = 0$.

Now one has to consider several cases.

If one of the roots α'_i, α'_j is orthogonal to $\mathfrak{h}'^*$ then the left hand side of the last equality is zero.

If $\alpha'_i, \alpha'_j \in \{\gamma_1, \ldots, \gamma_k\}$ then by (A1.10) the left hand side of (A1.11) is equal to ± 1.

If $\alpha'_i = \alpha'_{p+t}$, $1 < t < k$, $\alpha'_j = \alpha'_{p+k+1}$ then

$$(A1.12) \qquad \left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha'_i, \alpha'^\vee_j\right) = (-1)^t(\omega'_{t-1} + \omega'_{t+1}, \alpha'^\vee_{p+k+1}) = (-1)^t(\vartheta - \delta\frac{2t}{k+1}),$$

where $\delta = 2$ if $\alpha'^\vee_j = 2\varepsilon_{p+k+1}$ or $\alpha'^\vee_j = \varepsilon_{p+k} + \varepsilon_{p+k+1}$, $\vartheta = 0$ in the former case, and $\vartheta = 1$ in the latter case. In all other cases $\vartheta = 0$ and $\delta = 1$. Note that $\delta \neq 1$ only in case when Δ is of type B_n or D_n; for arbitrary s this situation can only be realized if an orbit with the smallest number of elements for the action of the group $\langle s \rangle$ on E corresponds to a block of type A_k and s does not fix any root from Δ. The denominator d of the number in the right hand side of (A1.12) is given by

$$(A1.13) \qquad d = \begin{cases} 2p+1 & \text{if } k = 2p \text{ is even;} \\ p+1 & \text{if } k = 2p+1, \, k \neq 4p-1 \text{ is odd;} \\ \frac{2p}{\delta} & \text{if } k = 4p-1 \text{ is odd.} \end{cases}$$

If $\alpha'_i = \alpha'_{p+1}$, $\alpha'_j = \alpha'_{p+k+1}$ then

$$\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha'_i, \alpha'^\vee_j\right) = -(\omega'_2, \alpha'^\vee_{p+k+1}) = \delta\frac{2}{k+1} - \vartheta,$$

where $\delta = 2$ if $\alpha'^\vee_j = 2\varepsilon_{p+k+1}$ or $\alpha'^\vee_j = \varepsilon_{p+k} + \varepsilon_{p+k+1}$, $\vartheta = 0$ in the former case, and $\vartheta = 1$ in the latter case if $k = 2$. In all other cases $\vartheta = 0$ and $\delta = 1$. We again obtain (A1.13).

If $\alpha'_i = \alpha'_{p+k}$, $\alpha'_j = \alpha'_{p+k+1}$ then

$$\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha'_i, \alpha'^\vee_j\right) = (-1)^k(\omega'_{k-1}, \alpha'^\vee_{p+k+1}) = -(-1)^k\delta\frac{k-1}{k+1},$$

and we obtain (A1.13).

If $\alpha'_i = \alpha'_p$, $\alpha'_j = \alpha'_{p+k+1}$ then

$$\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha'_i, \alpha'^\vee_j\right) = \sum_{t=1}^k(-1)^t(\omega'_t, \alpha'^\vee_p)(\omega'_{t-1} + \omega'_{t+1}, \alpha'^\vee_{p+k+1}) =$$

$$= -\sum_{t=1}^{k-1}(-1)^t\left(-1 + \frac{t}{k+1}\right)\frac{2t\delta}{k+1} - (-1)^k\left(-1 + \frac{k}{k+1}\right)\frac{k-1}{k+1}\delta + (-1)^{k-1}\vartheta\left(-1 + \frac{k-1}{k+1}\right).$$

Using the fact that

$$\sum_{r=1}^n(-1)^{r+1}r^2 = (-1)^{n+1}\frac{n(n+1)}{2} \quad \text{and} \quad \sum_{r=1}^n(-1)^{r+1}r = \begin{cases} \frac{n+1}{2} & \text{if } n \text{ is odd;} \\ -\frac{n}{2} & \text{if } n \text{ is even} \end{cases}$$

we obtain

$$\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha'_i, \alpha'^\vee_j\right) = \begin{cases} -\frac{\delta}{k+1} + \vartheta\frac{2}{k+1} & \text{if } k \text{ is even;} \\ -\vartheta\frac{2}{k+1} & \text{if } k \text{ is odd.} \end{cases}$$

The denominator d of the number in the right hand side of the last equality is given by

$$d = \begin{cases} 2p+1 & \text{if } k = 2p \text{ is even;} \\ 1 & \text{if } k = 2p+1, \, n \text{ is odd and } \vartheta = 0; \\ p+1 & \text{if } k = 2p+1 \text{ is odd and } \vartheta = 1. \end{cases}$$

Summarizing all cases considered above and adding the case $i > j$ (see (A1.9)) we arrive at (A1.4), (A1.5), (A1.6), (A1.7) and (A1.8).

Other cases can be treated in a similar way.

$$\square$$

APPENDIX 2. TRANSVERSAL SLICES FOR SIMPLE EXCEPTIONAL ALGEBRAIC GROUPS.

In this appendix, for simple exceptional algebraic groups we present the data related to the transversal slices Σ_s defined in Theorem 5.2. Let G be a connected simple algebraic group of an exceptional type over an algebraically closed field of characteristic good for G, and $\mathcal{O} \in \widehat{N}(G)$. Let H be a maximal torus of G, W the Weyl group of the pair (G, H), and $s \in W$ an element from the conjugacy class $\Psi^W(\mathcal{O})$. Let Δ be the root system of the pair (G, H) and Δ_+ the system of positive roots in Δ associated to $\Psi^W(\mathcal{O})$ and defined in Section 3 with the help of decomposition (3.13), where the subspaces $\mathfrak{h}_i$ are ordered in such a way that if $\mathfrak{h}_i = \mathfrak{h}^k_\lambda$, $\mathfrak{h}_j = \mathfrak{h}^l_\mu$ and

$0 \leq \lambda < \mu < 1$ then $i < j$. Let Σ_s be the corresponding transversal slice defined in Proposition 3.3. Then straightforward calculation shows that

$$\dim Z_{G_p}(n) = \dim \Sigma_s$$

for any $n \in \mathcal{O} \in \mathcal{N}(G_p) \subset \widehat{\mathcal{N}}(G)$. The numbers $\dim Z_{G_p}(n)$ can be found in [11], Chapter 22 (note, however, that the notation in [11] for some classes is different from ours; we follow [14, 22]). The numbers $\dim \Sigma_s$ are contained in the tables below. These two numbers coincide in all cases. The tables below contain also the following information for each $\mathcal{O} \in \widehat{\mathcal{N}}(G)$:

– The Weyl group conjugacy class $\Psi^W(\mathcal{O})$ which can be found in [14];

– The two involutions s^1 and s^2 in the decomposition $s = s^1 s^2 \in \Psi^W(\mathcal{O})$; they are represented by sets of natural numbers which are the numbers of roots appearing in decompositions $s^1 = s_{\gamma_1} \ldots s_{\gamma_n}$, $s^2 = s_{\gamma_{n+1}} \ldots s_{\gamma_{l'}}$, where the system of positive roots Δ_+ is chosen as in Theorem 5.2, and the numeration of positive roots is given in Appendix 3;

– The dimension of the fixed point space $\mathfrak{h}_0$ for the action of s on $\mathfrak{h}$;

– The number $|\overline{\Delta}_0|$ of roots fixed by s;

– The type of the root system $\overline{\Delta}_0$ fixed by s;

