GROUPS OF UNSTABLE ADAMS OPERATIONS ON p-LOCAL COMPACT GROUPS

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Abstract. A p-local compact group is an algebraic object modelled on the homotopy theory associated with p-completed classifying spaces of compact Lie groups and p-compact groups. In particular p-local compact groups give a unified framework in which one may study p-completed classifying spaces from an algebraic and homotopy theoretic point of view. Like connected compact Lie groups and p-compact groups, p-local compact groups admit unstable Adams operations - self equivalences that are characterised by their cohomological effect. Unstable Adams operations on p-local compact groups were constructed in a previous paper by F. Junod and the authors. In the current paper we study groups of unstable operations from a geometric and algebraic point of view. We give a precise description of the relationship between algebraic and geometric operations, and show that under some conditions unstable Adams operations are determined by their degree. We also examine a particularly well behaved subgroup of operations.

Let G be a connected compact Lie group. An unstable Adams operation of degree $k \geq 1$ on BG is a self equivalence $f\colon BG \to BG$ which induces multiplication by k^i on $H^{2i}(BG;\mathbb{Q})$ for every $i > 0$. In [JMO] unstable Adams operations for compact connected simple Lie groups G were classified. The analysis is centred around studying suitable self equivalences of the p-completion $BG^\wedge_p$. Later on, after p-compact groups were introduced by Dwyer and Wilkerson, their classification by Andersen-Grodal [AG09] and Andersen-Grodal-Møller-Viruel [AGMV08] relied on studying a suitable notion of unstable Adams operation of p-compact groups.

In this paper we study unstable Adams operations on p-local compact groups and their classifying spaces. In light of [BLO7, Theorem A] and the definition of unstable Adams operations for compact Lie groups, it is natural to define a *geometric* unstable Adams operation of degree $\zeta \in \mathbb{Z}_p^\times$ on a p-local compact group $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$ as a self equivalence f of $B\mathcal{G} \overset{\text{def}}{=} |\mathcal{L}|^\wedge_p$ which induces multiplication by ζ^k on $H^{2k}_{\mathbb{Q}_p}(B\mathcal{G})$ for every $k \geq 1$. Here $H^*_{\mathbb{Q}_p}(X) = H^*(X;\mathbb{Z}_p) \otimes \mathbb{Q}$.

This definition differs from the one we gave in [JLL]. Recall that $\psi \in \mathrm{Aut}(S)$ is called an *Adams automorphism* if it induces multiplication by some $\zeta \in \mathbb{Z}_p^\times$, called the degree of ψ, on the identity component S_0 of S. We say ψ is *normal* if it induces the identity on S/S_0. In [JLL, Def. 3.4] we define a *geometric Adams operation* on $\mathcal{G}$ as a self equivalence f of $B\mathcal{G}$ which renders the following square homotopy-commutative for some normal Adams automorphism ψ.

$$\begin{array}{ccc} BS & \xrightarrow{B\psi} & BS \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i} \\ B\mathcal{G} & \xrightarrow[f]{} & B\mathcal{G} \end{array} \tag{1.1}$$

The two definitions of "geometric unstable Adams operations" we have given above coincide under the assumption that T is self-centralising in S (and hence $\mathcal{F}$-centric) in which case we say that $\mathcal{F}$ is *weakly connected*. This is the content of [BLO7, Proposition 3.2]. We will write $\mathrm{Ad}^{\mathrm{geom}}(B\mathcal{G})$ for the group of the *homotopy classes* of the geometric unstable Adams operations as defined in [JLL].

Following [JLL, Definition 3.3], replicated as 4.2 in this paper, an *(algebraic) unstable Adams operation* on $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$ is a pair (Ψ, ψ) of an automorphism Ψ of the small category $\mathcal{L}$ and a fusion preserving

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normal Adams automorphism ψ of S such that: (a) $\pi \circ \Psi = \psi_* \circ \pi$ where $\pi\colon \mathcal{L} \to \mathcal{F}$ is the projection and ψ_* is the automorphism of $\mathcal{F}$ induced by ψ and, (b) for every $\mathcal{F}$-centric $P, Q \leq S$ and every $s \in N_S(P,Q)$ we have $\Psi(\hat{s}) = \widehat{\psi(s)}$. We notice in Remark 4.3 that ψ is determined by Ψ.

In this paper we will investigate the relationship between the algebraic and the geometric Adams operations. When we will write "Adams operation" we will always mean "algebraic Adams operation". Throughout, we will use the following notation. Let $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$ be a p-local compact group.

- $\mathrm{Ad}(S) \leq \mathrm{Aut}(S)$ is the group of all the *normal* Adams automorphisms of S (Definition 3.3).
- $\mathrm{Ad}_{\mathrm{fus}}(S) = \mathrm{Aut}_{\mathrm{fus}}(S) \cap \mathrm{Ad}(S)$.
- $\mathrm{OutAd}_{\mathrm{fus}}(S)$ is the image of $\mathrm{Ad}_{\mathrm{fus}}(S)$ in $\mathrm{Out}_{\mathrm{fus}}(S) = \mathrm{Aut}_{\mathrm{fus}}(S)/\mathrm{Aut}_{\mathcal{F}}(S)$. See Definition 3.4.
- $\mathrm{Ad}^{\mathrm{geom}}(B\mathcal{G}) \leq \mathrm{Out}(B\mathcal{G})$ is the group of the *homotopy classes* of the geometric unstable Adams operations on $B\mathcal{G}$.
- $\mathrm{Ad}(\mathcal{G}) \leq \mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L})$ is the group of algebraic unstable Adams operation of $\mathcal{G}$ as defined in [JLL], where $\mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L})$ is the group of inclusion preserving isotypical self equivalences of $\mathcal{L}$ [AOV, Lemma 1.14].
- $\mathrm{Inn}_T(\mathcal{G}) \leq \mathrm{Ad}(\mathcal{G})$ for the subgroup of operations $(c_{\widehat{t}}, c_t)$ of degree 1 where $c_t \in \mathrm{Inn}(S)$ is conjugation by t and $c_{\widehat{t}} \in \mathrm{Aut}(\mathcal{L})$ is "conjugation" by $\widehat{t} \in \mathrm{Aut}_{\mathcal{L}}(S)$. See equation (4.5) for more details.

Let G be a discrete group. We say that G is $\mathbb{Z}/p^\infty$*-free* (Definition 3.1) if it contains no subgroup isomorphic to $\mathbb{Z}/p^\infty$. We say that G has a *maximal discrete p-torus* if it contains a subgroup $T \leq G$ isomorphic to a discrete p-torus such that any discrete p-torus $U \leq G$ is conjugate to a subgroup of T. This subgroup of G is normal if and only if it is the unique maximal discrete p-torus in G, and in this case we denote it by G_0. Since an extension of two discrete p-tori is a discrete p-torus (this follows easily from [BLO3, Lemma 1.3]), it is clear that G/G_0 is $\mathbb{Z}/p^\infty$-free.

We say that G has a *Sylow p-subgroup* if it contains a subgroup P isomorphic to a discrete p-toral group such that any discrete p-toral subgroup of G is conjugate to a subgroup of P. See [BLO3, Section 8]. Clearly, if G is $\mathbb{Z}/p^\infty$-free then its Sylow p-subgroup, if it exists, is a finite p-group. The next result is closely related to [BLO3, Proposition 7.2].

1.2. **Proposition.** *Let $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$ be a p-local compact group. Then $\mathrm{Ad}^{\mathrm{geom}}(B\mathcal{G})$ is $\mathbb{Z}/p^\infty$-free and contains a normal Sylow p-subgroup. In addition there is a short exact sequence*

$$1 \to \varprojlim_{\mathcal{O}^c(\mathcal{F})}{}^1 \mathcal{Z} \to \mathrm{Ad}^{\mathrm{geom}}(B\mathcal{G}) \to \mathrm{OutAd}_{\mathrm{fus}}(S) \to 1.$$

If p is odd then $\mathrm{Ad}^{\mathrm{geom}}(B\mathcal{G}) \cong \mathrm{OutAd}_{\mathrm{fus}}(S)$. In addition $\mathrm{Ad}^{\mathrm{geom}}(B\mathcal{G})$ is solvable, in fact, $\mathrm{OutAd}_{\mathrm{fus}}(S)$ is solvable of class ≤ 2.

Geometric realisation gives rise to a homomorphism from the group of algebraic Adams operations to the group of geometric Adams operations, see (4.4)

$$\gamma\colon \mathrm{Ad}(\mathcal{G}) \xrightarrow{(\Psi,\psi)\mapsto |\Psi|^\wedge_p} \mathrm{Ad}^{\mathrm{geom}}(B\mathcal{G}).$$

Let $D(\mathcal{F})$ be the subgroup of $\mathbb{Z}_p^\times$ consisting of the degrees of all $\varphi \in \mathrm{Aut}_{\mathcal{F}}(S) \cap \mathrm{Ad}(S)$. In fact, $D(\mathcal{F})$ is a subgroup of the group U_p of the roots of unity in $\mathbb{Z}_p$ which, by [Ro, Sec. 6.7, Prop. 1,2], is isomorphic to the cyclic group C_{p-1} if p is odd and to C_2 if $p = 2$.

1.3. **Theorem.** *Let $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$ be a weakly connected p-local compact group (Definition 3.8). Let T denote the maximal torus of S. Then*

(i) $\mathrm{Ad}(\mathcal{G})$ *has a normal maximal discrete p-torus denoted* $\mathrm{Ad}(\mathcal{G})_0$*. It contains a Sylow p-subgroup which is normal if $p = 2$.*

(ii) $\mathrm{Ad}(\mathcal{G})_0$ *is contained in the kernel of γ (see (4.4)) and there is a short exact sequence*

$$1 \to D(\mathcal{F}) \to \mathrm{Ad}(\mathcal{G})/\mathrm{Ad}(\mathcal{G})_0 \xrightarrow{\bar{\gamma}} \mathrm{Ad}^{\mathrm{geom}}(B\mathcal{G}) \to 1$$

(iii) $\mathrm{Ad}(\mathcal{G})_0 = \mathrm{Inn}_T(\mathcal{G}) \cong T/Z(\mathcal{F})$ *where $Z(\mathcal{F})$ is the centre of $\mathcal{F}$.*

A natural question is whether an unstable Adams operation is determined by its degree. Such a question is too naïve. Looking at Diagram (1.1), and taking into account [BLO3, Theorem 6.3(a)], it is clear that ψ is only determined modulo $\mathrm{Aut}_{\mathcal{F}}(S) \cap \mathrm{Ad}(S)$. Hence, the degree of a geometric unstable Adams operation is, at best, only determined modulo $D(\mathcal{F})$. Also, all elements of $\mathrm{Inn}_T(\mathcal{G})$ are (distinct) algebraic unstable Adams operations of degree 1. The best we can hope for is to address the question of the injectivity of

$$\mathrm{Ad}^{\mathrm{geom}}(B\mathcal{G}) \xrightarrow{\deg} \mathbb{Z}_p^\times / D(\mathcal{F}) \qquad \text{and} \qquad \mathrm{Ad}(\mathcal{G})/\,\mathrm{Ad}(\mathcal{G})_0 \xrightarrow{\deg} \mathbb{Z}_p^\times.$$

The next result will be proven in section 5.

1.4. **Proposition.** *Suppose that $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$ is weakly connected and let $W = \mathrm{Aut}_{\mathcal{F}}(T)$ be its Weyl group. If $H^1(W, T) = 0$ then $\mathrm{OutAd}_{\mathrm{fus}}(S) \xrightarrow{\deg} \mathbb{Z}_p^\times / D(\mathcal{F})$ is injective and there are exact sequences*

$$(1) \qquad 1 \to \varprojlim_{\mathcal{O}(\mathcal{F}^c)}{}^1 \mathcal{Z} \to \mathrm{Ad}^{\mathrm{geom}}(B\mathcal{G}) \xrightarrow{\quad\deg\quad} \mathbb{Z}_p^\times / D(\mathcal{F})$$

$$(2) \qquad 1 \to \varprojlim_{\mathcal{O}(\mathcal{F}^c)}{}^1 \mathcal{Z} \to \mathrm{Ad}(\mathcal{G})/\,\mathrm{Ad}(\mathcal{G})_0 \xrightarrow{\quad\deg\quad} \mathbb{Z}_p^\times$$

If $p \neq 2$ and $H^1(W, T) = 0$, then the degree maps in (1) and (2) are injective.

In Proposition 5.8 we will present conditions under which $H^1(W, T) = 0$. For example, if p is odd and $D(\mathcal{F}) \neq 1$ then $H^1(W, T) = 0$ and as a consequence $\mathrm{Ad}^{\mathrm{geom}}(B\mathcal{G}) \xrightarrow{\deg} \mathbb{Z}_p^\times / D(\mathcal{F})$ is injective.

The next natural question to ask is what the possible degrees of unstable Adams operation of a given $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$ are. This is closely related to the problem of constructing unstable Adams operations which was our main goal in [JLL]. We will call an unstable Adams operation (Ψ, ψ) *special* relative to a collection $\mathcal{R}$ of $\mathcal{F}$-centric subgroups of S if for every $P \in \mathcal{R}$ there exists $\tau_P \in T$ and for every $\varphi \in \mathcal{L}(P, Q)$, where $P, Q \in \mathcal{R}$, there exists some $\tau_\varphi \in Q_0$ such that the following hold. First, $\psi(P) = \tau_P P \tau_P^{-1}$ for every $P \in \mathcal{R}$. Next, $\Psi(\varphi) = \widehat{\tau_Q} \circ \widehat{\tau_\varphi} \circ \varphi \circ \widehat{\tau_P}^{-1}$. See Definition 7.1. We will denote the group of all such Adams operations by $\mathrm{SpAd}(\mathcal{G}; \mathcal{R})$. It turns out that all the unstable Adams operations we constructed in [JLL] are special, and in this paper we will study this more general class of operations.

Of course $\mathrm{SpAd}(\mathcal{G}; \emptyset) = \mathrm{Ad}(\mathcal{G})$. It is reasonable that "a minimum requirement" from $\mathcal{R}$ is to have the property that $|\mathcal{L}^{\mathcal{R}}|_p^\wedge \simeq B\mathcal{G}$ where $\mathcal{L}^{\mathcal{R}}$ is the full subcategory of $\mathcal{L}$ with object set $\mathcal{R}$.

In section 6 we introduce a category $\mathcal{L}^{\mathcal{R}}_{/0}$ whose objects are the elements of $\mathcal{R}$ and morphisms are $\mathcal{L}^{\mathcal{R}}_{/0}(P, Q) = \mathcal{L}(P, Q)/Q_0$ for any $P, Q \in \mathcal{R}$ and where Q_0 acts on $\mathcal{L}(P, Q)$ by post-composition with the elements of $\widehat{Q_0} \leq \mathrm{Aut}_{\mathcal{L}}(Q)$. There is a naturally defined functor $\Phi \colon \mathcal{L}^{\mathcal{R}}_{/0} \to \mathbf{Ab}$ defined by $P \mapsto P_0$. By utilising a theory of extensions of categories, see e.g. [Ho], we show that the category $\mathcal{L}^{\mathcal{R}}$ corresponds to a well defined element $[\mathcal{L}^{\mathcal{R}}]$ in $H^2(\mathcal{L}^{\mathcal{R}}_{/0}, \Psi)$.

For every $k \geq 0$ set

$$(1.5) \qquad \Gamma_k(p) = \mathrm{Ker}\left(\mathbb{Z}_p^\times \xrightarrow{\quad x \mapsto (x \mod p^k) \quad} (\mathbb{Z}/p^k)^\times\right).$$

This group has finite index in $\mathbb{Z}_p^\times$, of course. Note that $\Gamma_0(p) = \mathbb{Z}_p^\times$.

The following result, Theorem 1.6, is the content of Lemma 7.17, Proposition 7.19 and Proposition 7.2. Observe that $\mathrm{Inn}_T(\mathcal{G})$ is a discrete p-torus and that under the hypotheses of the theorem $H^1(\mathcal{L}^{\mathcal{R}}_{/0}, \Phi)$ is discrete p-toral because $\mathcal{L}^{\mathcal{R}}_{/0}$ has finitely many isomorphism classes of objects and morphisms and therefore the cobar construction $C^*(\mathcal{L}^{\mathcal{R}}_{/0}, \Phi)$ consists of discrete p-tori.