– The Dynkin diagram Γ_0 of $\overline{\Delta}_0$, where the numbers at the vertices of Γ_0 are the numbers of simple roots in Δ_+ which appear in Γ_0; the numeration of simple roots is given in Appendix 3;

– The length $\underline{l}(s)$ of s with respect to the system of simple roots in Δ_+;

– $\dim \Sigma_s = \dim \mathfrak{h}_0 + |\overline{\Delta}_0| + \underline{l}(s)$;

– The lowest common multiple d of the denominators of the numbers $\frac{1}{d_j}\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha_i, \alpha_j\right)$, where $i, j = 1, \ldots, l$ and $i < j$ (or $i > j$); in the non simply–laced cases two numbers are given: the first one is for $i < j$ and the second one is for $i > j$;

– The number q; if an odd positive integer m is not divisible by q then for any integers $p_j \in \mathbb{Z}$, $j = 1, \ldots, l$ the system of equations $Y_j(\sum_{i=1}^{l'} m_i \gamma_i) = mp_j$, $j = 1, \ldots, l$ for unknowns $m_i \in \{-m+1, \ldots, m-1\}$, $i = 1, \ldots, l'$ has no nontrivial solutions; here Y_j are defined by $Y_j(\alpha_i) = d_i \delta_{ij}$.

The algorithm for computing all the data above is as follows. MAGMA software was used to realize the algorithm (see [1]).

(1) The input data are two sets of mutually orthogonal roots $\gamma_1, \ldots, \gamma_n$ and $\gamma_{n+1}, \ldots, \gamma_{l'}$ which appear in decomposition (3.1). These sets can be found in [3] for a representative s in each Weyl group conjugacy class. MAGMA associates to each positive root its position in the list of positive roots (see Appendix 3), and the roots $\gamma_1, \ldots, \gamma_n$ and $\gamma_{n+1}, \ldots, \gamma_{l'}$ are given in terms of their positions with respect to a system of positive roots Δ_+^0.

(2) The eigenvalues of s are computed in the cyclotomic field $\mathbb{Q}(\epsilon)$, where $\epsilon^h = 1$, and h is the order of s, $s^h = 1$.

(3) The matrix M has coefficients in the field $\mathbb{Q}$, $\mathbb{Q}(\sqrt{2})$ or $\mathbb{Q}(\sqrt{3})$. The eigenvalues of the corresponding matrix $I - M$ are computed in the splitting field $\mathbb{F}$ of its characteristic polynomial. Note that MAGMA does computations in cyclotomic fields and in algebraic number fields with infinite precision.

(4) The different eigenvalues $0 < \lambda_i < 1$ of $I - M$ are ordered in the following way

$$1 > \lambda_K > \lambda_{K-1} > \ldots > \lambda_1 > 0,$$

and the corresponding eigenvalues of s are ordered in the same way.

(5) A cycle over λ_j is run starting from λ_K. For λ_j an orthonormal basis of the corresponding eigenspace of $I - M$ is constructed in the space $\mathbb{F}^{l'}$.

(6) A cycle is run over all elements of the basis constructed at the previous step. For each element u of that basis the corresponding elements a_u and b_u are defined in the $\mathbb{F}$–form $\mathfrak{h}_{\mathbb{F}}^*$ of the Cartan subalgebra. $\mathfrak{h}_{\mathbb{F}}^*$ is spanned by simple roots over $\mathbb{F}$.

(7) Assuming that the set $\bigcup_{k \leq i} \overline{\Delta}_k$ had been defined at the previous step, $\bigcup_{k \leq i-1} \overline{\Delta}_k$ is defined as the subset of $\bigcup_{k \leq i} \overline{\Delta}_k$ which consists of roots orthogonal in $\mathfrak{h}_{\mathbb{F}}^*$ to a_u and b_u spanning $\mathfrak{h}_i$, and $\overline{\Delta}_i$ is defined as the subset of $\bigcup_{k \leq i} \overline{\Delta}_k$ which consists of roots not orthogonal to a_u and b_u in $\mathfrak{h}_{\mathbb{F}}^*$.

(8) If necessarily the signs of some of the roots in the set $\{\gamma_1, \ldots, \gamma_{l'}\} \cap \overline{\Delta}_i$ are changed so that the orthogonal projections of the roots from the sets $\{\gamma_1, \ldots, \gamma_n\} \cap \overline{\Delta}_i$ and $\{\gamma_{n+1}, \ldots, \gamma_{l'}\} \cap \overline{\Delta}_i$ onto the plane $\mathfrak{h}_i$ spanned by a_u and b_u have the same directions. This does not change s.

(9) If the angle between the rays given by the orthogonal projections of the roots from the sets $\{\gamma_1, \ldots, \gamma_n\} \cap \overline{\Delta}_i$ and $\{\gamma_{n+1}, \ldots, \gamma_{l'}\} \cap \overline{\Delta}_i$ onto the plane $\mathfrak{h}_i$ is less than $\frac{\pi}{2}$ then the signs of roots from the set $\{\gamma_{n+1}, \ldots, \gamma_{l'}\} \cap \overline{\Delta}_i$ are changed. It does not change s. This ensures that the angle between the rays v_1 and v_2 given by the

orthogonal projections of the roots from the sets $\{\gamma_1, \ldots, \gamma_n\} \cap \overline{\Delta}_i$ and $\{\gamma_{n+1}, \ldots, \gamma_{l'}\} \cap \overline{\Delta}_i$ onto the plane $\mathfrak{h}_i$ is greater than or equal to $\frac{\pi}{2}$ as shown at the figure below.

An invariant plane for the action of the Weyl group element s.

(10) A generic element x in the real form of $\mathfrak{h}_i$ is defined as a linear combination of a_u and b_u with coefficients transcendental to all elements of $\mathbb{F}$ (the coefficients contain numbers e and π). x is rotated by powers s^c, $c = 0, \ldots, f-1$, where f is the order of s in $\mathfrak{h}_i$, until the angles formed by x, v_1 and x, v_2 become acute (see Fig. 1). The order f is found as the order of the eigenvalue of s in $\mathbb{Q}(\epsilon)$ which corresponds to λ_j.

(11) Positive roots $\alpha \in (\overline{\Delta}_i)_+$ in $\overline{\Delta}_i$ are defined by the condition $(\alpha, x) > 0$. Since x is defined as a linear combination of a_u and b_u with coefficients transcendental to all elements of $\mathbb{F}$ the set $(\overline{\Delta}_i)_+$ is defined with infinite precision.

(12) The cycle over the elements u of the basis is terminated.

(13) The cycle over the elements λ_j is terminated.

(14) An orthonormal basis of the eigenspace corresponding to the eigenvalue -1 of s is constructed in the $\mathbb{G}$–form $\mathfrak{h}_{\mathbb{G}}^*$ of $\mathfrak{h}^*$, where $\mathbb{G} = \mathbb{Q}$ in the simply–laced cases and $\mathbb{G} = \mathbb{Q}(\sqrt{2})$ or $\mathbb{G} = \mathbb{Q}(\sqrt{3})$ in the non simply–laced cases. The basis is chosen in such a way that the one–dimensional subspaces spanned by the basic vectors are also invariant with respect to the involutions s^1 and s^2.

(15) A cycle over the elements v of the basis is run.

(16) Assuming that the set and $\bigcup_{k \leq i} \overline{\Delta}_k$ had been defined at the previous step, $\bigcup_{k \leq i-1} \overline{\Delta}_k$ is defined as the subset of $\bigcup_{k \leq i} \overline{\Delta}_k$ which consists of roots orthogonal to $v \in \mathfrak{h}_i$ in $\mathfrak{h}_{\mathbb{G}}^*$, and $\overline{\Delta}_i$ is defined as the subset of $\bigcup_{k \leq i} \overline{\Delta}_k$ which consists of roots not orthogonal to v in $\mathfrak{h}_{\mathbb{G}}^*$.

(17) If necessarily the signs of some of the roots in the set $\{\gamma_1, \ldots, \gamma_{l'}\} \cap \overline{\Delta}_i$ are changed so that the orthogonal projections of the roots from the set $\{\gamma_1, \ldots, \gamma_{l'}\} \cap \overline{\Delta}_i$ onto the line $\mathfrak{h}_i$ spanned by v have the same directions as v. This does not change s.

(18) Positive roots $\alpha \in (\overline{\Delta}_i)_+$ in $\overline{\Delta}_i$ are defined by the condition $(\alpha, v) > 0$.

(19) The cycle over the elements v of the basis is terminated.

(20) After running the previous steps we obtain the set of roots $\overline{\Delta}_0$ fixed by s and the set of positive roots

$$\Delta_+ = \bigcup_{i=0}^{I} (\overline{\Delta}_i)_+,$$

where $(\overline{\Delta}_0)_+ = \overline{\Delta}_0 \cap \Delta_+^0$. The decomposition $s = s^1 s^2$ becomes reduced since the roots from the sets $\Delta_{s^{1,2}} \cap \overline{\Delta}_i$ belong to the sectors labeled by $\overline{\Delta}_i^{1,2}$ at Fig. 1, and these sectors are disjoint. By construction the roots $\gamma_1, \ldots, \gamma_{l'}$ are positive with respect to Δ_+.

(21) Simple roots in Δ_+ are found using the property that α is simple if and only if for any $\beta \neq \alpha$, $\beta \in \Delta_+$ we have $s_\alpha(\beta) \in \Delta_+$.

(22) Denote by $\alpha_1^0, \ldots, \alpha_l^0$ the simple roots of Δ_+^0 numbered as in Appendix 3 and by $\alpha_1, \ldots, \alpha_l$ the simple roots of Δ_+. A permutation π of the set $1, \ldots, l$ is found such that if $w \in W$ and $w(\Delta_+) = \Delta_+^0$ then $w(\alpha_i) = \alpha_{\pi(i)}^0$ for all i. This is done by comparing the Cartan matrices with respect to the bases $\alpha_1^0, \ldots, \alpha_l^0$ and $\alpha_1, \ldots, \alpha_l$. Then the basis $\alpha_1, \ldots, \alpha_l$ is reordered in such a way that $\pi = id$.

(23) The coordinates of the roots $\gamma_1, \ldots, \gamma_{l'}$ are computed with respect to the basis $\alpha_1, \ldots, \alpha_l$ and the root positions of the roots $\gamma_1, \ldots, \gamma_{l'}$ are defined according to the MAGMA numeration of roots.

(24) Simple roots in $\overline{\Delta}_0$ are found as the intersection $\overline{\Delta}_0 \cap \{\alpha_1, \ldots, \alpha_l\}$.

(25) The type of the root system $\overline{\Delta}_0$ and its Coxeter graph are obtained.

(26) The basis $\gamma_1^*, \ldots, \gamma_{l'}^*$ of the rational form $\mathfrak{h}_\mathbb{Q}'^*$ of $\mathfrak{h}'^*$, dual to $\gamma_1, \ldots, \gamma_{l'}$, is obtained.

(27) The matrix

$$p_{ij} = \frac{1}{d_j}\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\alpha_i, \alpha_j\right) = \sum_{p,q=1}^{l'} \frac{1}{d_j}\left(\frac{1+s}{1-s}P_{\mathfrak{h}'^*}\gamma_p, \gamma_q\right)(\alpha_i, \gamma_p^*)(\alpha_j, \gamma_q^*) = \sum_{p,q=1}^{l'} \frac{1}{d_j}\varepsilon_{pq}(\gamma_p, \gamma_q)(\alpha_i, \gamma_p^*)(\alpha_j, \gamma_q^*)$$

is computed.

(28) The lowest common multiple of the denominators of the numbers p_{ij}, $i < j$ $(i > j)$ is computed.

(29) The matrix of the system $Y_j(\sum_{i=1}^{l'} x_i \gamma_i) = p_j$, $j = 1, \ldots, l$ is found and the system is reduced to the standard echelon form using Gaussian reduction. All computations are done over $\mathbb{Q}$.

(30) The reduced system is of the form

(A2.1)
$$\begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{l'} \end{pmatrix} = \begin{pmatrix} P_1 \\ P_2 \\ \vdots \\ P_{l'} \\ P_{l'+1} \\ \vdots \\ P_l \end{pmatrix},$$

where P_i are polynomials with rational coefficients in p_j, $j = 1, \ldots, l$. We are interested in the rational nontrivial solutions $-1 < x_i < 1$ with odd denominators (compare with the proof of Proposition A1.1).

Since $Y_j(\sum_{i=1}^{l'} x_i \gamma_i) = p_j$, the possible values of p_j are integers, and $-b_j < p_j < b_j$ $b_j = Y_j(\sum_{i=1}^{l'} \gamma_i)$.

Now it is verified if system (A2.1) has nontrivial rational solutions $-1 < x_i < 1$ with odd denominators by substituting $p_j = -b_j, \ldots, b_j$, $j = 1, \ldots, l$ into the right hand side of (A2.1). If yes, the lowest common multiple q of the denominators of the numbers x_i, $i = 1, \ldots, l'$ is computed.

(31) The dimension of $\mathfrak{h}_0$ is computed, $\dim \mathfrak{h}_0 = l - l'$.

(32) The number of roots in $\overline{\Delta}_0$ is computed.

(33) The length of s with respect to the system $\alpha_1, \ldots, \alpha_l$ is computed.

(34) The dimension of the slice $\dim \Sigma_s = \underline{l}(s) + \dim \mathfrak{h}_0 + |\overline{\Delta}_0|$ is computed.

Note that despite of the fact that a computer program is used to obtain the results summarized in the tables below, all computations are done with infinite precision since MAGMA does computations in algebraic number fields and in cyclotomic fields with infinite precision.

G_2.

$\mathcal{O}$	$\Psi^W(\mathcal{O})$	s^1	s^2	$\dim \mathfrak{h}_0$	$|\overline{\Delta}_0|$	$\overline{\Delta}_0$	Γ_0	$l(s)$	$\dim \Sigma_s$	d	q
A_1	A_1	6	–	1	2	A_1	$\overset{1}{\bullet}$	5	8	1,1	3
$(\tilde{A}_1)_3$	$\tilde{A}_1$	4	–	1	2	A_1	$\overset{2}{\bullet}$	5	8	1,1	–
$\tilde{A}_1$	$A_1 + \tilde{A}_1$	6	1	0	0	–	–	6	6	1,1	3
$G_2(a_1)$	A_2	5	2	0	0	–	–	4	4	3,1	3
G_2	G_2	1	2	0	0	–	–	2	2	1,1	3

F$_4$.