1.6. **Theorem.** *Let $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$ be a weakly connected p-local compact group. Suppose that a collection $\mathcal{R} \subseteq \mathcal{F}^c$ has finitely many $\mathcal{F}$-conjugacy classes. Then $\mathrm{SpAd}(\mathcal{G}; \mathcal{R})$ is a normal subgroup of $\mathrm{Ad}(\mathcal{G})$ of finite index. The quotient group is solvable of class at most 3.*

If in addition, $\mathcal{R} \supseteq \mathcal{H}^\bullet(\mathcal{F})$, see [BLO3, Sec. 3]*, then there is an exact sequence*

$$H^1(\mathcal{L}^{\mathcal{R}}_{/0}, \Phi) \to \operatorname{SpAd}(\mathcal{G};\mathcal{R})/\operatorname{Inn}_T(\mathcal{G}) \xrightarrow{\deg} \Gamma_m(p) \to 1$$

where p^m is the order of $[\mathcal{L}^{\mathcal{R}}]$ in $H^2(\mathcal{L}^{\mathcal{R}}_{/0}, \Phi)$. In particular, $\operatorname{SpAd}(\mathcal{G};\mathcal{R})$ *has a normal Sylow p-subgroup with* $\operatorname{Inn}_T(\mathcal{G})$ *as its maximal discrete p-torus.*

A corollary of this theorem is that the image of $\deg\colon \operatorname{Ad}(\mathcal{G}) \to \mathbb{Z}_p^\times$ contains $\Gamma_m(p)$ where p^m is the order of $[\mathcal{L}^{\mathcal{R}}]$ in $H^2(\mathcal{L}^{\mathcal{R}}_{/0}, \Phi)$. This gives a more precise result than the one obtained in [JLL, Theorem A].

Even though $\operatorname{SpAd}(\mathcal{G};\mathcal{R})$ has finite index in $\operatorname{Ad}(\mathcal{G})$, it is usually a proper subgroup. In fact, in Proposition 8.7 we give an example of a geometric unstable Adams operation that is not the geometric realisation of a special unstable Adams operation relative to the collection $\mathcal{R}$ of the $\mathcal{F}$-centric $\mathcal{F}$-radical subgroups.

The paper is organised as follows. In Section 2 we recall some definitions and useful facts on p-local compact groups. In Section 3 we discuss geometric unstable operations and prove Proposition 1.2. Section 4 is dedicated to the proof of Theorem 1.3, and in Section 5 we prove Proposition 1.4 and find conditions for the vanishing of $H^1(W,T)$. In Section 6 we recall some basic theory of extensions of categories and introduce the category $\mathcal{L}_{/0}$. Sections 7 is dedicated to what we call special Adams operations, and the proof of Theorem 1.6. Finally in Section 8 we show that there exist unstable Adams operations which are not special. We end with two appendices containing proofs of statements from Section 6 and the observation that the operations constructed in [JLL] are all special.

2. Recollections of p-local compact groups

This section briefly introduces p-local compact groups and collect some results about them that we will use. The reference is [BLO3].

Fix a prime p. Recall that $\mathbb{Z}/p^\infty \overset{\text{def}}{=} \cup_{n\geq 1}\mathbb{Z}/p^n$. This is a divisible group. A *discrete p-torus* of rank n is a group T isomorphic to $\bigoplus^n \mathbb{Z}/p^\infty$. A group P is called *discrete p-toral* if it contains a discrete p-torus T as a normal subgroup and P/T is a finite p-group. In this case T is characteristic in P and we write $T = P_0$ and call it the *identity component* of P. Every sub-quotient of P is a discrete p-toral group by [BLO3, Lemma 1.3]. Also, an extension of discrete p-toral groups is a discrete p-toral group. The *order* of a discrete p-toral group P is the pair $(\operatorname{rk}(P_0), |P/P_0|)$. These pairs are ordered lexicographically.

A fusion system $\mathcal{F}$ over a discrete p-toral group S is a category whose objects are all the subgroups of S. Morphisms between $P, Q \leq S$ are group monomorphisms and $\operatorname{Hom}_{\mathcal{F}}(P,Q)$ always contains $\operatorname{Hom}_S(P,Q)$, namely the homomorphisms $P \to Q$ induced by conjugation by elements of S. In addition every morphism in $\mathcal{F}$ can be factored as an isomorphism in $\mathcal{F}$ followed by an inclusion homomorphism. See [BLO3, Definition 2.1]

We say that $P, Q \leq S$ are $\mathcal{F}$-conjugate if they are isomorphic as objects of $\mathcal{F}$. A subgroup $P \leq S$ is called fully centralised (resp. fully normalised) if for every $P' \leq S$ which is $\mathcal{F}$-conjugate to P, the order of $C_S(P')$ (resp. $N_S(P')$) is at most the order of $C_S(P)$ (resp. $N_S(P)$).

A fusion system $\mathcal{F}$ over S is called *saturated* if

(1) Every fully normalised $P \leq S$ is fully centralised, $\operatorname{Out}_{\mathcal{F}}(P) \overset{\text{def}}{=} \operatorname{Aut}_{\mathcal{F}}(P)/\operatorname{Inn}(P)$ is finite and contains $\operatorname{Out}_S(P)$ as a Sylow p-subgroup.
(2) If $\varphi \in \operatorname{Hom}_{\mathcal{F}}(P,S)$ and if $\varphi(P)$ is fully centralised, then φ extends to $\psi \in \operatorname{Hom}_{\mathcal{F}}(N_\varphi, S)$ where
$$N_\varphi = \{g \in N_S(P) : \varphi \circ c_g \circ \varphi^{-1} \in \operatorname{Aut}_S(\varphi(P))\}.$$
(3) If $P_1 \leq P_2 \leq \dots$ is an increasing sequence of subgroups of S and $\varphi \in \operatorname{Hom}(\bigcup_n P_n, S)$ is a homomorphism such that $\varphi|_{P_n} \in \operatorname{Hom}_{\mathcal{F}}(P_n, S)$ then $\varphi \in \operatorname{Hom}_{\mathcal{F}}(\bigcup_n P_n, S)$.

A subgroup P of S is called *$\mathcal{F}$-centric* if $C_S(P') = Z(P')$ for every P' which is $\mathcal{F}$-conjugate to P. It is called *$\mathcal{F}$-radical* if $O_p(\operatorname{Out}_{\mathcal{F}}(P)) = 1$ where $O_p(K)$ denotes the largest normal p-subgroup of a finite group K.

If $P \leq S$ is $\mathcal{F}$-centric then it must contain $Z(S)$. The centre of $\mathcal{F}$ is defined by:

$$Z(\mathcal{F}) = \{x \in Z(S) : \varphi(x) = x \text{ for every } P \in \mathcal{F}^c \text{ and every } \varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S)\}.$$

In light of Alperin's fusion theorem [BLO3, Theorem 3.6] this is the same as the subgroup of $x \in Z(S)$ such that $\varphi(x) = x$ for any $P \leq S$ which contains x and every $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$.

The orbit category of $\mathcal{F}$ denoted $\mathcal{O}(\mathcal{F})$ has the same objects as $\mathcal{F}$ and morphism sets $\mathrm{Hom}_{\mathcal{O}(\mathcal{F})}(P, Q) = \mathrm{Hom}_{\mathcal{F}}(P, Q)/\,\mathrm{Inn}(Q)$. The full subcategory on the $\mathcal{F}$-centric subgroups is denoted $\mathcal{O}(\mathcal{F}^c)$. In [BLO3, proof of Theorem 7.1] the functor $\mathcal{Z} \colon \mathcal{O}(\mathcal{F}^c)^{\mathrm{op}} \to \mathbb{Z}_{(p)}$-**mod** was defined:

$$\mathcal{Z} \colon P \mapsto Z(P) = C_S(P).$$

A *linking system* $\mathcal{L}$ associated to a saturated fusion system $\mathcal{F}$ over S is a small category whose objects are the $\mathcal{F}$-centric subgroups of S. It is equipped with a surjective functor $\pi \colon \mathcal{L} \to \mathcal{F}^c$ which is the identity on objects and with monomorphisms of groups $\delta_P \colon P \to \mathrm{Aut}_{\mathcal{L}}(P)$, one for each $P \in \mathcal{F}^c$ such that the following hold.

(A) For each $P, Q \in \mathcal{L}$ the group $Z(P)$ acts freely on $\mathcal{L}(P, Q)$ via $\delta_P \colon P \to \mathrm{Aut}_{\mathcal{L}}(P)$ and precomposition of morphisms, and $\pi \colon \mathcal{L}(P, Q) \to \mathrm{Hom}_{\mathcal{F}}(P, Q)$ is the quotient map by this action.
(B) For any $P \in \mathcal{L}$ and $g \in P$, $\pi(\delta_P(g)) = c_g \in \mathrm{Aut}_{\mathcal{F}}(P)$.
(C) For any $\varphi \in \mathcal{L}(P, Q)$ and $g \in P$, the following diagram commutes in $\mathcal{L}$

$$\begin{tikzcd} P \arrow[r, "\varphi"] \arrow[d, "\delta_P(g)"'] & Q \arrow[d, "\delta_Q(\pi(\varphi)(g))"] \\ P \arrow[r, "\varphi"'] & Q. \end{tikzcd}$$

The morphisms $\delta_P(g)$ will be denoted $\widehat{g}$.

A *p-local compact group* is a triple $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$ where $\mathcal{F}$ is a saturated fusion system over S and $\mathcal{L}$ is an associated centric linking system. The *classifying space* of $\mathcal{G}$ denoted $B\mathcal{G}$ is by definition $|\mathcal{L}|^\wedge_p$.

It is shown in [JLL, Proposition 1.5] that the monomorphisms δ_P can be extended to monomorphisms $N_S(P) \xrightarrow{g \mapsto \widehat{g}} \mathrm{Aut}_{\mathcal{L}}(P)$ which satisfy (B), and more generally to injective functions $\delta_{P,Q} \colon N_S(P, Q) \xrightarrow{g \mapsto \widehat{g}} \mathcal{L}(P, Q)$ which satisfy (B). Moreover $\widehat{g} \circ \widehat{h} = \widehat{gh}$ whenever $h \in N_S(P, Q)$ and $g \in N_S(Q, R)$. Thus, the identity element $e \in S$ give rise to morphisms $\iota_P^Q \in \mathcal{L}(P, Q)$ for any $P \leq Q$ where $\iota_P^Q = \widehat{e}$.

By [JLL, Corollary 1.8] the category $\mathcal{L}$ has the property that every morphism in $\mathcal{L}$ is both a monomorphism and an epimorphism (but in general not an isomorphism). This allows us, by [JLL, Lemma 1.7(i)], to define "restrictions": if $\varphi \in \mathcal{L}(P, Q)$ and $P' \leq P$ and $Q' \leq Q$ are $\mathcal{F}$-centric subgroups such that $\pi(\varphi)(P') \leq Q'$ then there exists a unique morphism $\psi \in \mathcal{L}(P', Q')$ such that $\varphi \circ \iota_{P'}^P = \iota_{Q'}^Q \circ \psi$. We write $\varphi|_{P'}^{Q'}$ for this unique morphism and call it the restriction of φ.

In [BLO3, Section 3] a collection of subgroups of S denoted $\mathcal{H}^\bullet(\mathcal{F})$ was constructed. We will not recall the precise details of its construction here. The full subcategory of $\mathcal{F}$ on this set of objects is denoted $\mathcal{F}^\bullet$. There is a functor $\mathcal{F} \to \mathcal{F}^\bullet$ where $P \leq P^\bullet$ for any $P \leq S$. This functor is left adjoint to the inclusion $\mathcal{F}^\bullet \subseteq \mathcal{F}$. In addition $\mathcal{H}^\bullet(\mathcal{F})$ contains the collection $\mathcal{F}^{cr}$ of all $P \leq S$ which are both $\mathcal{F}$-centric and $\mathcal{F}$-radical.

The functor $P \mapsto P^\bullet$ "lifts" to a functor $\mathcal{L} \to \mathcal{L}^\bullet$ such that $\widehat{g}^\bullet = \widehat{g}$ for any $g \in N_S(P, Q)$. See [JLL, Proposition 1.12].

3. Geometric unstable Adams operations

In this section we recall the the concept of geometric unstable Adams operations and make some basic observations, ending with the proof of Proposition 1.2.

3.1. **Groups with maximal discrete p-tori.**

3.1. **Definition.** A group G is called $\mathbb{Z}/p^\infty$*-free* if it contains no subgroup isomorphic to $\mathbb{Z}/p^\infty$, or equivalently if every homomorphism $\mathbb{Z}/p^\infty \to G$ is trivial. We say that $T \leq G$ is a maximal discrete p-torus in G if $T \cong \bigoplus^n \mathbb{Z}/p^\infty$ and any other subgroup of G isomorphic to a discrete p-torus is conjugate to a subgroup of T. We say that $S \leq G$ is a Sylow p-subgroup if every discrete p-toral subgroup of G is conjugate to a subgroup of S.

3.2. **Lemma.** *Let $1 \to K \to G \to H \to 1$ be an exact sequence of group.*

(i) If H and K are $\mathbb{Z}/p^\infty$-free then G is $\mathbb{Z}/p^\infty$-free.
(ii) If G is $\mathbb{Z}/p^\infty$-free and K is finite then H is $\mathbb{Z}/p^\infty$-free.

Proof. (i). Suppose $T \leq G$ is a discrete p-torus. Its image in H must be trivial by assumption. Hence $T \leq K$ which implies $T = 1$.

(ii). Assume that H is not $\mathbb{Z}/p^\infty$-free. Then there exists a sequence $x_1, x_2, x_3, \dots$ of non-identity elements in H such that $x_i = (x_{i+1})^p$ for all $i \geq 1$. The preimages of these elements in G are cosets $X_1, X_2, X_3, \dots$ of K and hence finite subsets of G. The function (not a homomorphism) $G \xrightarrow{x \mapsto x^p} G$ restricts to functions $X_{i+1} \xrightarrow{x \mapsto x^p} X_i$ for all i and we obtain a tower

$$\cdots \to X_{i+1} \xrightarrow{x \mapsto x^p} X_i \xrightarrow{x \mapsto x^p} \dots \xrightarrow{x \mapsto x^p} X_2 \xrightarrow{x \mapsto x^p} X_1.$$

Since the sets X_i are finite, $\varprojlim X_i$ is not empty. An element in $\varprojlim X_i$ is a sequence $g_1, g_2, g_3, \dots$ of non-identity elements of G such that $g_i = {g_{i+1}}^p$. These elements generate a copy of $\mathbb{Z}/p^\infty$ in G which is a contradiction. □

3.2. **Adams automorphisms of discrete p-toral groups and fusion systems.** Let S be a discrete p-toral group and let T denote its maximal torus. Recall that $\mathrm{Aut}(T) \cong \mathrm{GL}_r(\mathbb{Z}_p)$ where $r = \mathrm{rk}(T)$. It contains a copy of $\mathbb{Z}_p^\times$ in its centre (the diagonal matrices).

3.3. **Definition.** An *Adams automorphism* of S of degree $\zeta \in \mathbb{Z}_p^\times$ is an automorphism φ of S such that $\varphi|_T$ is multiplication by ζ. We say that φ is a *normal* Adams automorphism if it induces the identity on the set of components S/T. Set

$$\mathrm{Ad}(S) = \{\text{all } \textit{normal} \text{ Adams automorphisms of } S\}.$$

This is a normal subgroup of $\mathrm{Aut}(S)$. Restriction $\mathrm{Aut}(S) \to \mathrm{Aut}(T)$ induces a "degree homomorphism"

$$\deg\colon\ \mathrm{Ad}(S) \to \mathbb{Z}_p^\times.$$

Let $\mathrm{Ad}^1(S)$ or $\mathrm{Ad}^{\{\deg=1\}}(S)$ be the subgroup of the normal Adams automorphisms of degree 1.

Recall that if $\mathcal{F}$ is a fusion system over S, then $\alpha \in \mathrm{Aut}(S)$ is called *fusion preserving* if $\alpha \circ \varphi$ is a homomorphism in $\mathcal{F}$ for every $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$. The group of fusion preserving automorphisms is denoted $\mathrm{Aut}_{\mathrm{fus}}(S)$. It clearly contains $\mathrm{Aut}_{\mathcal{F}}(S)$ as a normal subgroup. We define (see [BLO3, Section 7])

$$\mathrm{Out}_{\mathrm{fus}}(S) = \mathrm{Aut}_{\mathrm{fus}}(S)/\mathrm{Aut}_{\mathcal{F}}(S).$$

3.4. **Definition.** Let $\mathcal{F}$ be a saturated fusion system over S. Set

$$\mathrm{Ad}_{\mathrm{fus}}(S) = \mathrm{Ad}(S) \cap \mathrm{Aut}_{\mathrm{fus}}(S).$$

Let $\mathrm{OutAd}_{\mathrm{fus}}(S)$ be the image of $\mathrm{Ad}_{\mathrm{fus}}(S)$ in $\mathrm{Out}_{\mathrm{fus}}(S)$.