$\mathcal{O}$	$\Psi^W(\mathcal{O})$	s^1	s^2	$\dim \mathfrak{h}_0$	$|\overline{\Delta}_0|$	$\overline{\Delta}_0$	Γ_0	$l(s)$	$\dim \Sigma_s$	d	q
A_1	A_1	24	–	3	18	C_3	$\overset{4}{\bullet}\!-\!\overset{3}{\bullet}\!=\!\overset{2}{\bullet}$	15	36	1,1	–
$(\tilde{A}_1)_2$	$\tilde{A}_1$	21	–	3	18	B_3	$\overset{1}{\bullet}\!-\!\overset{2}{\bullet}\!=\!\overset{3}{\bullet}$	15	36	1,1	–
$\tilde{A}_1$	$2A_1$	24	16	2	8	B_2	$\overset{2}{\bullet}\!=\!\overset{3}{\bullet}$	20	30	1,1	–
$A_1+\tilde{A}_1$	$4A_1$	16 24	2 9	0	0	–	–	24	24	1,1	–
A_2	A_2	23	1	2	6	A_2	$\overset{3}{\bullet}\!-\!\overset{4}{\bullet}$	14	22	3,3	–
$\tilde{A}_2$	$\tilde{A}_2$	19	4	2	6	A_2	$\overset{1}{\bullet}\!-\!\overset{2}{\bullet}$	14	22	3,3	–
$(B_2)_2$	B_2	16	8	2	8	B_2	$\overset{2}{\bullet}\!=\!\overset{3}{\bullet}$	10	20	1,2	–
$A_2+\tilde{A}_1$	$A_2+\tilde{A}_1$	23 7	1	1	0	–	–	17	18	3,3	–
$(\tilde{A}_2+A_1)_2$	$\tilde{A}_2+A_1$	19 5	4	1	0	–	–	17	18	3,3	–
$\tilde{A}_2+A_1$	$A_2+\tilde{A}_2$	23 3	1 4	0	0	–	–	16	16	3,3	3
B_2	A_3	1 14	16	1	2	A_1	$\overset{3}{\bullet}$	13	16	1,2	–
$(C_3(a_1))_2$	B_2+A_1	16	8 9	1	2	A_1	$\overset{2}{\bullet}$	13	16	1,2	–
$C_3(a_1)$	$A_3+\tilde{A}_1$	1 14	3 16	0	0	–	–	14	14	1,2	–
$F_4(a_3)$	$D_4(a_1)$	16 2	5 11	0	0	–	–	12	12	1,1	–
B_3	D_4	16 9 2	1	0	0	–	–	10	10	1,1	–
C_3	C_3+A_1	3 14	4 1	0	0	–	–	10	10	1,1	–
$F_4(a_2)$	$F_4(a_1)$	1 3	9 10	0	0	–	–	8	8	1,1	–
$F_4(a_1)$	B_4	9 2	1 4	0	0	–	–	6	6	1,1	–
F_4	F_4	1 3	2 4	0	0	–	–	4	4	1,1	–

E$_6$.

$\mathcal{O}$	$\Psi^W(\mathcal{O})$	s^1	s^2	dim $\mathfrak{h}_0$	$|\overline{\Delta}_0|$	$\overline{\Delta}_0$	Γ_0	$l(s)$	dim Σ_s	d	q
A_1	A_1	36	–	5	30	A_5	$\underset{\bullet}{1}\!-\!\underset{\bullet}{3}\!-\!\underset{\bullet}{4}\!-\!\underset{\bullet}{5}\!-\!\underset{\bullet}{6}$	21	56	1	–
$2A_1$	$2A_1$	36	23	4	12	A_3	$\underset{\bullet}{3}\!-\!\underset{\bullet}{4}\!-\!\underset{\bullet}{5}$	30	46	1	–
$3A_1$	$4A_1$	36 23	4 15	2	0	–	–	36	38	1	–
A_2	A_2	35	2	4	12	$2A_2$	$\underset{\bullet}{1}\!-\!\underset{\bullet}{3}\ \ \underset{\bullet}{5}\!-\!\underset{\bullet}{6}$	20	36	3	–
A_2+A_1	A_2+A_1	35 11	2	3	6	A_2	$\underset{\bullet}{1}\!-\!\underset{\bullet}{3}$	23	32	3	–
$2A_2$	$2A_2$	35 6	2 5	2	6	A_2	$\underset{\bullet}{1}\!-\!\underset{\bullet}{3}$	22	30	3	–
A_2+2A_1	A_2+2A_1	35 11	2 7	2	0	–	–	26	28	3	–
A_3	A_3	24 2	23	3	4	$2A_1$	$\underset{\bullet}{3}\ \ \underset{\bullet}{5}$	19	26	2	–
$2A_2+A_1$	$3A_2$	35 1 5	2 3 6	0	0	–	–	24	24	3	3
A_3+A_1	A_3+2A_1	2 5 24	23 3	1	0	–	–	21	22	2	–
$D_4(a_1)$	$D_4(a_1)$	23 4	8 19	2	0	–	–	18	20	2	–
A_4	A_4	24 21	2 1	2	2	A_1	$\underset{\bullet}{5}$	14	18	5	–
D_4	D_4	23 4 15	2	2	0	–	–	16	18	1	–
A_4+A_1	A_4+A_1	24 21 5	2 1	1	0	–	–	15	16	5	–
A_5	A_5+A_1	14 13 15	1 6 4	0	0	–	–	14	14	1	–
$D_5(a_1)$	$D_5(a_1)$	2 7	15 12 16	1	0	–	–	13	14	2	–
A_5+A_1	$E_6(a_2)$	1 2 6	9 10 19	0	0	–	–	12	12	1	–
D_5	D_5	6 1 2	4 15	1	0	–	–	9	10	2	–
$E_6(a_1)$	$E_6(a_1)$	6 8 9	1 2 5	0	0	–	–	8	8	1	–
E_6	E_6	1 4 6	2 3 5	0	0	–	–	6	6	1	–

E₇.

$\mathcal{O}$	$\Psi^W(\mathcal{O})$	s^1	s^2	$\dim \mathfrak{h}_0$	$|\overline{\Delta}_0|$	$\overline{\Delta}_0$	Γ_0	$l(s)$	$\dim \Sigma_s$	d	q
A_1	A_1	63	–	6	60	D_6	nodes 7–6–5–4, 4–3, 4–2	33	99	1	–
$2A_1$	$2A_1$	63	49	5	26	A_1+D_4	nodes 7; 2–4, 4–3, 4–5	50	81	1	–
$(3A_1)''$	$(3A_1)'$	63	7 49	4	24	D_4	nodes 2–4, 4–3, 4–5	51	79	1	–
$(3A_1)'$	$(4A_1)''$	63 49	2 28	3	6	$3A_1$	nodes 3, 5, 7	60	69	1	–
A_2	A_2	62	1	5	30	A_5	nodes 2–4–5–6–7	32	67	3	–
$4A_1$	$7A_1$	41 63 40 19	2 3 6	0	0	–	–	63	63	1	–
A_2+A_1	A_2+A_1	30 62	1	4	12	A_3	nodes 4–5–6	41	57	3	–
A_2+2A_1	A_2+2A_1	30 62	1 18	3	2	A_1	node 5	46	51	3	–
A_2+3A_1	A_2+3A_1	30 62	1 5 18	2	0	–	–	47	49	3	–
$2A_2$	$2A_2$	23 62	1 7	3	6	A_2	nodes 4–5	40	49	3	–
A_3	A_3	1 37	49	4	14	A_1+A_3	nodes 7; 2–4–5	31	49	2	–
$(A_3+A_1)''$	$(A_3+A_1)'$	1 37	7 49	3	12	A_3	nodes 2–4–5	32	47	2	–
$2A_2+A_1$	$3A_2$	5 25 62	1 2 6	1	0	–	–	42	43	3	3
$(A_3+A_1)'$	$(A_3+2A_1)''$	1 16 37	7 49	2	2	A_1	node 4	37	41	2	–
A_3+2A_1	A_3+3A_1	14 26 28	3 7 49	1	0	–	–	38	39	2	–