The goal of this subsection is to prove

3.5. **Proposition.** *Let $\mathcal{F}$ be a saturated fusion system over S and let T be the identity component of S. Then*

(i) $\mathrm{OutAd}_{\mathrm{fus}}(S)$ is $\mathbb{Z}/p^\infty$-free and solvable of class at most 2.
(ii) It contains a normal Sylow p-subgroup which is abelian if p is odd or if $p = 2$ and the order of the extension class of S in $H^2(S/T; T)$ is at least 4.

Proof. The torsion elements in $\mathbb{Z}_p^\times$ form the group U_p of roots of unity in $\mathbb{Z}_p$, and by [Ro, Sec. 6.7, Prop. 1,2]

$$U_p \cong \begin{cases} C_2 = \{\pm 1\} & \text{if } p = 2 \\ C_{p-1} & \text{if } p > 2 \end{cases}$$

In particular U_p is finite and therefore $\mathbb{Z}_p^\times$ is $\mathbb{Z}/p^\infty$-free, hence so are all of $\Gamma_m(p)$. In fact, if $p > 2$ then $\Gamma_k(p)$ contains no finite p-subgroup, and if $p = 2$ then $\Gamma_k(2)$ contains no finite 2-subgroup if $k \geq 2$. By [JLL, Proposition 2.8] there is a short exact sequence

$$1 \to H^1(S/T;T) \to \mathrm{Ad}(S)/\mathrm{Aut}_T(S) \xrightarrow{\deg} \Gamma_m(p) \to 1$$

where p^m is the order of the extension class of S in $H^2(S/T,T)$. Since S/T is a finite p-group, $H^1(S/T,T)$ is a finite abelian p-group by a transfer argument. It follows that $\mathrm{Ad}(S)/\mathrm{Aut}_T(S)$ is solvable of class ≤ 2, and from Lemma 3.2(i) that it is $\mathbb{Z}/p^\infty$-free. Let P be the preimage of the normal Sylow p-subgroup of $\Gamma_m(p)$. Then P is a normal Sylow p-subgroup of $\mathrm{Ad}(S)/\mathrm{Aut}_T(S)$ and it is abelian if either p is odd or if $p = 2$ and $m \geq 2$ since in this case $U_p \cap \Gamma_m(p) = 1$, hence $P \cong H^1(S/T,T)$.

Observe that $\mathrm{Aut}_\mathcal{F}(S) \leq \mathrm{Aut}_{\mathrm{fus}}(S)$ and that $\mathrm{Aut}_T(S) \leq \mathrm{Ad}(S)$. Hence $\mathrm{Ad}_{\mathrm{fus}}(S) \cap \mathrm{Aut}_\mathcal{F}(S) = \mathrm{Ad}(S) \cap \mathrm{Aut}_\mathcal{F}(S)$ and by Definition 3.4 we obtain the short exact sequence

$$1 \to \frac{\mathrm{Aut}_\mathcal{F}(S) \cap \mathrm{Ad}(S)}{\mathrm{Aut}_T(S)} \to \frac{\mathrm{Ad}_{\mathrm{fus}}(S)}{\mathrm{Aut}_T(S)} \to \mathrm{OutAd}_{\mathrm{fus}}(S) \to 1.$$

From the results above about $\mathrm{Ad}(S)/\mathrm{Aut}_T(S)$ it follows that $\mathrm{OutAd}_{\mathrm{fus}}(S)$ is solvable of class at most 2. The first group in this exact sequence is finite since it is a subgroup of $\mathrm{Aut}_\mathcal{F}(S)/\mathrm{Aut}_T(S)$ which is finite since S/T is finite and $\mathrm{Out}_\mathcal{F}(S) = \mathrm{Aut}_\mathcal{F}(S)/\mathrm{Inn}(S)$ is finite by [BLO3, Lemma 2.5]. Lemma 3.2(ii) now implies that $\mathrm{OutAd}_{\mathrm{fus}}(S)$ is $\mathbb{Z}/p^\infty$-free and this completes the proof of (i). Recall that $\mathrm{Ad}(S)/\mathrm{Aut}_T(S)$ has a normal Sylow p-subgroup. Let P be the normal Sylow p-subgroup of $\mathrm{Ad}_{\mathrm{fus}}(S)/\mathrm{Aut}_T(S)$ and let $\bar{P}$ be its image in $\mathrm{OutAd}_{\mathrm{fus}}(S)$. Then $\bar{P}$ is normal and it is abelian if either p is odd or if $p = 2$ and $m \geq 2$ namely if the extension class of S in $H^2(S/T,T)$ has order at least 4. It remains to show $\bar{P}$ is a Sylow p-subgroup of $\mathrm{OutAd}_{\mathrm{fus}}(S)$. If $Q \leq \mathrm{OutAd}_{\mathrm{fus}}(S)$ is a discrete p-toral group, it is finite since $\mathrm{OutAd}_{\mathrm{fus}}(S)$ is $\mathbb{Z}/p^\infty$-free and its preimage H in $\mathrm{Ad}_{\mathrm{fus}}(S)/\mathrm{Aut}_T(S)$ is therefore finite. If $H_p \in \mathrm{Syl}_p(H)$ then $H_p \leq P$ since P is normal and its image in $\mathrm{OutAd}_{\mathrm{fus}}(S)$ is Q. Hence $Q \leq \bar{P}$ as needed. □

3.3. **Geometric Adams operations.** The goal of this subsection is to prove Theorem 1.3. Let $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$ be a p-local compact group.

3.6. **Definition** (Compare [JLL, Definition 3.4])**.** A *geometric unstable Adams operation* of degree $\zeta \in \mathbb{Z}_p^\times$ is a self equivalence $f\colon B\mathcal{G} \to B\mathcal{G}$ such that there exists $\varphi \in \mathrm{Ad}(S)$ of degree ζ which renders the following diagram commutative

$$\begin{tikzcd} BS \arrow[r, "B\varphi"] \arrow[d] & BS \arrow[d] \\ B\mathcal{G} \arrow[r, "f"'] & B\mathcal{G}. \end{tikzcd}$$

We will write

$$\mathrm{Ad}^{\mathrm{geom}}(B\mathcal{G})$$

for the group of *homotopy classes* of geometric unstable Adams operations.

This definition differs slightly from [JLL] where we wrote $\mathrm{Ad}^g(\mathcal{G})$ for the topological group of unstable Adams operations and used $\pi_0 \mathrm{Ad}^g(\mathcal{G})$ for what we now denote by $\mathrm{Ad}^{\mathrm{geom}}(B\mathcal{G})$.

At this point the reader may wonder about the restriction to normal Adams automorphisms of S (by virtue of $\mathrm{Ad}(S)$). Indeed, in general there are fusion preserving Adams automorphisms which are not normal. Recall that $\mathcal{F}$ is called *connected* if every element of S is $\mathcal{F}$-conjugate to an element of T.

3.7. **Definition.** Let $\mathcal{F}$ be a saturated fusion system over S. We say that $\mathcal{F}$ is *connected* if every element of S is $\mathcal{F}$-conjugate to an element of its maximal torus S_0. A p-local compact group $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$ is *connected* if its underlying fusion system $\mathcal{F}$ is connected.

3.8. **Definition.** A saturated fusion system $\mathcal{F}$ over S is *weakly connected* if S_0 is self-centralising in S, namely $C_S(S_0) = S_0$.

The definitions are motivated by the fact that a compact Lie group G is connected if and only if every element of G is conjugate to an element of its maximal torus and in this case the maximal torus is self-centralising.

3.9. **Lemma.** *If $\mathcal{F}$ is connected then it is weakly connected, namely S_0 is self centralising. If S_0 is self centralising in S then every Adams automorphism of S is normal.*

Proof. The second assertion is [JLL, Lemma 2.5]. To prove the first, we first claim that any $x \in S \setminus T$ is not fully centralised, namely the subgroup it generates is not fully centralised. If it is, then by assumption there is some $t \in T$ and a morphism $\varphi \in \mathcal{F}$ such that $\varphi(t) = x$. By the extension axiom φ extends to $\psi \colon C_S(T) \to S$. But $T \subseteq C_S(T)$ and $\psi(T) = T$ which is impossible since $\psi(t) = \varphi(t) = x$.

Now suppose that $x \in C_S(T) \setminus T$. Then there is a morphism $\varphi \in \mathcal{F}$ such that $\varphi(x) = t \in T$ is fully centralised. By the extension axion φ extends to $\psi \colon C_S(T) \to S$ which carries x to $t \in T$. But ψ must induce an isomorphism of T, a contradiction. Hence $C_S(T) = T$ as claimed. □

We recall the following from [BLO3]. For any discrete p-toral group Q define an equivalence relation on the set $\mathrm{Hom}(Q, S)$ where $\rho \sim \rho'$ if there exists some $\mathcal{F}$-isomorphism $\chi \colon \rho(Q) \to \rho'(Q)$ such that $\rho' = \chi \circ \rho$. Set $\mathrm{Rep}(Q, \mathcal{L}) = \mathrm{Hom}(Q, S)/\sim$. Notice that $\mathrm{Rep}(S, \mathcal{L}) = \mathrm{Aut}(S)/\mathrm{Aut}_{\mathcal{F}}(S)$. Recall from [BLO3, Theorem 6.3] that there is a natural bijection $[BQ, B\mathcal{G}] \cong \mathrm{Rep}(Q, \mathcal{L})$. In particular there is a natural bijection of sets

$$[BS, B\mathcal{G}] \cong \mathrm{Aut}(S)/\mathrm{Aut}_{\mathcal{F}}(S) \qquad \text{(left cosets of } \mathrm{Aut}_{\mathcal{F}}(S)). \tag{3.10}$$

The map $\mathrm{Res} \colon [B\mathcal{G}, B\mathcal{G}] \to [BS, B\mathcal{G}]$ therefore gives a homomorphism

$$\mathrm{Out}(B\mathcal{G}) \xrightarrow{\mu} \mathrm{Out}_{\mathrm{fus}}(S) \tag{3.11}$$

and by [BLO3, Proposition 7.2, Proposition 5.8] there is an exact sequence

$$0 \to \varprojlim_{\mathcal{O}^c(\mathcal{F})}{}^1 \mathcal{Z} \to \mathrm{Out}(B\mathcal{G}) \xrightarrow{\mu} \mathrm{Out}_{\mathrm{fus}}(S) \to \varprojlim_{\mathcal{O}^c(\mathcal{F})}{}^2 \mathcal{Z} \tag{3.12}$$

where the groups at the ends are finite abelian p-groups by [BLO3, Proposition 5.8] and since $\mathcal{O}^c(\mathcal{F})$ is equivalent to a finite subcategory.

Proof of Proposition 1.2. As we have seen above, by [BLO3] there is a commutative square

$$\begin{array}{ccc} \mathrm{Out}(B\mathcal{G}) & \xrightarrow{\mu} & \mathrm{Out}_{\mathrm{fus}}(S) \\ {\scriptstyle\mathrm{Res}}\downarrow & & \downarrow \\ [BS, B\mathcal{G}] & \xrightarrow[\cong]{} & \mathrm{Aut}(S)/\mathrm{Aut}_{\mathcal{F}}(S) \end{array}$$

Hence, if $f \colon B\mathcal{G} \to B\mathcal{G}$ is a geometric unstable Adams operation then by definition there exists $\varphi \in \mathrm{Ad}(S)$ such that the diagram in Definition 3.6 commutes up to homotopy and since the right arrow in the diagram above is injective, $\mu([f]) = [\varphi]$. In particular φ is fusion preserving and hence $\mu([f]) \in \mathrm{OutAd}_{\mathrm{fus}}(S)$. Conversely, suppose that $f \colon B\mathcal{G} \to B\mathcal{G}$ is a self equivalence such that $\mu([f]) \in \mathrm{OutAd}_{\mathrm{fus}}(S)$. Then there exists some $\varphi \in \mathrm{Ad}_{\mathrm{fus}}(S)$ such that $\mu([f]) = [\varphi]$. Since the bottom arrow in the diagram above is bijective it follows that the diagram in Definition 3.6 commutes up to homotopy and therefore f is a geometric unstable Adams operation. We have thus shown that

$$\mathrm{Ad}^{\mathrm{geom}}(B\mathcal{G}) = \mu^{-1}(\mathrm{OutAd}_{\mathrm{fus}}(S)).$$

By [LL, Theorem B] for any p-local compact group $\varprojlim_{\mathcal{O}^c(\mathcal{F})}{}^2 \mathcal{Z} = 0$ and, if p is odd then $\varprojlim_{\mathcal{O}^c(\mathcal{F})}{}^1 \mathcal{Z} = 0$. In light of (3.12), the vanishing of $\varprojlim{}^2 \mathcal{Z}$ implies the exactness of the sequence in the statement of this proposition. The vanishing of $\varprojlim{}^1 \mathcal{Z}$, provided p is odd, establishes the isomorphism $\mathrm{Ad}^{\mathrm{geom}}(B\mathcal{G}) \cong \mathrm{OutAd}_{\mathrm{fus}}(S)$.

It follows from Proposition 3.5 that $\mathrm{OutAd}_{\mathrm{fus}}(S)$ is solvable of class at most 2 and since $\varprojlim^1_{\mathcal{O}^c(\mathcal{F})} \mathcal{Z}$ is a finite abelian p-group, $\mathrm{Ad}^{\mathrm{geom}}(B\mathcal{G})$ has a normal Sylow p-subgroup. Lemma 3.2(i) shows that it is also $\mathbb{Z}/p^\infty$-free. $\square$

4. Algebraic unstable Adams operations

In this section we will prove Theorem 1.3. Throughout we will fix a p-local compact group $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$.

4.1. **Definition** (Compare [JLL, Definition 1.11])**.** Let $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$ be a p-local compact group and $\phi\colon S \to S$ be a fusion preserving automorphism. Let $\mathcal{L}', \mathcal{L}''$ be subcategories of $\mathcal{L}$ and $\Phi\colon \mathcal{L}' \to \mathcal{L}''$ be a functor. We say that Φ *covers* ϕ if the following square is commutative

$$\begin{tikzcd} \mathcal{L}' \arrow[r, "\Phi"] \arrow[d, "\pi"'] & \mathcal{L}'' \arrow[d, "\pi"] \\ \mathcal{F} \arrow[r, "\phi_*"'] & \mathcal{F} \end{tikzcd}$$

and if for every $P, Q \in \mathcal{L}'$ and every $g \in N_S(P,Q)$ the morphisms $\widehat{g} \in \mathcal{L}(P,Q)$ and $\widehat{\phi(g)} \in \mathcal{L}(\phi(P), \phi(Q))$ belong to $\mathcal{L}'$ and $\mathcal{L}''$ respectively, and moreover $\Phi(\widehat{g}) = \widehat{\phi(g)}$.

4.2. **Definition** ([JLL, Definition 3.3])**.** An algebraic unstable Adams operation on $\mathcal{G}$, or simply an *unstable Adams operation on* $\mathcal{G}$, is a pair (Ψ, ψ), where $\psi \in \mathrm{Ad}_{\mathrm{fus}}(S)$ and $\Psi\colon \mathcal{L} \to \mathcal{L}$ is a functor which *covers* ψ. In other words,

(a) The following diagram is commutative where ψ_* is the automorphism of $\mathcal{F}$ induced by ψ

$$\begin{tikzcd} \mathcal{L} \arrow[r, "\Psi"] \arrow[d, "\pi"'] & \mathcal{L} \arrow[d, "\pi"] \\ \mathcal{F} \arrow[r, "\psi_*"'] & \mathcal{F} \end{tikzcd}$$

(b) For every $\mathcal{F}$-centric $P, Q \leq S$ and every $g \in N_S(P,Q)$ we have $\Psi(\widehat{g}) = \widehat{\psi(g)}$.

Define

$$\mathrm{Ad}(\mathcal{G}) = \{ \text{ all unstable Adams operations } (\Psi, \psi) \text{ on } \mathcal{G} \ \}.$$

If $H \leq \mathbb{Z}_p^\times$ we will write $\mathrm{Ad}^H(\mathcal{G})$ for those (Ψ, ψ) such that $\deg(\psi) \in H$.