$\mathcal{O}$	$\Psi^W(\mathcal{O})$	s^1	s^2	dim $\mathfrak{h}_0$	$|\overline{\Delta}_0|$	$\overline{\Delta}_0$	Γ_0	$l(s)$	dim Σ_s	d	q
$D_4(a_1)$	$D_4(a_1)$	3 49	8 32	3	6	$3A_1$	$\overset{2}{\bullet}\quad\overset{5}{\bullet}\quad\overset{7}{\bullet}$	30	39	2	–
$D_4(a_1)+A_1$	$D_4(a_1)+A_1$	3 7 49	8 32	2	4	$2A_1$	$\overset{2}{\bullet}\quad\overset{5}{\bullet}$	31	37	2	–
D_4	D_4	3 28 49	1	3	6	$3A_1$	$\overset{2}{\bullet}\quad\overset{5}{\bullet}\quad\overset{7}{\bullet}$	28	37	1	–
$(A_3+A_2)_2$	A_3+A_2	20 21 22	4 49	2	2	A_1	$\overset{7}{\bullet}$	33	37	6	–
A_3+A_2	$D_4(a_1)+2A_1$	3 7 49	8 2 32	1	2	A_1	$\overset{5}{\bullet}$	32	35	2	–
$A_3+A_2+A_1$	$2A_3+A_1$	4 7 16 49	3 20 21	0	0	–	–	33	33	2	–
A_4	A_4	37 45	1 6	3	6	A_2	$\overset{2}{\bullet}\text{—}\overset{4}{\bullet}$	24	33	5	–
A_5''	A_5'	1 7 37	23 24	2	6	A_2	$\overset{4}{\bullet}\text{—}\overset{5}{\bullet}$	23	31	3	–
D_4+A_1	D_4+3A_1	28 49 3	1 7 2 5	0	0	–	–	31	31	1	–
A_4+A_1	A_4+A_1	9 37 45	1 6	2	0	–	–	27	29	5	–
A_4+A_2	A_4+A_2	4 37 45	1 2 6	1	0	–	–	26	27	15	–
$D_5(a_1)$	$D_5(a_1)$	3 28 49	1 30	2	2	A_1	$\overset{5}{\bullet}$	23	27	2	–
$(A_5+A_1)''$	A_5+A_2	1 5 7 37	4 23 24	0	0	–	–	25	25	3	3
A_5'	$(A_5+A_1)''$	4 16 37	1 12 13	1	0	–	–	24	25	3	–
$D_5(a_1)+A_1$	$D_5(a_1)+A_1$	3 28 49	1 5 30	1	0	–	–	24	25	2	–
$D_6(a_2)$	$D_6(a_2)+A_1$	2 7 8 32	5 23 24	0	0	–	–	23	23	1	–
$(A_5+A_1)'$	$E_6(a_2)$	1 4 16	28 29 31	1	0	–	–	22	23	3	–

$\mathcal{O}$	$\Psi^W(\mathcal{O})$	s^1	s^2	dim $\mathfrak{h}_0$	$|\overline{\Delta}_0|$	$\overline{\Delta}_0$	Γ_0	$l(s)$	dim Σ_s	d	q
D_5	D_5	3 28	1 6 19	2	2	A_1	2 ●	17	21	2	–
$D_6(a_2) + A_1$	$E_7(a_4)$	1 2 12 13	15 17 32	0	0	–	–	21	21	1	–
$D_5 + A_1$	$D_5 + A_1$	2 3 28	1 6 19	1	0	–	–	18	19	2	–
A_6	A_6	11 19 26	6 9 10	1	0	–	–	18	19	7	–
$D_6(a_1)$	$D_6(a_1)$	3 5 28	1 12 13	1	2	A_1	2 ●	16	19	1	–
$D_6(a_1) + A_1$	A_7	4 16 15 17	1 12 13	0	0	–	–	17	17	1	–
D_6	$D_6 + A_1$	1 6 9	7 10 11 22	0	0	–	–	15	15	1	–
$E_6(a_1)$	$E_6(a_1)$	8 19 22	1 4 6	1	0	–	–	14	15	3	–
E_6	E_6	3 6 19	1 9 11	1	0	–	–	12	13	3	–
$D_6 + A_1$	$E_7(a_3)$	7 10 11 22	1 2 6	0	0	–	–	13	13	1	–
$E_7(a_2)$	$E_7(a_2)$	2 3 12 13	1 9 11	0	0	–	–	11	11	1	–
$E_7(a_1)$	$E_7(a_1)$	6 9 10	1 2 5 7	0	0	–	–	9	9	1	–
E_7	E_7	1 4 6	2 3 5 7	0	0	–	–	7	7	1	–

E₈.

$\mathcal{O}$	$\Psi^W(\mathcal{O})$	s^1	s^2	dim $\mathfrak{h}_0$	$|\overline{\Delta}_0|$	$\overline{\Delta}_0$	Γ_0	$l(s)$	dim Σ_s	d	q
A_1	A_1	120	–	7	126	E_7	E_7: nodes 1–3–4–5–6–7, with 2 below 4	57	190	1	–
$2A_1$	$2A_1$	120	97	6	60	D_6	D_6: 7–6–5–4, with 2 and 3 branching from 4	90	156	1	–
$3A_1$	$(4A_1)'$	97 120	7 61	4	24	D_4	D_4: 2–4, with 3 and 5 branching from 4	108	136	1	–
A_2	A_2	119	8	6	72	E_6	E_6: nodes 1–3–4–5–6, with 2 below 4	56	134	3	–
$4A_1$	$8A_1$	74 118 104 32	2 3 5 8	0	0	–	–	120	120	1	–
A_2+A_1	A_2+A_1	69 119	8	5	30	A_5	A_5: nodes 1–3–4–5–6	77	112	3	–
A_2+2A_1	A_2+2A_1	69 119	8 31	4	12	A_3	A_3: nodes 3–4–5	86	102	3	–
A_3	A_3	8 74	97	5	40	D_5	D_5: 6–5–4, with 2 and 3 branching from 4	55	100	2	–
A_2+3A_1	A_2+4A_1	31 69 119	4 8 19	2	0	–	–	92	94	3	–
$2A_2$	$2A_2$	63 119	2 8	4	12	$2A_2$	$2A_2$: nodes 1–3 5–6	76	92	3	–
$2A_2+A_1$	$3A_2$	6 63 119	2 5 8	2	6	A_2	A_2: nodes 1–3	78	86	3	3
A_3+A_1	$(A_3+2A_1)'$	8 48 74	6 97	3	12	A_3	A_3: nodes 2–4–3	69	84	2	–

$\mathcal{O}$	$\Psi^W(\mathcal{O})$	s^1	s^2	dim $\mathfrak{h}_0$	$|\overline{\Delta}_0|$	$\overline{\Delta}_0$	Γ_0	$l(s)$	dim Σ_s	d	q
$D_4(a_1)$	$D_4(a_1)$	7 97	15 68	4	24	D_4	(nodes 2—4, 3, 5)	54	82	2	–
D_4	D_4	7 61 97	8	4	24	D_4	(nodes 2—4, 3, 5)	52	80	1	–
$2A_2 + 2A_1$	$4A_2$	3 5 63 119	1 2 6 8	0	0	–	–	80	80	3	3
$A_3 + 2A_1$	$A_3 + 4A_1$	8 74 48 17	4 6 97	1	0	–	–	75	76	2	–
$D_4(a_1) + A_1$	$D_4(a_1) + A_1$	7 32 97	15 68	3	6	$3A_1$	2 $\bullet$ 3 $\bullet$ 5 $\bullet$	63	72	2	–
$(A_3 + A_2)_2$	$A_3 + A_2$	29 55 56	6 97	3	4	$2A_1$	2 $\bullet$ 3 $\bullet$	65	72	6	–
$A_3 + A_2$	$(2A_3)'$	13 22 40 62	7 97	2	4	$2A_1$	2 $\bullet$ 3 $\bullet$	64	70	2	–
A_4	A_4	74 93	1 8	4	20	A_4	2 $\bullet$—4 $\bullet$—5 $\bullet$—6 $\bullet$	44	68	5	–
$A_3 + A_2 + A_1$	$2A_3 + 2A_1$	62 22 13 40	97 7 3 2	0	0	–	–	66	66	2	–
$D_4(a_1) + A_2$	$D_4(a_1) + A_2$	7 25 97	4 15 68	2	0	–	–	62	64	6	–
$D_4 + A_1$	$D_4 + 4A_1$	97 61 7 32	8 5 2 3	0	0	–	–	64	64	1	–
$2A_3$	$2D_4(a_1)$	2 3 7 97	11 12 15 68	0	0	–	–	60	60	1	–
$A_4 + A_1$	$A_4 + A_1$	26 74 93	1 8	3	6	A_2	4 $\bullet$—5 $\bullet$	51	60	5	–
$D_5(a_1)$	$D_5(a_1)$	8 31	39 61 75	3	12	A_3	3 $\bullet$—4 $\bullet$—5 $\bullet$	43	58	2	–

$\mathcal{O}$	$\Psi^W(\mathcal{O})$	s^1	s^2	dim $\mathfrak{h}_0$	$|\overline{\Delta}_0|$	$\overline{\Delta}_0$	Γ_0	$l(s)$	dim Σ_s	d	q
$(D_4+A_2)_2$	D_4+A_2	49 59 63 71	2 8	2	0	–		54	56	3	–
A_4+2A_1	A_4+2A_1	26 74 93	1 8 12	2	0	–		54	56	5	–
A_4+A_2	A_4+A_2	18 74 93	1 6 8	2	2	A_1	$\overset{4}{\bullet}$	50	54	15	–
$A_4+A_2+A_1$	$A_4+A_2+A_1$	18 74 93	1 4 6 8	1	0	–		51	52	15	–
A_5	$(A_5+A_1)'$	22 61 62	7 23 24	2	6	A_2	$\overset{3}{\bullet}\!-\!\overset{4}{\bullet}$	44	52	3	–
$D_5(a_1)+A_1$	$D_5(a_1)+A_1$	8 19 31	39 61 75	2	2	A_1	$\overset{4}{\bullet}$	48	52	2	–
D_4+A_2	D_4+A_3	31 46 61 70	8 10 25	1	0	–		49	50	2	–
$(A_5+A_1)''$	$E_6(a_2)$	1 8 44	34 35 68	2	6	A_2	$\overset{4}{\bullet}\!-\!\overset{5}{\bullet}$	42	50	3	–
D_5	D_5	1 8 44	7 61	3	12	A_3	$\overset{2}{\bullet}\!-\!\overset{4}{\bullet}\!-\!\overset{5}{\bullet}$	33	48	2	–
A_4+A_3	$2A_4$	4 6 74 93	1 2 5 8	0	0	–		48	48	5	5
$D_5(a_1)+A_2$	$D_5(a_1)+A_3$	4 8 17 18	53 64 61 31	0	0	–		46	46	2	–
$(A_5+A_1)'$	$A_5+A_2+A_1$	3 22 61 62	4 7 23 24	0	0	–		46	46	3	3
$D_6(a_2)$	$2D_4$	8 4 19 31	59 58 61 2	0	0	–		44	44	1	–
A_5+2A_1	$E_6(a_2)+A_2$	1 4 8 44	5 34 35 68	0	0	–		44	44	3	3
A_5+A_2	$E_7(a_4)+A_1$	8 37 9 5	59 61 58 2	0	0	–		42	42	1	–

$\mathcal{O}$	$\Psi^W(\mathcal{O})$	s^1	s^2	dim $\mathfrak{h}_0$	$|\overline{\Delta}_0|$	$\overline{\Delta}_0$	Γ_0	$l(s)$	dim Σ_s	d	q
$D_5 + A_1$	$D_5 + 2A_1$	8 1 44 18	4 7 61	1	0	–	–	39	40	2	–
$2A_4$	$E_8(a_8)$	2 3 6 8	19 41 59 76	0	0	–	–	40	40	1	–
$D_6(a_1)$	$D_6(a_1)$	3 7 61	8 9 37	2	4	$2A_1$	$\underset{\bullet}{2} \quad \underset{\bullet}{5}$	32	38	2	–
A_6	A_6	40 44 62	1 13 14	2	2	A_1	$\underset{\bullet}{2}$	34	38	7	–
$A_6 + A_1$	$A_6 + A_1$	2 40 44 62	1 13 14	1	0	–	–	35	36	7	–
$D_6(a_1) + A_1$	A_7'	26 28 41 27	8 9 37	1	2	A_1	$\underset{\bullet}{5}$	33	36	2	–
$(D_5 + A_2)_2$	$D_5 + A_2$	8 23 24 25	4 7 61	1	0	–	–	35	36	6	–
$D_5 + A_2$	$A_7 + A_1$	26 28 41 27	5 8 9 37	0	0	–	–	34	34	2	–
$E_6(a_1)$	$E_6(a_1)$	15 44 55	1 6 8	2	6	A_2	$\underset{\bullet}{2} \text{—} \underset{\bullet}{4}$	26	34	3	–
D_6	$D_6 + 2A_1$	3 49 21 48	1 2 6 8	0	0	–	–	32	32	1	–
$D_7(a_2)$	$D_7(a_2)$	4 7 61	3 8 16 30	1	0	–	–	31	32	2	–
E_6	E_6	1 7 44	8 26 27	2	6	A_2	$\underset{\bullet}{4} \text{—} \underset{\bullet}{5}$	24	32	3	–
$(A_7)_3$	A_7''	21 22 26 27	16 30 32	1	0	–	–	31	32	4	–
A_7	$D_8(a_3)$	3 5 7 61	4 8 23 24	0	0	–	–	30	30	1	–

$\mathcal{O}$	$\Psi^W(\mathcal{O})$	s^1	s^2	dim $\mathfrak{h}_0$	$|\overline{\Delta}_0|$	$\overline{\Delta}_0$	Γ_0	$l(s)$	dim Σ_s	d	q
$E_6(a_1) + A_1$	$E_6(a_1) + A_1$	15 44 55	1 6 8 10	1	0	–	–	29	30	3	–
$D_8(a_3)$	A_8	22 23 26 31	7 11 12 25	0	0	–	–	28	28	3	3
$D_6 + A_1$	$E_7(a_3)$	27 28 32 41	1 2 8	1	2	A_1	5 ●	25	28	1	–
$(D_7(a_1))_2$	$D_7(a_1)$	48 3 49 21	1 8 26	1	0	–	–	27	28	2	–
$D_7(a_1)$	$D_8(a_2)$	1 6 8	12 21 25 27 49	0	0	–	–	26	26	1	–
$E_6 + A_1$	$E_6 + A_2$	1 5 7 44	4 8 26 27	0	0	–	–	26	26	3	3
$E_7(a_2)$	$E_7(a_2) + A_1$	7 18 23 24	4 8 26 27	0	0	–	–	24	24	1	–
A_8	$E_8(a_6)$	1 2 5 8	21 25 27 49	0	0	–	–	24	24	1	–
D_7	$D_8(a_1)$	7 17 18 19	1 20 21 22	0	0	–	–	22	22	1	–
$E_7(a_2) + A_1$	$E_8(a_7)$	4 7 23 24	5 8 20 33	0	0	–	–	22	22	1	–
$E_7(a_1)$	$E_7(a_1)$	1 13 14	3 5 8 32	1	2	A_1	2 ●	17	20	1	–
$D_8(a_1)$	$E_8(a_3)$	3 7 23 24	4 8 26 27	0	0	–	–	20	20	1	–
$E_7(a_1) + A_1$	D_8	4 8 17 26 27	1 5 7	0	0	–	–	18	18	1	–