4.3. **Remark.** Recall that the inclusion $BS \xrightarrow{\text{incl}} B\mathcal{G}$ is induced by the geometric realisation of the inclusion of categories $\mathcal{B}S \subseteq \mathcal{B}\mathrm{Aut}_{\mathcal{L}}(S) \subseteq \mathcal{G}$ via $s \mapsto \hat{s}$. By definition of an unstable Adams operation (Ψ, ψ), upon taking geometric realisations the following diagram commutes on the nose

$$\begin{tikzcd} BS \arrow[r, "B\psi"] \arrow[d] & BS \arrow[d] \\ B\mathcal{G} \arrow[r, "|\Psi|^\wedge_p"'] & B\mathcal{G} \end{tikzcd}$$

hence $|\Psi|^\wedge_p$ is a geometric unstable Adams operation (Definition 3.6). We obtain a homomorphism

$$\gamma\colon \mathrm{Ad}(\mathcal{G}) \xrightarrow{(\Psi,\psi) \mapsto |\Psi|^\wedge_p} \mathrm{Ad}^{\mathrm{geom}}(B\mathcal{G}). \tag{4.4}$$

Recall from [BLO3, Section 7] that an automorphism $\Phi\colon \mathcal{L} \to \mathcal{L}$ is called *isotypical* if for every $\mathcal{F}$-centric subgroup $P \leq S$ the isomorphism $\mathrm{Aut}_{\mathcal{L}}(P) \to \mathrm{Aut}_{\mathcal{L}}(\Phi(P))$ induced by Φ carries $\delta(P)$ to $\delta(\Phi(P))$. The collection of all isotypical self equivalences of $\mathcal{L}$ that preserve inclusions is a group [AOV, Lemma 1.14], and we denote it here by $\mathrm{Aut}_{\mathrm{typ}}(\mathcal{L})$ (The notation in [AOV] is $\mathrm{Aut}^I_{\mathrm{typ}}(\mathcal{L})$). It is clear from the definition that if (Ψ,ψ) is an unstable Adams operation then $\Psi \in \mathrm{Aut}_{\mathrm{typ}}(\mathcal{L})$. Moreover, ψ is completely determined by Ψ because for every $s \in S$ we have $\Psi(\hat{s}) = \widehat{\psi(s)}$ and since $\delta\colon S \to \mathrm{Aut}_{\mathcal{L}}(S)$ is injective. Therefore

$$\mathrm{Ad}(\mathcal{G}) \xrightarrow{(\Psi,\psi)\mapsto\Psi} \mathrm{Aut}_{\mathrm{typ}}(\mathcal{L}).$$

is injective and we can, and will, identify $\mathrm{Ad}(\mathcal{G})$ with a subgroup of $\mathrm{Aut}_{\mathrm{typ}}(\mathcal{L})$.

Geometric realisation of isotypical equivalences gives rise to a homomorphism [BLO3, Section 7]

$$\Omega\colon \mathrm{Aut}_{\mathrm{typ}}(\mathcal{L}) \xrightarrow{\Phi\mapsto[|\Phi|^\wedge_p]} \mathrm{Out}(B\mathcal{G}).$$

By Remark 4.3 the restriction of Ω to $\mathrm{Ad}(\mathcal{G})$ is the map γ in (4.4). Among all isotypical equivalences of $\mathcal{L}$ there are the ones that are induced by "conjugation" by automorphisms of S. To be precise, there is a homomorphism

$$\tau\colon \mathrm{Aut}_{\mathcal{L}}(S) \to \mathrm{Aut}_{\mathrm{typ}}(\mathcal{L}) \tag{4.5}$$

defined for every $\varphi \in \mathrm{Aut}_{\mathcal{L}}(S)$ as follows. On object, $\tau(\varphi)(P) = \pi(\varphi)(P)$ where $\pi\colon \mathcal{L} \to \mathcal{F}$ is the projection. If $\alpha \in \mathcal{L}(P,Q)$ is a morphism, set $\tau(\varphi)(\alpha) = (\varphi|_Q^{\varphi(Q)}) \circ \alpha \circ (\varphi|_P^{\varphi(P)})^{-1}$ where we write $\varphi(P)$ instead of $\pi(\varphi)(P)$ etc. It is easy to check, using the axioms of linking systems, that $\tau(\varphi)$ is isotypical. In fact, $\mathrm{Aut}_{\mathcal{L}}(S) \trianglelefteq \mathrm{Aut}_{\mathrm{typ}}(\mathcal{L})$ and the quotient group is denoted $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L})$. It is shown in [BLO3, Theorem 7.1] that there is an isomorphism

$$\mathrm{Out}_{\mathrm{typ}}(\mathcal{L}) \cong \mathrm{Out}(|\mathcal{L}|^\wedge_p) \tag{4.6}$$

via the assignment $[\Phi] \mapsto [|\Phi|^\wedge_p]$. The definition of $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L})$ in [BLO3] and our definition coincide by [AOV, Lemma 1.14].

4.7. **Lemma.** *Let $\mathcal{F}$ be a weakly connected fusion system over S. If $\psi \in \mathrm{Aut}_{\mathcal{F}}(S)$ is an Adams automorphism of degree 1, then ψ is conjugation by t for some $t \in T$.*

Proof. Since $\mathcal{F}$ is weakly connected, T is $\mathcal{F}$-centric. Now, $\psi|_S = \mathrm{id}_S|_T$ and therefore [BLO3, Prop. 2.8] implies that ψ is conjugation by t for some $t \in Z(T) = T$. □

4.8. **Lemma.** *Let $\mathcal{G} = (S,\mathcal{F},\mathcal{L})$ be a weakly connected p-local compact group, and suppose that*

$$(\Psi, Id_S) \ \in\ \mathrm{Ker}\,(\gamma\colon\ \mathrm{Ad}(\mathcal{G}) \to \mathrm{Ad}^{\mathrm{geom}}(\mathcal{G}))\,.$$

Then $\Psi = \tau(\hat{t})$ for some $t \in T$.

Proof. By [BLO3, Theorem 7.1] the map γ is the restriction to $\mathrm{Ad}(\mathcal{G})$ of the composition of homomorphisms

$$\mathrm{Aut}_{\mathrm{typ}}(\mathcal{L}) \xrightarrow{\mathrm{proj}} \mathrm{Out}_{\mathrm{typ}}(\mathcal{L}) \xrightarrow{\cong} \mathrm{Out}(B\mathcal{G}).$$

Note that $\Psi(P) = \mathrm{Id}_S(P) = P$ for every $\mathcal{F}$-centric $P \le S$ and that given any $\varphi \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$ the images of φ and $\Psi(\varphi)$ in $\mathrm{Hom}_{\mathcal{F}}(P,Q)$ are equal because Ψ covers $(\mathrm{Id}_S)_* = \mathrm{Id}_{\mathcal{F}}$. Also, since $(\Psi,\mathrm{Id}_S) \in \mathrm{Ker}(\gamma)$, Ψ is conjugation by some $\rho \in \mathrm{Aut}_{\mathcal{L}}(S) \cong \mathrm{Aut}_{\mathcal{L}}(S)$. In particular for every $\varphi \in \mathrm{Aut}_{\mathcal{L}}(S)$, $\Psi(\varphi) \circ \rho = \rho \circ \varphi$.

Since (Ψ,Id_S) is an Adams operation, $\Psi(\hat{s}) = \hat{s}$ for every $s \in S$, and so $\rho_S \in C_{\mathrm{Aut}_{\mathcal{L}}(S)}(S) = Z(S)$, where S is considered as a subgroup of $\mathrm{Aut}_{\mathcal{L}}(S)$ via the canonical inclusion. Since $\mathcal{F}$ is weakly connected $T = C_S(T) \ge Z(S)$, and therefore $\rho = \hat{t}$ for some $t \in Z(S) \le T$. Thus, by the definition of τ in (4.5), $\Psi = \tau(\hat{t})$ as claimed. □

We will now consider the following subgroup of $\mathrm{Aut}_{\mathcal{L}}(S)$

$$\mathrm{Aut}_{\mathcal{L}}^{\mathrm{Ad}}(S) = \{\varphi \in \mathrm{Aut}_{\mathcal{L}}(S) : \pi(\varphi) \in \mathrm{Aut}_{\mathcal{F}}(S) \cap \mathrm{Ad}(S)\}.$$

Recall that the centre of $\mathcal{F}$ is the subgroup $Z(\mathcal{F})$ of $Z(S)$ of all $x \in Z(S)$ such that $\varphi(z) = z$ for any $\mathcal{F}$-homomorphism which contains x in its domain. Equivalently, since $Z(S)$ is contained in every $\mathcal{F}$-centric subgroup it follows from [BLO3, Theorem 3.6] that $Z(\mathcal{F})$ is the set of all $x \in Z(S)$ such that $\varphi(x) = x$ for any $\mathcal{F}$-centric $P \le S$ and any $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P,S)$.

4.9. **Lemma.** *There is an exact sequence*

$$1 \to Z(\mathcal{F}) \xrightarrow{z \mapsto \hat{z}} \mathrm{Aut}_{\mathcal{L}}^{\mathrm{Ad}}(S) \xrightarrow{\tau} \mathrm{Ad}(\mathcal{G}) \xrightarrow{\gamma} \mathrm{Ad}^{\mathrm{geom}}(B\mathcal{G}) \to 1.$$

Proof. The homomorphism γ defined in (4.4) is surjective by [JLL, Prop. 3.5]. We have seen above that if $\varphi \in \mathrm{Aut}_{\mathcal{L}}^{\mathrm{Ad}}(S)$ then $\tau(\varphi)$ is an unstable Adams operation whose geometric realisation is homotopic to the identity, so $\mathrm{im}(\tau) \leq \ker(\gamma)$. Consider some (Ψ, ψ) in the kernel of γ. We will show that Ψ is in the image of τ. The bijection (4.6) and by the definition of $\mathrm{Out}_{\mathrm{typ}}(\mathcal{L})$ it follows that there exists a natural isomorphism $\rho\colon \mathrm{Id}_{\mathcal{L}} \to \Psi$. In particular, at the object $S \in \mathcal{L}$ we obtain a morphism $\rho_S\colon S \to S$ in $\mathcal{L}$, i.e. $\rho_S \in \mathrm{Aut}_{\mathcal{L}}(S)$. In fact, ρ_S determines ρ completely because for every $P \in \mathcal{L}$ we have the following commutative square

$$\begin{tikzcd} P \arrow[r, "\iota_P^S"] \arrow[d, "\rho_P"'] & S \arrow[d, "\rho_S"] \\ \psi(P) \arrow[r, "\Psi(\iota_P^S)"'] & S. \end{tikzcd}$$

Since $\iota_P^S = \widehat{e}$ where $e \in N_S(P,S)$ and since $\Psi(\widehat{e}) = \widehat{e}$, we see that $\Psi(\iota_P^S) = \iota_{\psi(P)}^S$. Since ι_P^S is a monomorphism we deduce that ρ_P is determined by ρ_S, in fact $\rho_P = \rho_S|_P^{\psi(P)}$ for any $P \in \mathcal{L}$. For every $s \in S$ we obtain the following commutative square

$$\begin{tikzcd} S \arrow[r, "\hat{s}"] \arrow[d, "\rho_S"'] & S \arrow[d, "\rho_S"] \\ S \arrow[r, "\widehat{\psi(s)}"'] & S. \end{tikzcd}$$

It follows from Axiom (C) of linking systems [BLO3, Def. 4.1] that $\pi(\rho_S) = \psi$, in particular $\rho_S \in \mathrm{Aut}_{\mathcal{L}}^{\mathrm{Ad}}(S)$. Now we claim that $\Psi = \tau(\rho_S)$ First, for every $\mathcal{F}$-centric $P \leq S$,

$$\Psi(P) = \psi(P) = \pi(\rho_S)(P) = \tau(\rho_S)(P).$$

So Ψ and $\tau(\rho_S)$ agree on objects of $\mathcal{L}$. Next, suppose that $\alpha \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$. Since ρ is a natural transformation there is a commutative diagram

$$\begin{tikzcd} P \arrow[r, "\alpha"] \arrow[d, "\rho_P"'] & Q \arrow[d, "\rho_Q"] \\ \psi(P) \arrow[r, "\Psi(\alpha)"'] & \psi(Q). \end{tikzcd}$$

Since ρ_P and ρ_Q are restrictions of ρ_S we get

$$\Psi(\alpha) = \rho_Q \circ \alpha \circ \rho_P^{-1} = (\rho_S|_Q^{\psi(Q)}) \circ \alpha \circ (\rho_S|_P^{\psi(P)})^{-1} = \tau(\rho_S)(\alpha).$$

This completes the proof the the sequence is exact at $\mathrm{Ad}(\mathcal{G})$.

We now show exactness at $\mathrm{Aut}_{\mathcal{L}}^{\mathrm{Ad}}(S)$. If $z \in Z(\mathcal{F})$ then for any $\varphi \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$ we have $z \in Z(S) \leq P, Q$ and

$$\tau(\hat{z})(\varphi) = \hat{z}|_Q^Q \circ \varphi \circ (\hat{z}|_P^P)^{-1} = \hat{z}|_Q^Q \circ (\widehat{\pi(\varphi)(z)}|_Q^Q)^{-1} \circ \varphi = \hat{z}|_Q^Q \circ (\widehat{z}|_Q^Q)^{-1} \circ \varphi = \varphi.$$

Therefore $Z(\mathcal{F}) \to \mathrm{Aut}_{\mathcal{L}}^{\mathrm{Ad}}(S) \xrightarrow{\tau} \mathrm{Ad}(\mathcal{G})$ is trivial. Now suppose $\alpha \in \mathrm{Ker}(\tau)$. Then in particular $\alpha \in C_{\mathrm{Aut}_{\mathcal{L}}(S)}(S) = Z(S)$, namely $\alpha = \hat{z}$ for some $z \in Z(S)$. In addition for any $\varphi \in \mathrm{Mor}_{\mathcal{L}}(P,Q)$ we must have $\hat{z}|_Q^Q \circ \varphi \circ (\hat{z}|_P^P)^{-1} = \varphi$. Axiom (C) of linking systems and the fact that φ is an epimorphism in $\mathcal{L}$ imply that $\pi(\varphi)(z) = z$. It follows that $z \in Z(\mathcal{F})$ and this shows the exactness at $\mathrm{Aut}_{\mathcal{L}}^{\mathrm{Ad}}(S)$. □

Composition of the degree map with the projection $\pi\colon \mathcal{L} \to \mathcal{F}$ gives rise to a degree homomorphism

$$\deg\colon \mathrm{Aut}_{\mathcal{L}}^{\mathrm{Ad}}(S) \xrightarrow{\pi} \mathrm{Aut}_{\mathcal{F}}(S) \cap \mathrm{Ad}(S) \xrightarrow{\deg} \mathbb{Z}_p^\times.$$

Recall that $\mathrm{Out}_{\mathcal{F}}(S)$ is finite, hence every element of $\mathrm{Aut}_{\mathcal{F}}(S)$ has finite order. Therefore the image of $\mathrm{Aut}_{\mathcal{F}}(S)\cap \mathrm{Ad}(S)$ under the degree map $\deg\colon \mathrm{Ad}(S)\to \mathbb{Z}_p^\times$ must be contained in the group U_p of the roots of unity in $\mathbb{Z}_p$. We will set

$$D(\mathcal{F}) = \mathrm{Im}\ \Big(\mathrm{Ad}(S)\cap \mathrm{Aut}_{\mathcal{F}}(S) \xrightarrow{\deg} U_p\Big). \tag{4.10}$$

4.11. **Lemma.** *Let $\mathcal{G}=(S,\mathcal{F},\mathcal{L})$ be a weakly connected p-local compact group. There is a short exact sequence*

$$1\to T\to \mathrm{Aut}^{\mathrm{Ad}}_{\mathcal{L}}(S)\xrightarrow{\deg} D(\mathcal{F})\to 1.$$

Proof. By definition, deg is onto $D(\mathcal{F})$. Also $T\le \mathrm{Ker}(\deg)$ since T is abelian so $\mathrm{Aut}_T(S)\le \mathrm{Ad}^{\deg=1}(S)$. Suppose that $\varphi\in\mathrm{Ker}(\deg)$. Lemma 4.7 shows that $\pi(\varphi)$ is conjugation by some $t'\in T$. Since $Z(S)\le C_S(T)=T$ and T is $\mathcal{F}$-centric by the assumption of weak connectedness, $\varphi=\hat{t}$ for some $t\in T$. □

Proof of Theorem 1.3. Clearly $T\le \mathrm{Aut}^{\mathrm{Ad}}_{\mathcal{L}}(S)$, since $\mathrm{Aut}_T(S)\le \mathrm{Aut}_{\mathcal{F}}(S)\cap\mathrm{Ad}(S)$. Since $Z(\mathcal{F})\le Z(S)\le T$, The exact sequences in Lemmas 4.11 and 4.9 yield the exact sequence

$$1\to D(\mathcal{F})\xrightarrow{\tau}\mathrm{Ad}(\mathcal{G})/\tau(T)\xrightarrow{\bar{\gamma}}\mathrm{Ad}^{\mathrm{geom}}(B\mathcal{G})\to 1.$$

Since $D(\mathcal{F})$ is finite and $\mathrm{Ad}^{\mathrm{geom}}(B\mathcal{G})$ is $\mathbb{Z}/p^\infty$-free by Proposition 1.2, Lemma 3.2(i) implies that $\mathrm{Ad}(\mathcal{G})/\tau(T)$ is $\mathbb{Z}/p^\infty$-free. It follows that $\tau(T)$ is a normal maximal discrete p-torus in $\mathrm{Ad}(\mathcal{G})$ denoted $\mathrm{Ad}(\mathcal{G})_0$.