$\mathcal{O}$	$\Psi^W(\mathcal{O})$	s^1	s^2	dim $\mathfrak{h}_0$	$|\overline{\Delta}_0|$	$\overline{\Delta}_0$	Γ_0	$l(s)$	dim Σ_s	d	q
D_8	$E_8(a_5)$	10 16 20 22	2 3 5 7	0	0	–	–	16	16	1	–
E_7	$E_7 + A_1$	1 6 8 10	7 11 12 25	0	0	–	–	16	16	1	–
$E_7 + A_1$	$E_8(a_4)$	7 11 12 25	1 2 6 8	0	0	–	–	14	14	1	–
$E_8(a_2)$	$E_8(a_2)$	2 3 5 7	1 8 10 20	0	0	–	–	12	12	1	–
$E_8(a_1)$	$E_8(a_1)$	6 8 10 11	1 2 5 7	0	0	–	–	10	10	1	–
E_8	E_8	1 4 6 8	2 3 5 7	0	0	–	–	8	8	1	–

APPENDIX 3. IRREDUCIBLE ROOT SYSTEMS OF EXCEPTIONAL TYPES.

In this Appendix we give the lists of positive roots in irreducible root systems of exceptional types. All simple roots are numbered as shown at the Dynkin diagrams. The other roots in each list are given in terms of their coordinates with respect to the basis of simple roots. The coordinates are indicated in the brackets ( ). Each set of coordinates is preceded by the number of the corresponding root. These numbers are used to indicate roots which appear in the columns s^1, s^2 and Γ_0 in the tables in Appendix 2.

G$_2$.

1 (1 0)
2 (0 1)
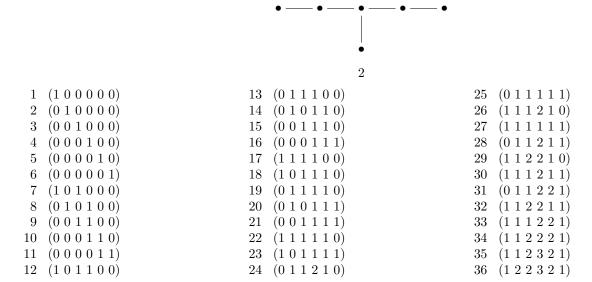3 (1 1)
4 (2 1)
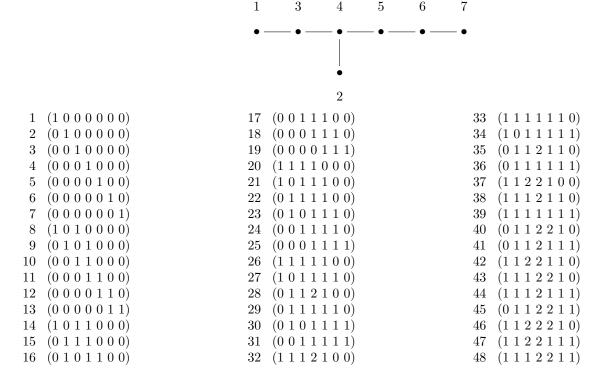5 (3 1)
6 (3 2)

F4.

1 (1 0 0 0)   5 (1 1 0 0)   9 (0 1 2 0)
2 (0 1 0 0)   6 (0 1 1 0)   10 (0 1 1 1)
3 (0 0 1 0)   7 (0 0 1 1)   11 (1 1 2 0)
4 (0 0 0 1)   8 (1 1 1 0)   12 (1 1 1 1)

13 (0 1 2 1)   17 (1 2 2 1)   21 (1 2 3 2)
14 (1 2 2 0)   18 (1 1 2 2)   22 (1 2 4 2)
15 (1 1 2 1)   19 (1 2 3 1)   23 (1 3 4 2)
16 (0 1 2 2)   20 (1 2 2 2)   24 (2 3 4 2)

E₆.

```
        1        3        4        5        6
        •————————•————————•————————•————————•
                          |
                          •
                          2
```

1   (1 0 0 0 0 0)   13 (0 1 1 1 0 0)   25 (0 1 1 1 1 1)
2   (0 1 0 0 0 0)   14 (0 1 0 1 1 0)   26 (1 1 1 2 1 0)
3   (0 0 1 0 0 0)   15 (0 0 1 1 1 0)   27 (1 1 1 1 1 1)
4   (0 0 0 1 0 0)   16 (0 0 0 1 1 1)   28 (0 1 1 2 1 1)
5   (0 0 0 0 1 0)   17 (1 1 1 1 0 0)   29 (1 1 2 2 1 0)
6   (0 0 0 0 0 1)   18 (1 0 1 1 1 0)   30 (1 1 1 2 1 1)
7   (1 0 1 0 0 0)   19 (0 1 1 1 1 0)   31 (0 1 1 2 2 1)
8   (0 1 0 1 0 0)   20 (0 1 0 1 1 1)   32 (1 1 2 2 1 1)
9   (0 0 1 1 0 0)   21 (0 0 1 1 1 1)   33 (1 1 1 2 2 1)
10 (0 0 0 1 1 0)   22 (1 1 1 1 1 0)   34 (1 1 2 2 2 1)
11 (0 0 0 0 1 1)   23 (1 0 1 1 1 1)   35 (1 1 2 3 2 1)
12 (1 0 1 1 0 0)   24 (0 1 1 2 1 0)   36 (1 2 2 3 2 1)

E₇.

```
        1        3        4        5        6        7
        •————————•————————•————————•————————•————————•
                          |
                          •
                          2
```

1   (1 0 0 0 0 0 0)   17 (0 0 1 1 1 0 0)   33 (1 1 1 1 1 1 0)
2   (0 1 0 0 0 0 0)   18 (0 0 0 1 1 1 0)   34 (1 0 1 1 1 1 1)
3   (0 0 1 0 0 0 0)   19 (0 0 0 0 1 1 1)   35 (0 1 1 2 1 1 0)
4   (0 0 0 1 0 0 0)   20 (1 1 1 1 0 0 0)   36 (0 1 1 1 1 1 1)
5   (0 0 0 0 1 0 0)   21 (1 0 1 1 1 0 0)   37 (1 1 2 2 1 0 0)
6   (0 0 0 0 0 1 0)   22 (0 1 1 1 1 0 0)   38 (1 1 1 2 1 1 0)
7   (0 0 0 0 0 0 1)   23 (0 1 0 1 1 1 0)   39 (1 1 1 1 1 1 1)
8   (1 0 1 0 0 0 0)   24 (0 0 1 1 1 1 0)   40 (0 1 1 2 2 1 0)
9   (0 1 0 1 0 0 0)   25 (0 0 0 1 1 1 1)   41 (0 1 1 2 1 1 1)
10 (0 0 1 1 0 0 0)   26 (1 1 1 1 1 0 0)   42 (1 1 2 2 1 1 0)
11 (0 0 0 1 1 0 0)   27 (1 0 1 1 1 1 0)   43 (1 1 1 2 2 1 0)
12 (0 0 0 0 1 1 0)   28 (0 1 1 2 1 0 0)   44 (1 1 1 2 1 1 1)
13 (0 0 0 0 0 1 1)   29 (0 1 1 1 1 1 0)   45 (0 1 1 2 2 1 1)
14 (1 0 1 1 0 0 0)   30 (0 1 0 1 1 1 1)   46 (1 1 2 2 2 1 0)
15 (0 1 1 1 0 0 0)   31 (0 0 1 1 1 1 1)   47 (1 1 2 2 1 1 1)
16 (0 1 0 1 1 0 0)   32 (1 1 1 2 1 0 0)   48 (1 1 1 2 2 1 1)