Let P be the unique Sylow p-subgroup of $\mathrm{Ad}^{\mathrm{geom}}(B\mathcal{G})$ guaranteed in Proposition 1.2 and let H be its preimage in $\mathrm{Ad}(\mathcal{G})/\mathrm{Ad}(\mathcal{G})_0$. If $p=2$ then $D(\mathcal{F})\le U_p\cong C_2$, so H is a finite 2-group and it is clearly the Sylow 2-subgroup of $\mathrm{Ad}(\mathcal{G})/\mathrm{Ad}(\mathcal{G})_0$. In particular $\mathrm{Ad}(\mathcal{G})$ has a normal Sylow 2-group. If p is odd, any Sylow p-subgroup of (the finite) group H is a Sylow p-subgroup of $\mathrm{Ad}(\mathcal{G})/\mathrm{Ad}(\mathcal{G})_0$. This completes the proof. □

5. The degree of an unstable Adams operation (uniqueness)

Unstable Adams operations of connected compact Lie groups have the pleasant property that their degree determines them completely up to homotopy. That is, given a connected compact Lie group G and an integer k, up to homotopy there is at most one unstable Adams operation on BG of degree k. This was shown in [JMO, Theorem 1]. In the set up of p-local compact groups, Proposition 1.2 makes it clear that this does not hold in general. First, at least when $p=2$, any non-trivial element in $\varprojlim^1\mathcal{Z}$ gives rise to a geometric unstable Adams operation whose underlying automorphism of S is the identity. Secondly, since $\mathrm{OutAd}_{\mathrm{fus}}(S)\cong \mathrm{Ad}_{\mathrm{fus}}(S)/(\mathrm{Aut}_{\mathcal{F}}(S)\cap\mathrm{Ad}(S))$, the degree can only be defined modulo $D(\mathcal{F})$. That is, the degree map that we must consider is

$$\deg\colon\ \mathrm{Ad}^{\mathrm{geom}}(B\mathcal{G})\to \mathrm{OutAd}_{\mathrm{fus}}(S)\xrightarrow{\deg}\mathbb{Z}_p^\times/D(\mathcal{F}).$$

From the algebraic point of view we can define a degree homomorphism

$$\deg\colon\ \mathrm{Ad}(\mathcal{G})\xrightarrow{(\Psi,\psi)\mapsto\psi}\mathrm{Ad}_{\mathrm{fus}}(S)\xrightarrow{\deg}\mathbb{Z}_p^\times.$$

But observe that every $t\in T\setminus Z(\mathcal{F})$ gives an unstable Adams operation $\tau(\widehat{t})$ of degree 1, hence the kernel of this degree map is in general not trivial. In this section we find conditions which guarantee that the degree does determine the unstable Adams operation in a suitable sense.

5.1. **Definition.** Let $\mathcal{F}$ be a saturated fusion system over S. The Weyl group of $\mathcal{F}$ is $W(\mathcal{F})=\mathrm{Aut}_{\mathcal{F}}(T)$.

Recall that $\mathrm{Aut}^{\mathrm{Ad}}_{\mathcal{L}}(S)$ is the subgroup of $\mathrm{Aut}_{\mathcal{L}}(S)$ of the morphisms which project to $\mathrm{Aut}_{\mathcal{F}}(S)\cap\mathrm{Ad}(S)$. There is therefore a degree homomorphism $\deg\colon \mathrm{Aut}^{\mathrm{Ad}}_{\mathcal{L}}(S)\to\mathrm{Aut}_{\mathcal{F}}(S)\cap\mathrm{Ad}(S)\xrightarrow{\deg}\mathbb{Z}_p^\times$. Its image is $D(\mathcal{F})$ by (4.10).

5.2. **Lemma.** *Let $\mathcal{G}=(S,\mathcal{F},\mathcal{L})$ be a weakly connected p-local compact group, and suppose that $H^1(W,T)=0$. Then there is a short exact sequence*

$$1\to T\xrightarrow{\delta|_T}\mathrm{Aut}^{\mathrm{Ad}}_{\mathcal{L}}(S)\xrightarrow{\deg} D(\mathcal{F})\to 1$$

Proof. Clearly $\delta|_T$ is injective and the degree map is surjective. Also $\deg \circ \delta|_T$ is the trivial homomorphism since T is abelian. Suppose that $\varphi \in \mathrm{Ker}(\deg)$. Then $\psi(\varphi)|_T = \mathrm{id}_T$ and since T is $\mathcal{F}$-centric, it follows from [BLO3, Proposition 2.8] that $\pi(\varphi) = c_z$ for some $z \in Z(T) = T$. Since $Z(S) \leq T$ we now deduce that $\varphi = \hat{t}$ for some $t \in T$. $\square$

5.3. **Lemma.** *The homomorphism*

$$\mathrm{Ad}(\mathcal{G}) \xrightarrow{(\Psi,\psi)\mapsto\psi} \mathrm{Ad}_{\mathrm{fus}}(S)$$

is surjective.

Proof. Consider some $\varphi \in \mathrm{Ad}_{\mathrm{fus}}(S)$ and let $[\varphi]$ denote its class in $\mathrm{OutAd}_{\mathrm{fus}}(S)$. The exact sequences in Proposition 1.2 and Theorem 1.3 show that there exists $(\Psi, \psi) \in \mathrm{Ad}(\mathcal{G})$ such that $[\varphi] = [\psi]$. Hence $\varphi = \alpha \circ \psi$ for some $\alpha \in \mathrm{Aut}_{\mathcal{F}}(S) \cap \mathrm{Ad}(S)$. Let $\tilde{\alpha} \in \mathrm{Aut}_{\mathcal{L}}^{\mathrm{Ad}}(S)$ be a lift for α. Then $\tau(\tilde{\alpha}) \circ \Psi$ is an unstable Adams operation with an underlying automorphism ψ. $\square$

5.4. **Lemma.** *Let $\mathcal{F}$ be weakly connected and assume that $H^1(W, T) = 0$ where $W = W(\mathcal{F})$. Then $\mathrm{OutAd}_{\mathrm{fus}}(S) \xrightarrow{\deg} \mathbb{Z}_p^\times/D(\mathcal{F})$ is injective.*

Proof. Consider some $\varphi \in \mathrm{Ad}_{\mathrm{fus}}(S)$ such that $\deg(\varphi) \in \mathcal{D}(\mathcal{F})$. We will show it represents the identity element in $\mathrm{OutAd}_{\mathrm{fus}}(S)$, namely $\varphi \in \mathrm{Aut}_{\mathcal{F}}(S) \cap \mathrm{Ad}(S)$. By the definition of $\mathcal{D}(\mathcal{F})$ in (4.10), there is $\psi \in \mathrm{Aut}_{\mathcal{F}}(S) \cap \mathrm{Ad}(S)$ such that $\deg(\psi) = \deg(\varphi)$. Therefore $\varphi \circ \psi^{-1}$ is a normal Adams automorphism of S of degree 1, namely is induces the identity on T and on S/T. Since $\mathcal{F}$ is weakly connected T is $\mathcal{F}$-centric and we set $G = \mathrm{Aut}_{\mathcal{L}}(T)$. Notice that $W = \mathrm{Aut}_{\mathcal{F}}(T) \cong G/T$. By [JLL, Lemma 2.2] there is an isomorphism $\mathrm{Aut}(G; \mathrm{Id}_T, \mathrm{Id}_{S/T})/\mathrm{Aut}_T(G) \cong H^1(W; T)$ which vanishes by assumption. Therefore $\varphi = c_t \circ \psi$ for some $t \in T$ and in particular $\varphi \in \mathrm{Ad}(S) \cap \mathrm{Aut}_{\mathcal{F}}(S)$. $\square$

Proof of Proposition 1.4. The exact sequence (1) follows from the exact sequence in Proposition 1.2 and Lemma 5.4. Lemma 4.9 together with Lemma 4.11 and the fact that the image of $T \xrightarrow{\tau} \mathrm{Ad}(\mathcal{G})$ is $\mathrm{Ad}(\mathcal{G})_0$ by Theorem 1.3, give rise to the following commutative diagram with short exact rows

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{Aut}_{\mathcal{L}}^{\mathrm{Ad}}(S)/T \arrow[r, "\tau"] \arrow[d, "\deg", "\cong"'] & \mathrm{Ad}(\mathcal{G})/\mathrm{Ad}(\mathcal{G})_0 \arrow[r, "\bar{\gamma}"] \arrow[d, "\deg"] & \mathrm{Ad}^{\mathrm{geom}}(B\mathcal{G}) \arrow[r] \arrow[d, "\deg"] & 1 \\
1 \arrow[r] & D(\mathcal{F}) \arrow[r] & \mathbb{Z}_p^\times \arrow[r] & \mathbb{Z}_p^\times/D(\mathcal{F}) \arrow[r] & 1
\end{tikzcd}$$

The snake lemma together with the exact sequence (1) yield the exact sequence (2). If $p \neq 2$ then by [LL] the group $\varprojlim^1 \mathcal{Z}$ vanishes and therefore the degree maps in (1) and (2) are injective. $\square$

We are led to find conditions under which $H^1(W, T) = 0$. Recall that $U_p \leq \mathbb{Z}_p^\times$ is the group of p-adic units. It acts in the natural way on any discrete p-torus T via central automorphisms.

5.5. **Lemma.** *Let $1 \neq G \leq U_p$ be a subgroup which acts on a discrete p-torus T or rank $r \geq 1$ in the natural way. If $p \neq 2$ then $H^n(G, T) = 0$ for all $n \geq 0$. If $p = 2$ then $H^{2n}(G, T) = (\mathbb{Z}/2)^r$ and $H^{2n+1}(G, T) = 0$ for all $n \geq 0$.*

Proof. If $p \neq 2$ then $U_p \cong C_{p-1}$ and T is p-torsion, hence $H^{*\geq 1}(G, T) = 0$. Choose some $1 \neq \zeta \in G$. Then $\zeta \mod p \neq 1$ so $\zeta - 1$ is an invertible element in $\mathbb{Z}_p$ and therefore it must act without fixed points on T. This show that $H^0(G, T) = 0$.

Now suppose that $p = 2$. Then $G = U_p = C_2$. If $t \in G$ is the non-trivial element, a projective resolution of $\mathbb{Z}$ is given by

$$\cdots \to \mathbb{Z}[G] \xrightarrow{1-t} \mathbb{Z}[G] \xrightarrow{1+t} \mathbb{Z}[G] \xrightarrow{1-t} \mathbb{Z}[G] \to \mathbb{Z}.$$

Applying $\mathrm{Hom}_{\mathbb{Z}G}(-, T)$, one obtains the cochain complex

$$T \xrightarrow{\cdot 2} T \xrightarrow{0} T \xrightarrow{2} T \xrightarrow{0} \cdots$$

whose homology groups are $H^*(G, T)$. The description of $H^*(G, T)$ now follows. $\square$

We recall that $\mathbb{Q}_p = \mathbb{Z}_p \otimes \mathbb{Z}[\frac{1}{p}]$. We obtain a short exact sequence

$$1 \to \mathbb{Z}_p \to \mathbb{Q}_p \to \mathbb{Z}/p^\infty \to 1.$$

We also recall that $\mathrm{Aut}(\mathbb{Z}/p^\infty) \cong \mathbb{Z}_p^\times$ where every $\zeta \in \mathbb{Z}_p$ acts on $\mathbb{Z}/p^k \subseteq \mathbb{Z}/p^\infty$ via multiplication by ζ mod p^k. Using the inclusion $\mathbb{Z}[\frac{1}{p}] \leq \mathbb{Q}_p$ it is easy to check that for any $\zeta \in \mathbb{Z}_p^\times$, multiplication by ζ in $\mathbb{Q}_p$ induces the automorphism ζ on $\mathbb{Z}_p$. Thus the sequence is a short exact sequence of $\mathbb{Z}_p^\times$-modules. More generally, if T is a discrete p-torus of rank $r > 0$, any decomposition $T \cong \oplus_r \mathbb{Z}/p^\infty$ gives rise to a short exact sequence of $\mathbb{Z}_p$-modules

$$1 \to L \to V \to T \to 1 \tag{5.6}$$

where $V = \oplus_r \mathbb{Q}_p$ and $L = \oplus_r \mathbb{Z}_p$. Also, $\mathrm{Aut}(T) \cong \mathrm{GL}_r(\mathbb{Z}_p)$ which acts naturally on L and V, and the sequence becomes a short exact sequence of $\mathrm{GL}_r(\mathbb{Z}_p)$-modules. We notice that different decompositions of T give rise to isomorphic $\mathrm{GL}_r(\mathbb{Z}_p)$-modules V and L.

5.7. **Lemma.** *Let T be a discrete p-torus, and let $G \leq \mathrm{Aut}(T) \cong \mathrm{GL}_r(\mathbb{Z}_p)$ be a finite subgroup. Let V and L be the associated $\mathbb{Q}_p$ representation and the corresponding integral lattice respectively as above. Then*

$$H^i(G, L) \cong \begin{cases} H^{i-1}(G,T) & i > 1, \\ \mathrm{Coker}(H^0(G,V) \to H^0(G,T)) & i = 1 \end{cases}$$

Proof. Since G is finite and V is a rational vector space, $H^i(G, V) = 0$ for all $i > 0$. Thus the claim follows at once from the standard dimension shifting argument (see for instance [Br, III.7]). □

Recall that an element $w \in \mathrm{GL}_r(\mathbb{Q}_p)$ is called a pseudo-reflection if $\mathrm{rk}(w - 1) = 1$, namely w fixes a hyperplane of dimension $r - 1$. A subgroup $W \leq \mathrm{GL}_r(\mathbb{Q}_p)$ is called a pseudo-reflection group if it is generated by pseudo-reflections.

We notice that $\mathrm{Ad}(S) \xrightarrow{\deg} \mathbb{Z}_p^\times \hookrightarrow Z(\mathrm{Aut}(T))$ is a factorisation of the restriction map $\mathrm{Ad}(S) \xrightarrow{\varphi \mapsto \varphi|_T} \mathrm{Aut}(T)$. In light of (4.10), we see that $\mathcal{D}(\mathcal{F})$ can be identified with a subgroup of $Z(\mathrm{Aut}_\mathcal{F}(T))$ which acts in the natural way on T considered as as a subgroup of $U_p \leq \mathbb{Z}_p^\times$.

5.8. **Proposition.** *Let $\mathcal{F}$ be a weakly connected saturated fusion system and let $W = \mathrm{Aut}_\mathcal{F}(T) \leq \mathrm{GL}_r(\mathbb{Z}_p)$ be its Weyl group. Assume that either one of the following conditions holds:*

(a) p is odd and $D(\mathcal{F}) \neq 1$, or
(b) $p = 2$, $D(\mathcal{F}) \neq 1$ and $H^1(W/D(\mathcal{F}), T^{D(\mathcal{F})}) = 0$, or
(c) p is odd and the Weyl group $W = \mathrm{Out}_\mathcal{F}(T)$ is a pseudo-reflection group.

Then $H^1(W,T) = 0$.

Proof. By the remarks above there is a central extension

$$1 \to D(\mathcal{F}) \to W \to W/D(\mathcal{F}) \to 1$$

and $D(\mathcal{F}) \leq U_p$ acts on T in the natural way. The associated Lyndon-Hochschild-Serre spectral sequence takes the form

$$E_2^{i,j} \;=\; H^i\Big(W/D(\mathcal{F})\,,\, H^j(D(\mathcal{F}),T)\Big) \Rightarrow H^{i+j}(W,T).$$

If $p \neq 2$ and $D(\mathcal{F}) \neq 1$ then Lemma 5.5 implies that $H^j(D(\mathcal{F}), T) = 0$ for all $j \geq 0$, and hence that $E^{i,j} = 0$ for all i, j. Thus $H^k(W, T) = 0$ for all $k \geq 0$.

If $p = 2$ and $D(\mathcal{F}) \neq 1$ and $H^1(W/D(\mathcal{F}), T^{D(\mathcal{F})}) = 0$ then $E_2^{i,j} = 0$ for j odd since in this case $H^j(D(\mathcal{F}), T) = 0$ by Lemma 5.5. Hence the only, potentially non-zero contribution to $H^1(W,T)$ come from $H^1(W/D(\mathcal{F}), H^0(D(\mathcal{F}), T)) = H^1(W/D(\mathcal{F}), T^{D(\mathcal{F})}) = 0$ so $H^1(W,T) = 0$.

Finally, suppose that $p \neq 2$ and $W \leq \mathrm{Aut}(T) \cong \mathrm{GL}_r(\mathbb{Z}_p)$ is a pseudo-reflection group. Let L and V be the associated p-adic lattice and $\mathbb{Q}_p$-vector space associated to T as in (5.6). This is a short exact sequence of $\mathbb{Z}_p[W]$-modules. Now, $W \leq \mathrm{GL}_r(\mathbb{Z}_p)$ acts faithfully on L so [A, Theorem 3.3] applies and it follows that $H^2(W, L) = 0$. Hence, $H^1(W,T) = 0$ by Lemma 5.7. □

6. Extensions of categories

In this section we will study extensions of categories and their automorphisms. A large portion of the content of this section is contained in a different form in [Ho].

6.1. **Definition.** Let $\mathcal{C}$ be a small and let $\Phi\colon \mathcal{C} \to \mathbf{Ab}$ be a functor. An *extension* of $\mathcal{C}$ by Φ is a small category $\mathcal{D}$ with the same object set as that of $\mathcal{C}$, together with a functor $\mathcal{D} \xrightarrow{\pi} \mathcal{C}$, and for every $X \in \mathcal{C}$ a "distinguished" monomorphism $\delta_X\colon \Phi(X) \to \mathrm{Aut}_{\mathcal{D}}(X)$, such that the following hold.

1. (1) The functor π is the identity on the objects and is surjective on morphism sets. Furthermore, for each $X, Y \in \mathcal{C}$, the action of $\Phi(Y)$ on $\mathrm{Mor}_{\mathcal{D}}(X, Y)$ via δ_Y by left composition is free, and the projection
$$\mathrm{Mor}_{\mathcal{D}}(X, Y) \to \mathrm{Mor}_{\mathcal{C}}(X, Y).$$
is the quotient map by this action.
2. (2) For any $d \in \mathrm{Mor}_{\mathcal{D}}(X, Y)$ and any $g \in \Phi(X)$ the following square commutes in $\mathcal{D}$.
$$\begin{tikzcd} X \arrow[r, "d"] \arrow[d, "\delta_X(g)"'] & Y \arrow[d, "\delta_Y(\Phi(\pi(d))(g))"] \\ X \arrow[r, "d"] & Y \end{tikzcd}$$

We will write $\mathcal{E} = (\mathcal{D}, \mathcal{C}, \Phi, \pi, \delta)$ for the extension.

To simplify notation throughout, if d is a morphism in $\mathcal{D}$, then $\pi(d) \in \mathcal{C}$ will be denoted by $[d]$. If $g \in \Phi(X)$, we denote $\delta_X(g)$ by $[\![g]\!]$. In addition for any $c \in \mathcal{C}(X, Y)$ we will write $c_*\colon \Phi(X) \to \Phi(Y)$ for the homomorphism $\Phi(c)$. Thus, the relation in Definition 6.1(2) can be written

$$d \circ [\![g]\!] = [\![\Phi([d])(g)]\!] \circ d \quad \text{or simply} \quad d \circ [\![g]\!] = [\![[d]_*(g)]\!] \circ d. \tag{6.2}$$

We remark that the restriction to functors $\Phi\colon \mathcal{C} \to \mathbf{Ab}$ is only made for the sake of simplification. In fact, we could have considered functors into the category of groups in which case we would have recovered Hoff's results [Ho] in full generality.

6.3. **Example.** Let $N \xrightarrow{i} G \xrightarrow{\pi} H$ be an extension of groups with N abelian. Thus, N becomes an H-module. Every group Γ gives rise to a category $\mathcal{B}\Gamma$ with one object whose set of automorphisms is Γ. We then obtain an extension of categories $\mathcal{E} = (\mathcal{B}G, \mathcal{B}H, \Phi, \mathcal{B}\pi, \mathcal{B}i)$ where $\Phi\colon \mathcal{B}H \to \mathbf{Ab}$ is the functor representing the H-module N.

6.4. **Example.** Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local compact group. Let $\mathcal{F}^c$ be the full subcategory of $\mathcal{F}$ of the $\mathcal{F}$-centric subgroups. There is a functor $\mathcal{Z}\colon \mathcal{F}^{c\text{op}} \xrightarrow{P \mapsto Z(P)} \mathbf{Ab}$. Also the distinguished homomorphisms $\delta_P\colon P \to \mathrm{Aut}_{\mathcal{L}}(P)$ restrict to $\delta_P\colon \mathcal{Z}(P) \to \mathrm{Aut}_{\mathcal{L}^{\text{op}}}(P)$. It follows directly from the definitions of linking systems that $\mathcal{L}^{\text{op}}$ is an extension of $\mathcal{F}^{c\text{op}}$ by $\mathcal{Z}$ with structure maps $\mathcal{L}^{\text{op}} \xrightarrow{\pi^{\text{op}}} \mathcal{F}^{c\text{op}}$ and $\delta_P\colon \mathcal{Z}(P) \to \mathrm{Aut}_{\mathcal{L}^{op}}(P)$. Here we used the fact that if Γ is an abelian group then $B\Gamma \cong \mathcal{B}\Gamma^{\text{op}}$ via the identity on objects and morphisms.

The next example is the one that the next sections will build on. Due to its importance we give it as Definition 6.5. Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local compact group. Let P, Q be subgroups of S and suppose that $f\colon P \to Q$ is a homomorphism. Then $f(P_0) \le Q_0$ because $f(P_0)$ is a discrete p-torus.

6.5. **Definition.** Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local compact group. Define the category $\mathcal{L}_{/0}$ as follows. First, $\mathrm{Obj}(\mathcal{L}_{/0}) = \mathrm{Obj}(\mathcal{L})$. For any $P, Q \in \mathcal{L}$ set

$$\mathrm{Mor}_{\mathcal{L}_{/0}}(P, Q) = \mathrm{Mor}_{\mathcal{L}}(P, Q)/\widehat{Q_0}.$$

where $\widehat{Q_0} = \delta_Q(Q_0)$ acts on $\mathrm{Mor}_{\mathcal{L}}(P, Q)$ by postcomposition. We will write $\bar{\varphi}$ for the equivalence class (orbit) of $\varphi \in \mathcal{L}(P, Q)$. We write

$$\pi_{/0}\colon \mathcal{L} \to \mathcal{L}_{/0}$$

for the projection functor.

This defines a category because for any $P \xrightarrow{\alpha} Q \xrightarrow{\beta} R$ and any $t \in Q_0$ and $u \in R_0$ we have

$$\hat{u} \circ \beta \circ \hat{t} \circ \alpha = \hat{u} \cdot \widehat{\pi(\beta)(t)} \circ \beta \circ \alpha$$

and $\pi(\beta)(t) \in R_0$ by the observation above.

Any homomorphism $f \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ restricts to $\varphi \colon P_0 \to Q_0$ of the maximal tori. Also, if $t \in Q_0$ then c_t induces the identity on Q_0 and therefore $\varphi|_{P_0}^{Q_0} = (c_t \circ \varphi)|_{P_0}^{Q_0}$. This justifies the following definition.

6.6. **Definition.** Let $\Phi \colon \mathcal{L}_{/0} \to \mathbf{Ab}$ be the functor which on objects is defined by $\Phi \colon P \mapsto P_0$. For a morphisms $\bar{\varphi} \in \mathcal{L}_{/0}(P, Q)$ set $\Phi(\bar{\varphi}) = \pi(\varphi)|_{P_0}^{Q_0}$.

6.7. **Proposition.** *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group. Then $\mathcal{L}$ is an extension (6.1) of $\mathcal{L}_{/0}$ by the functor $\Phi \colon \mathcal{L}_{/0} \to \mathbf{Ab}$. The structure maps are given by $\pi_{/0} \colon \mathcal{L} \to \mathcal{L}_{/0}$ and $\delta_{/0} \overset{\text{def}}{=} \delta_P|_{P_0} \colon P_0 \to \mathrm{Aut}_{\mathcal{L}}(P)$.*

Proof. The functor $\pi_{/0} \colon \mathcal{L} \to \mathcal{L}_{/0}$ is clearly the identity on objects and is surjective on morphism sets. The group $\delta_Q(Q_0) \leq \mathrm{Aut}_{\mathcal{L}}(Q)$ acts freely on $\mathcal{L}(P, Q)$ because all morphisms in $\mathcal{L}$, in particular the morphisms in $\mathcal{L}(P, Q)$, are epimorphisms by [JLL, Corollary 1.8]. So condition (1) of Definition 6.1 holds. Condition (2) follows from axiom (C) of linking systems. $\square$

6.8. **Definition.** Let $\mathcal{E} = (\mathcal{D}, \mathcal{C}, \Phi, \pi, \delta)$ and $\mathcal{E}' = (\mathcal{D}', \mathcal{C}', \Phi', \pi', \delta')$ be extensions. A morphism $\mathcal{E} \to \mathcal{E}'$ is a functor $\Psi \colon \mathcal{D} \to \mathcal{D}'$ such that there exists a functor $\overline{\Psi} \colon \mathcal{C} \to \mathcal{C}'$ which satisfies $\pi' \circ \Psi = \overline{\Psi} \circ \pi$.

An automorphism of extensions is therefore an isomorphism of categories $\alpha \colon \mathcal{D} \to \mathcal{D}$ such that both α and α^{-1} are morphisms of the extension $\mathcal{E}$. Among these there are the inner automorphisms of the extension $\mathcal{E}$.

6.9. **Definition.** Let $\mathcal{E} = (\mathcal{D}, \mathcal{C}, \Phi, \pi, \delta)$ be an extension. Given a choice of elements $u(X) \in \Phi(X)$ for every $X \in \mathcal{C}$, we obtain an equivalence $\tau_u \colon \mathcal{E} \to \mathcal{E}$ where τ_u is the identity on objects and for any $d \in \mathcal{D}(X, Y)$ we define

$$\tau_u(d) = [\![u(Y)]\!] \circ d \circ [\![u(X)]\!]^{-1}.$$

An automorphism of $\mathcal{E}$ is called *inner* if it is equal to some τ_u. The collection of all the inner automorphisms of $\mathcal{E}$ is denoted $\mathrm{Inn}(\mathcal{E})$.

We remark that the functor $\overline{\Psi} \colon \mathcal{C} \to \mathcal{C}'$ in Definition 6.8, if it exists then it is unique because π and π' are surjective. In addition there is no condition on the functors Φ and Φ' in the definition of morphisms because of Lemma 6.10 below.

6.10. **Lemma.** *Let $\mathcal{E} = (\mathcal{D}, \mathcal{C}, \Phi, \pi, \delta)$ and $\mathcal{E}' = (\mathcal{D}', \mathcal{C}', \Phi', \pi', \delta')$ extensions. Then any morphism $\Psi \colon \mathcal{E} \to \mathcal{E}'$ gives rise to a unique natural transformation*

$$\eta(\Psi) \colon \Phi \to \Phi' \circ \overline{\Psi}$$

which takes an object $X \in \mathcal{C}$ to the homomorphism $\eta_X \colon \Phi(X) \to \Phi' \circ \overline{\Psi}(X)$ which is determined by the formula

$$[\![\eta_X(g)]\!] = \Psi([\![g]\!]), \tag{6.11}$$

for any $g \in \Phi(X)$. (Here η denotes $\eta(\Psi)$.

Proof. For every object $X \in \mathcal{C}$ the functors Ψ and $\overline{\Psi}$ give rise to a morphism of exact sequences

$$\begin{tikzcd}
1 \arrow[r] & \Phi(X) \arrow[r, "\delta_X"] \arrow[d, dotted, "\eta_X"'] & \mathrm{Aut}_{\mathcal{D}}(X) \arrow[r, "\pi"] \arrow[d, "\Psi"] & \mathrm{Aut}_{\mathcal{C}}(X) \arrow[d, "\overline{\Psi}"] \\
1 \arrow[r] & \Phi'(\overline{\Psi}(X)) \arrow[r, "\delta'_{\overline{\Psi}(X)}"] & \mathrm{Aut}_{\mathcal{D}'}(\overline{\Psi}(X)) \arrow[r, "\pi'"] & \mathrm{Aut}_{\mathcal{C}'}(\overline{\Psi}(X))
\end{tikzcd}$$

This defines η_X, which satisfies (6.11) by definition. It remains to prove that the homomorphisms η_X define a natural transformation $\eta\colon \Phi \to \Phi' \circ \overline{\Psi}$. For any $c \in \mathrm{Mor}_{\mathcal{C}}(X,Y)$, we need to show that

$$\begin{tikzcd} \Phi(X) \arrow[r, "\Phi(c)"] \arrow[d, "\eta_X"'] & \Phi(Y) \arrow[d, "\eta_Y"] \\ \Phi'(\overline{\Psi}(X)) \arrow[r, "\Phi'(\overline{\Psi}(c))"'] & \Phi'(\overline{\Psi}(Y)). \end{tikzcd}$$

Choose $d \in \mathcal{D}(X,Y)$ such that $c = [d]$. By (6.11) and (6.2)

$$\Psi(d) \circ [\![\eta_X(g)]\!] = \Psi(d) \circ \Psi([\![g]\!]) = \Psi(d \circ [\![g]\!]) = \Psi([\![\Phi(c)(g)]\!] \circ d) = \\ \Psi([\![\Phi(c)(g)]\!]) \circ \Psi(d) = [\![\eta_Y(\Phi(c)(g))]\!] \circ \Psi(d).$$

On the other hand, by applying (6.2) directly to the left hand side of this equality and noticing that $[\Psi(d)] = \overline{\Psi}(c)$ we get

$$\Psi(d) \circ [\![\eta_X(g)]\!] = [\![\Phi'(\overline{\Psi}(c))(\eta_X(g))]\!] \circ \Psi(d).$$

By comparing the right hand sides of these equalities and using the free action of $\Phi'(Y)$ on $\mathcal{D}'(X,Y)$ where $\Psi(d)$ belongs, we see that $\eta_Y(\Phi(c)(g)) = (\Phi' \circ \overline{\Psi})(c)(\eta_X(g))$ as needed. □

6.12. **Lemma.** *Let $\mathcal{E} = (\mathcal{D}, \mathcal{C}, \Phi, \pi, \delta)$ be an extension. Then any $\Theta \in \mathrm{Inn}(\mathcal{E})$ induces the identity on $\mathcal{C}$ and $\eta(\Theta) = Id$.*

Proof. Using the notation of Definition 6.9 we write $\Theta = \tau_u$. Then τ_u induces the identity on $\mathcal{C}$ because $[[\![u(Y)]\!] \circ \varphi \circ [\![u(X)^{-1}]\!]] = [[\![u(Y) \cdot \varphi_*(u(X)^{-1})]\!] \circ \varphi] = [\varphi]$. This shows that $\overline{\tau_u} = \mathrm{Id}_{\mathcal{C}}$. By Lemma 6.10, for any $X \in \mathcal{C}$ and any $x \in \Phi(X)$ we have $[\![\eta(\tau_u)(x)]\!] = \tau_u([\![x]\!]) = [\![u(X)]\!] \circ [\![x]\!] \circ [\![u(X)^{-1}]\!] = [\![x]\!]$ because $\Phi(X)$ is abelian. This shows that $\eta(\tau_u) = \mathrm{Id}$. □

At this stage it is useful to remark about which functors $\Psi\colon \mathcal{D} \to \mathcal{D}'$ give rise to a morphism of extensions $\mathcal{E} \to \mathcal{E}'$ as in Definition 6.8.