49 (0 1 1 2 2 2 1)   54 (1 1 2 3 2 1 1)   59 (1 1 2 3 3 2 1)
50 (1 1 2 3 2 1 0)   55 (1 1 2 2 2 2 1)   60 (1 2 2 3 3 2 1)
51 (1 1 2 2 2 1 1)   56 (1 2 2 3 2 1 1)   61 (1 2 2 4 3 2 1)
52 (1 1 1 2 2 2 1)   57 (1 1 2 3 2 2 1)   62 (1 2 3 4 3 2 1)
53 (1 2 2 3 2 1 0)   58 (1 2 2 3 2 2 1)   63 (2 2 3 4 3 2 1)

E$_8$.

$$
\begin{array}{ccccccc}
1 & 3 & 4 & 5 & 6 & 7 & 8 \\
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet
\end{array}
$$

(vertex 2 below vertex 4)

2

1 (1 0 0 0 0 0 0 0)	41 (0 1 1 1 1 1 1 0)	81 (1 2 2 3 2 1 1 1)
2 (0 1 0 0 0 0 0 0)	42 (0 1 0 1 1 1 1 1)	82 (1 1 2 3 3 2 1 0)
3 (0 0 1 0 0 0 0 0)	43 (0 0 1 1 1 1 1 1)	83 (1 1 2 3 2 2 1 1)
4 (0 0 0 1 0 0 0 0)	44 (1 1 2 2 1 0 0 0)	84 (1 1 2 2 2 2 2 1)
5 (0 0 0 0 1 0 0 0)	45 (1 1 1 2 1 1 0 0)	85 (1 2 2 3 3 2 1 0)
6 (0 0 0 0 0 1 0 0)	46 (1 1 1 1 1 1 1 0)	86 (1 2 2 3 2 2 1 1)
7 (0 0 0 0 0 0 1 0)	47 (1 0 1 1 1 1 1 1)	87 (1 1 2 3 3 2 1 1)
8 (0 0 0 0 0 0 0 1)	48 (0 1 1 2 2 1 0 0)	88 (1 1 2 3 2 2 2 1)
9 (1 0 1 0 0 0 0 0)	49 (0 1 1 2 1 1 1 0)	89 (1 2 2 4 3 2 1 0)
10 (0 1 0 1 0 0 0 0)	50 (0 1 1 1 1 1 1 1)	90 (1 2 2 3 3 2 1 1)
11 (0 0 1 1 0 0 0 0)	51 (1 1 2 2 1 1 0 0)	91 (1 2 2 3 2 2 2 1)
12 (0 0 0 1 1 0 0 0)	52 (1 1 1 2 2 1 0 0)	92 (1 1 2 3 3 2 2 1)
13 (0 0 0 0 1 1 0 0)	53 (1 1 1 2 1 1 1 0)	93 (1 2 3 4 3 2 1 0)
14 (0 0 0 0 0 1 1 0)	54 (1 1 1 1 1 1 1 1)	94 (1 2 2 4 3 2 1 1)
15 (0 0 0 0 0 0 1 1)	55 (0 1 1 2 2 1 1 0)	95 (1 2 2 3 3 2 2 1)
16 (1 0 1 1 0 0 0 0)	56 (0 1 1 2 1 1 1 1)	96 (1 1 2 3 3 3 2 1)
17 (0 1 1 1 0 0 0 0)	57 (1 1 2 2 2 1 0 0)	97 (2 2 3 4 3 2 1 0)
18 (0 1 0 1 1 0 0 0)	58 (1 1 2 2 1 1 1 0)	98 (1 2 3 4 3 2 1 1)
19 (0 0 1 1 1 0 0 0)	59 (1 1 1 2 2 1 1 0)	99 (1 2 2 4 3 2 2 1)
20 (0 0 0 1 1 1 0 0)	60 (1 1 1 2 1 1 1 1)	100 (1 2 2 3 3 3 2 1)
21 (0 0 0 0 1 1 1 0)	61 (0 1 1 2 2 2 1 0)	101 (2 2 3 4 3 2 1 1)
22 (0 0 0 0 0 1 1 1)	62 (0 1 1 2 2 1 1 1)	102 (1 2 3 4 3 2 2 1)
23 (1 1 1 1 0 0 0 0)	63 (1 1 2 3 2 1 0 0)	103 (1 2 2 4 3 3 2 1)
24 (1 0 1 1 1 0 0 0)	64 (1 1 2 2 2 1 1 0)	104 (2 2 3 4 3 2 2 1)
25 (0 1 1 1 1 0 0 0)	65 (1 1 2 2 1 1 1 1)	105 (1 2 3 4 3 3 2 1)
26 (0 1 0 1 1 1 0 0)	66 (1 1 1 2 2 2 1 0)	106 (1 2 2 4 4 3 2 1)
27 (0 0 1 1 1 1 0 0)	67 (1 1 1 2 2 1 1 1)	107 (2 2 3 4 3 3 2 1)
28 (0 0 0 1 1 1 1 0)	68 (0 1 1 2 2 2 1 1)	108 (1 2 3 4 4 3 2 1)
29 (0 0 0 0 1 1 1 1)	69 (1 2 2 3 2 1 0 0)	109 (2 2 3 4 4 3 2 1)
30 (1 1 1 1 1 0 0 0)	70 (1 1 2 3 2 1 1 0)	110 (1 2 3 5 4 3 2 1)
31 (1 0 1 1 1 1 0 0)	71 (1 1 2 2 2 2 1 0)	111 (2 2 3 5 4 3 2 1)
32 (0 1 1 2 1 0 0 0)	72 (1 1 2 2 2 1 1 1)	112 (1 3 3 5 4 3 2 1)
33 (0 1 1 1 1 1 0 0)	73 (1 1 1 2 2 2 1 1)	113 (2 3 3 5 4 3 2 1)
34 (0 1 0 1 1 1 1 0)	74 (0 1 1 2 2 2 2 1)	114 (2 2 4 5 4 3 2 1)
35 (0 0 1 1 1 1 1 0)	75 (1 2 2 3 2 1 1 0)	115 (2 3 4 5 4 3 2 1)
36 (0 0 0 1 1 1 1 1)	76 (1 1 2 3 2 2 1 0)	116 (2 3 4 6 4 3 2 1)
37 (1 1 1 2 1 0 0 0)	77 (1 1 2 3 2 1 1 1)	117 (2 3 4 6 5 3 2 1)
38 (1 1 1 1 1 1 0 0)	78 (1 1 2 2 2 2 1 1)	118 (2 3 4 6 5 4 2 1)
39 (1 0 1 1 1 1 1 0)	79 (1 1 1 2 2 2 2 1)	119 (2 3 4 6 5 4 3 1)
40 (0 1 1 2 1 1 0 0)	80 (1 2 2 3 2 2 1 0)	120 (2 3 4 6 5 4 3 2)

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