6.13. **Definition.** Let $F\colon \mathbf{Gps} \to \mathbf{Gps}$ be a functor such that $F(G) \leq G$ for any group G and these inclusions induce a natural transformation of functors $F \to \mathrm{Id}$. An extension $\mathcal{E} = (\mathcal{D}, \mathcal{C}, \Phi, \pi, \delta)$ is called *F-rigid* if δ_X induces an isomorphism $\Phi(X) \cong F(\mathrm{Aut}_{\mathcal{D}}(X))$ for every $X \in \mathrm{Obj}(\mathcal{D})$.

Here is an example of a functor F as in the definition above which will play a role in this paper.

6.14. **Definition.** Let $\Lambda\colon \mathbf{Gps} \to \mathbf{Gps}$ be the functor which assigns to every group G the subgroup $\Lambda(G)$ generated by the images of all homomorphisms $\varphi\colon \mathbb{Z}/p^\infty \to G$.

Recall that a group G is called virtually discrete p-toral if it is an extension of a finite group by a discrete p-torus. In this case $\Lambda(G) = G_0$ is the identity component of G and is a discrete p-torus. Hence, the restriction of Λ to the full subcategory of virtually discrete p-toral groups factors through the category **Ab**.

The reason we consider F-rigid extension is that morphisms between them (Definition 6.8) are just functors between the categories. This is the content of the next proposition.

6.15. **Proposition.** *Suppose that $\mathcal{E} = (\mathcal{D}, \mathcal{C}, \Phi, \pi, \delta)$ and $\mathcal{E}' = (\mathcal{D}', \mathcal{C}', \Phi', \pi', \delta')$ are F-rigid extensions. Then any functor $\Psi\colon \mathcal{D} \to \mathcal{D}'$ is a morphism of extensions $\mathcal{E} \to \mathcal{E}'$.*

Proof. For any $C \in \mathrm{Obj}(\mathcal{D})$ the functor Ψ induces a homomorphism $\Psi\colon \mathrm{Aut}_{\mathcal{D}}(C) \to \mathrm{Aut}_{\mathcal{D}'}(\Psi(C))$. By applying F and using the natural transformation $\mathrm{Id} \to F$ we get

$$\Psi(\Phi(C)) = \Psi(F(\mathrm{Aut}_{\mathcal{D}}(C)) \leq F(\mathrm{Aut}_{\mathcal{D}'}(\Psi(C))) = \Phi'(\Psi(C)).$$

In view of this we define $\psi\colon \mathcal{C} \to \mathcal{C}'$ as follows. On objects $\psi\colon C \mapsto \Psi(C)$. Fix $c \in \mathcal{C}(C_0, C_1)$ and choose a lift $d \in \mathcal{D}(C_0, C_1)$. Define $\psi\colon c \mapsto [\Psi(d)]$. Now, ψ is well defined on morphisms since if d' is another lift for c then $d' = [\![x]\!] \circ d$ for some $x \in \Phi(C)$ and therefore

$$[\Psi([\![x]\!] \circ d)] = [\Psi([\![x]\!])] \circ [\Psi(d)] = [\Psi(d)]$$

because $\Psi(\llbracket x \rrbracket \in \Phi'(\Psi(C))$ as we have seen above. The verification that ψ respects identities and compositions is straightforward and the equality $\pi' \circ \Psi = \psi \circ \pi$ holds by the way we defined ψ. $\square$

6.16. **Proposition.** *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local compact group. The extension $\mathcal{E} = (\mathcal{L}, \mathcal{L}_{/0}, \Phi, \pi_{/0}, \delta_{/0})$ in Proposition 6.7 is Λ-rigid (Definitions 6.13, 6.14).*

Proof. It follows from [BLO3, Lemma 2.5] that for every $\mathcal{F}$-centric $P \leq S$ the group $\mathrm{Aut}_{\mathcal{L}}(P)$ is an extension of P_0 by a finite group and therefore $\Lambda(\mathrm{Aut}_{\mathcal{L}}(P)) = P_0$. $\square$

Fix a small category $\mathcal{C}$. An *n-chain* in $\mathcal{C}$ is a sequence $X_0 \xrightarrow{c_0} X_1 \xrightarrow{c_1} \dots \xrightarrow{c_{n-1}} X_n$ of composable morphisms. We write $\mathcal{C}_n$ for the set of n-chains. Now consider a functor $\Phi \colon \mathcal{C} \to \mathbf{Ab}$. Recall that the cobar construction is the cochain complex $C^*(\mathcal{C}, \Phi)$ where

$$C^n(\mathcal{C}, \Phi) = \prod_{X_0 \xrightarrow{c_1} \dots \xrightarrow{c_{n-1}} X_n} \Phi(X_n).$$

We view it as a set of functions $u \colon \mathcal{C}_n \to \coprod_X \Phi(X)$ such that $u(X_0 \xrightarrow{c_0} \dots \xrightarrow{c_{n-1}} X_n) \in \Phi(X_n)$. The differential $\delta \colon C^n(\mathcal{C}, \Phi) \to C^{n+1}(\mathcal{C}, \Phi)$ is defined on the factor $X_0 \xrightarrow{c_0} \dots \xrightarrow{c_n} X_{n+1}$ of the target by

$$\delta(u)(X_\bullet) = \sum_{j=0}^{n} (-1)^j u(\delta_i(X_\bullet)) + (-1)^{n+1} \Phi(c_n)(\delta_{n+1}(X_\bullet))$$

where $\delta_i(X_\bullet)$ is the n-chain obtained by deleting X_i from $X_\bullet$.

The cohomology groups of $C^*(\mathcal{C}, \Phi)$ are isomorphic to $\varprojlim^* \Phi$ [GZ, Appendix II, Section 3]. The following facts are elementary and are left to the reader.

6.17. **Lemma.** *Let $\mathcal{C}^*(C, \Phi)$ be the cochain complex defined above. Then the following hold.*

(a) *Any 1-cocycle $z \in C^1(\mathcal{C}, \Phi)$ satisfies $z(1_X) = 1$ for any $X \in \mathcal{C}$.*
(b) *A 2-cocycle z is called* regular *if $z(1_{X_1}, c) = 1 = z(c, 1_{X_0})$ for any $c \in \mathcal{C}(X_0, X_1)$. Every 2-cocycle $z' \in C^2(\mathcal{C}, \Phi)$ is cohomologous to a regular 2-cocycle z.*

6.18. **Definition.** Let $\mathcal{E} = (\mathcal{D}, \mathcal{C}, \Phi, \pi, \delta)$ be an extension (Definition 6.1). A *section* is a function $\sigma \colon \mathrm{Mor}(\mathcal{C}) \to \mathrm{Mor}(\mathcal{D})$ such that $[\sigma(c)] = c$ for every $c \in \mathrm{Mor}(\mathcal{C})$. We say that a section σ is *regular* if $\sigma(1_X) = 1_X$ for every $X \in \mathcal{C}$.

The following definition is an analogue of the well known construction of the 2-cocycles associated with extensions of groups. Compare with [Ho].

6.19. **Definition.** Let $\mathcal{E} = (\mathcal{D}, \mathcal{C}, \Phi, \pi, \delta)$ be an extension and $\sigma \colon \mathcal{C}_1 \to \mathcal{D}_1$ a regular section. Define a 2-cochain as follows. Given a 2-chain $X_0 \xrightarrow{c_0} X_1 \xrightarrow{c_1} X_2$ notice that $[\sigma(c_1) \circ \sigma(c_0)] = c_1 \circ c_0 = [\sigma(c_1 \circ c_0)]$ and therefore there exists a unique element $z_\sigma(c_1, c_0)$ in $\Phi(X_2)$ such that

$$\sigma(c_1) \circ \sigma(c_0) = \llbracket z_\sigma(c_1, c_0) \rrbracket \circ \sigma(c_1 \circ c_0).$$

The next lemma is analogous the the well known result about the 2-cocycles associated to a given extension of groups. We leave the details to Appendix A.

6.20. **Lemma.** *The 2-cochain z_σ defined above is a regular 2-cocycle. Moreover, if σ' is another regular section then $z_{\sigma'}$ and z_σ are cohomologous.*

This lemma justifies the following definition.

6.21. **Definition.** Let $\mathcal{E} = (\mathcal{D}, \mathcal{C}, \Phi, \pi, \delta)$ be an extension. Let $[\mathcal{D}]$ denote the element of $\mathcal{H}^2(\mathcal{C}, \Phi)$ defined by the 2-cocycle z_σ associated with a section $\sigma \colon \mathcal{C}_1 \to \mathcal{D}_1$.

Here is a simple consequence of the definitions analogous to the statement that an extension of groups is split if and only if the associated 2-cohomology class is trivial. The proof is given in Appendix A.

6.22. **Lemma.** *Let $\mathcal{E} = (\mathcal{D}, \mathcal{C}, \Phi, [-], [\![-]\!])$ be an extension. Then $[\mathcal{D}] = 0$ if and only if there exists a functor $s\colon \mathcal{C} \to \mathcal{D}$ which is a right inverse to $\mathcal{D} \xrightarrow{[-]} \mathcal{C}$.*

The next definition is analogous to that of an equivalence of extensions of groups.

6.23. **Definition.** Let $\mathcal{C}$ be a small category and $\Phi\colon \mathcal{C} \to \mathbf{Ab}$ a functor. Two extensions $\mathcal{E} = (\mathcal{D}, \mathcal{C}, \Phi, \pi, \delta)$ and $\mathcal{E}' = (\mathcal{D}', \mathcal{C}, \Phi, \pi', \delta)$ are called *equivalent* if there exists an isomorphism $\Psi\colon \mathcal{E} \to \mathcal{E}'$ such that $\overline{\Psi} = \mathrm{Id}_{\mathcal{C}}$ and the natural transformation $\eta(\Psi)$ is the identity transformation $\Phi \to \Phi \circ \overline{\Psi} = \Phi$.

This defines an equivalence relation on the class of all extensions of $\mathcal{C}$ by Φ. The equivalence class of an extension $\mathcal{E}$ is denoted $\{\mathcal{E}\}$. Let $\mathrm{Ext}(\mathcal{C}, \Phi)$ denote the collection of equivalence classes of these extensions.

The next result is a special case of the results in [Ho]. It's proof is deferred to Appendix A.

6.24. **Lemma.** *Fix a small category $\mathcal{C}$ and a functor $\Phi\colon \mathcal{C} \to \mathbf{Ab}$. Then there exists a one-to-one correspondence*

$$\Gamma : \quad \mathrm{Ext}(\mathcal{C}, \Phi) \xrightarrow{\{\mathcal{E}\} \mapsto [\mathcal{E}]} H^2(\mathcal{C}, \Phi).$$

Now we deal with constructing morphisms of extensions. The next result should be compared with [JLL, Lemma 2.2(i)]. It's proof is deferred to Appendix A.

6.25. **Proposition.** *Let $\mathcal{E} = (\mathcal{D}, \mathcal{C}, \Phi, \pi, \delta)$ and $\mathcal{E}' = (\mathcal{D}', \mathcal{C}', \Phi', \pi', \delta')$ be extensions. Let $\psi\colon \mathcal{C} \to \mathcal{C}'$ be a functor, and let $\eta\colon \Phi \to \Phi' \circ \psi$ be a natural transformation. Then the following are equivalent.*

(i) There exists a morphism of extensions $\Psi\colon \mathcal{E} \to \mathcal{E}'$ such that $\psi = \overline{\Psi}$ and $\eta = \eta(\Psi)$.
(ii) The homomorphisms in cohomology induced by ψ and η

$$H^2(\mathcal{C}; \Phi) \xrightarrow{\eta_*} H^2(\mathcal{C}; \Phi' \circ \psi) \xleftarrow{\psi^*} H^2(\mathcal{C}', \Phi')$$

satisfy $\eta_*([\mathcal{D}]) = \psi^*([\mathcal{D}'])$.

For the sake of completeness we prove the following result. We will not use it in this paper. Let $\mathrm{Aut}(\mathcal{E}; 1_{\mathcal{C}}, 1_{\Phi})$ denote the group of all automorphisms Ψ of $\mathcal{E}$ such that $\overline{\Psi} = \mathrm{Id}_{\mathcal{C}}$ and $\eta(\Psi) = \mathrm{Id}_{\Phi}$.

6.26. **Proposition.** *Let $\mathcal{E} = (\mathcal{D}, \mathcal{C}, \Phi, \pi, \delta)$ be an extension. Then there is a one-to-one correspondence*

$$\Gamma\colon H^1(\mathcal{C}, \Phi) \to \mathrm{Aut}(\mathcal{E}; 1_{\mathcal{C}}, 1_{\Phi}) / \mathrm{Inn}(\mathcal{E}).$$

7. Special Adams operations

Let $\mathcal{F}$ be a saturated fusion system over S. A set $\mathcal{R}$ of subgroups of S is called an *$\mathcal{F}$-collection* or simply a *collection* if it is closed under conjugacy in $\mathcal{F}$, namely it is the union of isomorphism classes of objects in $\mathcal{F}$. We will write $\mathcal{F}^{\mathcal{R}}$ for the full subcategory of $\mathcal{F}$ with object set $\mathcal{R}$. If $\mathcal{R} \subseteq \mathcal{F}^c$ we let $\mathcal{L}^{\mathcal{R}}$ be the full subcategory of $\mathcal{L}$ on the object set $\mathcal{R}$. In this section we will make use of the structure map δ of a linking system $\mathcal{L}$ as well as the structure map $\delta_{/0}$ of $\mathcal{L}$ regarded as an extension of the associated quotient $\mathcal{L}_{/0}$, see Proposition 6.7. While $\delta_{/0}\colon P_0 \to \mathrm{Aut}_{\mathcal{L}}(P)$ is merely the restriction of $\delta\colon P \to \mathrm{Aut}_{\mathcal{L}}(P)$ to P_0 these morphisms play different roles in their respective contexts. To emphasize this, for $x \in P$ and $t \in P_0$, we will denote $\delta(x)$ by $\widehat{x}$ and $\delta_{/0}(t)$ by $[\![t]\!]$. This is consistent with the notation we have established in previous sections. An equality of the form $[\![x]\!] = \widehat{x}$ for $x \in P_0$ will simply mean that the corresponding elements in $\mathrm{Aut}_{\mathcal{L}}(P)$ coincide. Notice that $[\![x]\!]$ only makes sense when $x \in P_0$ wheras $\widehat{x}$ is defined for any $x \in P$, or indeed, $x \in N_S(P, Q)$. Also, we will use the symbol $[\varphi]$ for image in $\mathcal{L}_{/0}$ of a morphism φ in $\mathcal{L}$, and $\pi(\varphi)$ for the image of that morphism in $\mathcal{F}$.

7.1. **Definition.** Let $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$ be a p-local compact group and let $\mathcal{R}$ be a collection of $\mathcal{F}$-centric subgroups. We say that $(\Psi, \psi) \in \mathrm{Ad}(\mathcal{G})$ is a *special* unstable Adams operation *relative to* $\mathcal{R}$ if there exists a choice of

(a) $\tau_P \in T$ for every $P \in \mathcal{R}$ and
(b) $\tau_\varphi \in Q_0$ for every $\varphi \in \mathcal{L}^{\mathcal{R}}(P, Q)$

such that the following hold

(1) $\psi(P) = \tau_P P \tau_P^{-1}$ for every $P \in \mathcal{R}$ and
(2) $\Psi(\varphi) = \widehat{\tau_Q} \circ \widehat{\tau_\varphi} \circ \varphi \circ \widehat{\tau_P}^{-1}$ for every $\varphi \in \mathcal{L}^{\mathcal{R}}(P,Q)$.

The subset of $\mathrm{Ad}(\mathcal{G})$ of all the special unstable Adams operation relative to $\mathcal{R}$ is denoted $\mathrm{SpAd}(\mathcal{G};\mathcal{R})$.

In [JLL], given a p-local compact group $\mathcal{G} = (S,\mathcal{F},\mathcal{L})$, we find an integer $m \geq 0$ such that every $\zeta \in \Gamma_m(p)$ is the degree of some unstable Adams operation which we construct. The construction, however, involves many choices and the number m is quite mysterious. It turns out that all the unstable Adams operation we constructed in [JLL] are special relative to the collection $\mathcal{H}^\bullet(\mathcal{F}^c)$. We will show this in Appendix B.

Recall that $\mathcal{G}$ gives rise to an extension of categories $\mathcal{E} = (\mathcal{L}^{\mathcal{R}}, \mathcal{L}^{\mathcal{R}}_{/0}, \Phi|_{\mathcal{R}}, \pi_{/0}, \delta)$, as in Definitions 6.5, 6.6 and Proposition 6.7. The main purpose of this section is to relate special unstable Adams operations relative to $\mathcal{R}$ to the extension $\mathcal{E}$. The main result of this section is Proposition 7.2 which gives a conceptual meaning to the integer m above. It also implies that if $H^1(\mathcal{L}^{\mathcal{R}}_{/0};\Phi) = 0$ then special unstable Adams operations relative to $\mathcal{R}$ are determined, modulo $\mathrm{Aut}_T(\mathcal{G})$, by their degree. Here $\mathrm{Aut}_T(\mathcal{G})$ means automorphisms of $\mathcal{G}$ induced by elements of $\widehat{T} = \delta(T) \leq \mathrm{Aut}_{\mathcal{L}}(S)$ (see (4.5)).

7.2. **Proposition.** *Let $\mathcal{G} = (S,\mathcal{F},\mathcal{L})$ be a p-local compact group and $\mathcal{R} \subseteq \mathcal{F}^c$ be a collection which contains $\mathcal{H}^\bullet(\mathcal{F}^c)$. Let $\mathcal{E}$ denote the extension $(\mathcal{L}^{\mathcal{R}}, \mathcal{L}^{\mathcal{R}}_{/0}, \Phi, [-], [\![-]\!])$ and suppose that the order of $[\mathcal{L}^{\mathcal{R}}]$ in $H^2(\mathcal{L}^{\mathcal{R}}_{/0};\Phi)$ is p^m for some $m \geq 0$. Then there is an exact sequence*

$$H^1(\mathcal{L}^{\mathcal{R}}_{/0};\Phi) \to \mathrm{SpAd}(\mathcal{G};\mathcal{R})/\mathrm{Aut}_T(\mathcal{G}) \xrightarrow{\deg} \Gamma_m(p) \to 1.$$

Proposition 7.2 gives a sufficient condition for the existence of special Adams operations. The next statement is a necessary condition.

7.3. **Proposition.** *Let $\mathcal{G} = (S,\mathcal{F},\mathcal{L})$ be a p-local compact group and $\mathcal{R} \subseteq \mathcal{F}^c$ a collection. If $(\Psi,\psi) \in \mathrm{SpAd}(\mathcal{G};\mathcal{R})$ is of degree ζ then $\zeta \cdot [\mathcal{L}^{\mathcal{R}}] = [\mathcal{L}^{\mathcal{R}}]$ in $H^2(\mathcal{L}^{\mathcal{R}}_{/0},\Phi)$.*

Let R be a ring and let ξ be a central element in R. Let $\mathcal{C}$ a small category and $F\colon \mathcal{C} \to R$-**mod** be a functor. Then ξ induces a natural transformation $F \to F$ given by $F(c) \xrightarrow{x \mapsto \xi \cdot x} F(c)$ for every object $c \in \mathcal{C}$ and $x \in F(c)$. We call this natural transformation *multiplication by* ξ and we will denote it by $\xi\colon F \to F$. Proposition 7.3 will follow as a corollary from the next lemma.

7.4. **Lemma.** *Let $\mathcal{G} = (S,\mathcal{F},\mathcal{L})$ be a p-local compact group and $\mathcal{R}$ a collection. Let (Ψ,ψ) be a special unstable Adams operation relative to $\mathcal{R}$ of degree ζ. Let $\mathcal{E}$ denote the extension $(\mathcal{L}^{\mathcal{R}}, \mathcal{L}^{\mathcal{R}}_{/0}, \Phi, [-], [\![-]\!])$. Then Ψ restricts to a functor $\Psi|_{\mathcal{R}}\colon \mathcal{L}^{\mathcal{R}} \to \mathcal{L}^{\mathcal{R}}$. This gives rise to a homomorphism*

$$\mathrm{Res}\ :\ \mathrm{SpAd}(\mathcal{G};\mathcal{R}) \xrightarrow{(\Psi,\psi)\mapsto\Psi|_{\mathcal{R}}} \mathrm{Aut}(\mathcal{E}).$$

As an automorphism of the extension $\mathcal{E}$ (see Definition 6.8), $\Psi|_{\mathcal{R}}\colon \mathcal{L}^{\mathcal{R}} \to \mathcal{L}^{\mathcal{R}}$ has the following properties.

(1) $\Phi \circ \overline{\Psi} = \Phi$,
(2) $\eta(\Psi)\colon \Phi \to \Phi \circ \overline{\Psi} = \Phi$ is multiplication by ζ (see Lemma 6.10) , and
(3) $\overline{\Psi}^*([\mathcal{L}^{\mathcal{R}}]) = [\mathcal{L}^{\mathcal{R}}]$ in $H^2(\mathcal{L}^{\mathcal{R}}_{/0},\Phi)$.

Proof. Since $\mathcal{R}$ is closed under $\mathcal{F}$-conjugation, it is invariant under conjugation by T. Since Ψ is special, $\Psi(P) = \psi(P)$ is a T-conjugate of P for any $P \in \mathcal{R}$ and therefore Ψ restricts to a functor on the object set $\mathcal{R} \subseteq \mathrm{Obj}(\mathcal{L})$ and hence gives a functor $\Psi|_{\mathcal{R}}\colon \mathcal{L}^{\mathcal{R}} \to \mathcal{L}^{\mathcal{R}}$ whose inverse is $\Psi^{-1}|_{\mathcal{R}}$. Since the extension $\mathcal{E}$ is Λ-rigid by Proposition 6.16, it follows that $\Psi|_R$ is an automorphism of the extension $\mathcal{E}$. It is clear that the assignment $\mathrm{Res}\colon \Psi \mapsto \Psi|_{\mathcal{R}}$ is a homomorphism (since composition of fucntors is the group operation).

Now consider some $(\Psi,\psi) \in \mathrm{SpAd}(\mathcal{G};\mathcal{R})$. By definition of the functor $\overline{\Psi}\colon \mathcal{L}^{\mathcal{R}}_{/0} \to \mathcal{L}^{\mathcal{R}}_{/0}$ and since (Ψ,ψ) is an Adams operation, for any $P \in \mathcal{R}$

$$\overline{\Psi}(P) = [\Psi(P)] = \psi(P).$$

Suppose that $P, Q \in \mathcal{R}$ and that $\varphi \in \mathcal{L}(P,Q)$ and consider its image $[\varphi] \in \mathcal{L}^{\mathcal{R}}_{/0}(P,Q)$. Then by definition of $\overline{\Psi}$

$$\overline{\Psi}([\varphi]) = [\Psi(\varphi)].$$

Proof of (1). Let $\pi \colon \mathcal{L} \to \mathcal{F}$ denote the projection. Since ψ is an Adams automorphism of S then $\psi|_T$ is multiplication by ζ, hence it leaves any subgroup of T invariant. Also note that if $P \in \mathcal{R}$ then P_0 is a characteristic subgroup of P. This shows that

$$\Phi(\overline{\Psi}(P)) = \overline{\Psi}(P))_0 = \psi(P)_0 = \psi(P_0) = P_0 = \Phi(P).$$

So $\Phi \circ \overline{\Psi}$ and Φ attain the same values on objects. Now suppose that $P \xrightarrow{[\varphi]} Q$ is a morphism in $\mathcal{L}_{/0}$ where $P, Q \in \mathcal{R}$ and $\varphi \in \mathcal{L}(P,Q)$. Notice that $\pi(\varphi)(P_0)$ is a discrete p-torus and it is therefore a subgroup of T. Since (Ψ, ψ) is a special unstable Adams operation relative to $\mathcal{R}$,

$$\begin{aligned}\Phi(\overline{\Psi}([\varphi])) &\overset{(6.8)}{=} \Phi([\Psi(\varphi)]) \overset{(7.1)}{=} \Phi([\widehat{\tau_Q} \circ \widehat{\tau_\varphi} \circ \varphi \circ \widehat{\tau_P}^{-1}]) \\ &\overset{(6.6)}{=} c_{\tau_Q} \circ c_{\tau_\varphi} \circ \pi(\varphi) \circ c_{\tau_P}{}^{-1}|_{P_0} = \pi(\varphi)|_{P_0} \overset{(6.6)}{=} \Phi([\varphi]),\end{aligned}$$

where each equality follows from the definition indicated above it, and the fourth equality holds since T is abelian. This shows that $\Phi \circ \overline{\Psi} = \Phi$.

Proof of (2). Fix some $P \in \mathcal{R}$ and $x \in \Phi(P_0) = P_0$. Then, since $\psi|_T$ is multiplication by ζ, one has

$$[\![\eta(\Psi)(x)]\!] = \Psi([\![x]\!]) = \Psi(\widehat{x}) = \widehat{\psi(x)} = \widehat{\zeta \cdot x} = [\![\zeta \cdot x]\!],$$

where the first quality follows from Lemma 6.10, and the third from Definition 4.2. Thus $\eta(\Psi)$ is multiplication by $\zeta \in \mathbb{Z}_p$.

Proof of (3). Let $\{\tau_P\}_{P \in \mathcal{R}}$ and $\{\tau_\varphi\}_{\varphi \in \mathrm{Mor}(\mathcal{L}^{\mathcal{R}})}$ be as in Definition 7.1. Choose a regular section

$$\sigma \colon \mathrm{Mor}(\mathcal{L}^{\mathcal{R}}_{/0}) \to \mathrm{Mor}(\mathcal{L}^{\mathcal{R}}).$$

Define a 1-cochain $t \in C^1(\mathcal{L}^{\mathcal{R}}_{/0}, \Phi)$ by setting

$$t(c) \overset{\text{def}}{=} \tau_{\sigma(c)}, \qquad c \in \mathrm{Mor}(\mathcal{L}^{\mathcal{R}}_{/0}).$$

Observe that for every $c \in \mathcal{L}^{\mathcal{R}}_{/0}(P,Q)$,

$$[\Psi(\sigma(c))] = \overline{\Psi}([\sigma(c)]) = \overline{\Psi}(c)$$

and therefore there exists a unique element $v(c) \in Q_0$ such that

$$\sigma(\overline{\Psi}(c)) = [\![v(c)]\!] \circ \Psi(\sigma(c)). \tag{7.5}$$

We obtain a 1-cochain $v \in C^1(\mathcal{L}^{\mathcal{R}}_{/0}, \Phi)$. Recall that we use multiplicative notation for the group operation in Φ and hence in the cochain complex $C^*(\mathcal{L}^{\mathcal{R}}_{/0}, \Phi)$. Set

$$u = v \cdot t \in C^1(\mathcal{L}^{\mathcal{R}}_{/0}, \Phi).$$

Since Ψ is special relative to $\mathcal{R}$, for any $c \in \mathcal{L}^{\mathcal{R}}_{/0}(P,Q)$,

$$\Psi(\sigma(c)) = \widehat{\tau_Q} \circ [\![t(c)]\!] \circ \sigma(c) \circ \widehat{\tau_P}^{-1}.$$

Together with (7.5) and the commutativity of T we obtain

$$\sigma(\overline{\Psi}(c)) = [\![v(c)]\!] \circ \Psi(\sigma(c)) = \widehat{\tau_Q} \circ [\![u(c)]\!] \circ \sigma(c) \circ \widehat{\tau_P}^{-1}. \tag{7.6}$$

Let $P \xrightarrow{c_0} Q \xrightarrow{c_1} R$ be a 2-chain in $\mathcal{L}^{\mathcal{R}}_{/0}$. By definition of z_σ (Definition 6.19) and the functoriality of $\overline{\Psi}$,

$$[\![z_\sigma(\overline{\Psi}(c_1), \overline{\Psi}(c_0))]\!] \circ \sigma(\overline{\Psi}(c_1 \circ c_0)) = \sigma(\overline{\Psi}(c_1)) \circ \sigma(\overline{\Psi}(c_0)). \tag{7.7}$$

Thus we obtain the following sequence of equalities:

$$\begin{aligned}
&[\![z_\sigma\Big(\overline{\Psi}(c_1),\overline{\Psi}(c_0)\Big)]\!] \circ \widehat{\tau_R} \circ [\![u(c_1\circ c_0)]\!] \circ \sigma(c_1\circ c_0) \circ \widehat{\tau_P}^{-1} = \\
&\quad \widehat{\tau_R} \circ [\![u(c_1)]\!] \circ \sigma(c_1) \circ \widehat{\tau_Q}^{-1} \circ \widehat{\tau_Q} \circ [\![u(c_0)]\!] \circ \sigma(c_0) \circ \widehat{\tau_P}^{-1} = \\
&\quad \widehat{\tau_R} \circ [\![u(c_1)]\!] \circ \sigma(c_1) \circ [\![u(c_0)]\!] \circ \sigma(c_0) \circ \widehat{\tau_P}^{-1} = \\
&\quad \widehat{\tau_R} \circ [\![u(c_1)]\!] \circ [\![\Phi(c_1)(u(c_0))]\!] \circ \sigma(c_1) \circ \sigma(c_0) \circ \widehat{\tau_P}^{-1} = \\
&\quad \widehat{\tau_R} \circ [\![u(c_1)\cdot\Phi(c_1)(u(c_0))]\!] \circ [\![z_\sigma(c_1,c_0)]\!] \circ \sigma(c_1\circ c_0) \circ \widehat{\tau_P}^{-1},
\end{aligned}$$

where the first equality follows from (7.6) applied to both sides of (7.7), the third by Definition (6.6) and Axiom (C) of linking systems, and the fourth by Definition (6.19).

Now, $\widehat{\tau_P}\colon P \to \psi(P)$ and $\widehat{\tau_R}\colon R \to \psi(R)$ are both isomorphisms, and $\sigma(c_1 \circ c_0)$ is an epimorphism in $\mathcal{L}$ by [JLL, Corollary 1.8]. Hence,

$$[\![z_\sigma\Big(\overline{\Psi}(c_1),\overline{\Psi}(c_0)\Big)\cdot u(c_1\circ c_0)]\!] = [\![u(c_1)\cdot\Phi(c_1)(u(c_0))\cdot z_\sigma(c_1,c_0)]\!],$$

and so we deduce that

$$z_\sigma(\overline{\Psi}(c_1),\overline{\Psi}(c_0)) = z_\sigma(c_1,c_0)\cdot u(c_1)\cdot u(c_1\circ c_0)^{-1}\cdot\Phi(c_1)(u(c_0)) = z_\sigma(c_1,c_0)\cdot\delta(u)(c_1,c_0),$$

where δ is the differential in $C^*(\mathcal{L}^{\mathcal{R}}_{/0},\Phi)$. This shows that z_σ and $\overline{\Psi}^*(z_\sigma)$ are cohomologous. □

Proof of Proposition 7.3. Proposition 6.25 and Lemma 7.4 show that $[\mathcal{L}^{\mathcal{R}}] = \overline{\Psi}^*([\mathcal{L}^{\mathcal{R}}]) = \eta(\Psi)_*([\mathcal{L}^{\mathcal{R}}]) = \zeta\cdot[\mathcal{L}^{\mathcal{R}}]$. □

Next we turn to Proposition 7.2. This require some preparation.

7.8. **Lemma** ([JLL, Proposition 1.14]). *Let $\mathcal{G} = (S,\mathcal{F},\mathcal{L})$ be a p-local compact group and let $\mathcal{R}$ be a collection which contains $\mathcal{H}^\bullet(\mathcal{F})$. Let $\psi\colon S\to S$ be a fusion preserving Adams automorphism. Then any functor $\Psi'\colon \mathcal{L}^{\mathcal{R}} \to \mathcal{L}^{\mathcal{R}}$ which covers ψ in the sense of Definition 4.2, extends uniquely to an unstable Adams operation (Ψ,ψ).*

7.9. **Definition.** Let $\mathcal{G} = (S,\mathcal{F},\mathcal{L})$ be a p-local finite group and $\mathcal{R}\subseteq\mathcal{F}^c$ a collection. Let $\mathcal{E}$ denote the extension $(\mathcal{L}^{\mathcal{R}},\mathcal{L}^{\mathcal{R}}_{/0},\Phi,[-],[\![-]\!])$. Let $Z\subseteq\mathbb{Z}_p^\times$ be a subgroup. Let

$$\mathrm{Aut}(\mathcal{E};\mathrm{Id}_{\mathcal{L}^{\mathcal{R}}_{/0}},Z)\le\mathrm{Aut}(\mathcal{E})$$

denote the subgroup of the automorphisms Θ, such that $\overline{\Theta} = \mathrm{Id}_{\mathcal{L}^{\mathcal{R}}_{/0}}$ and $\eta(\Theta)$ is multiplication by some $\zeta\in Z$.

Observe that $\mathrm{Aut}(\mathcal{E};\mathrm{Id}_{\mathcal{L}^{\mathcal{R}}_{/0}},Z)$ is indeed a subgroup, since for any $P\in\mathcal{R}$ and any $x\in P_0$, the definition of $\eta(-)$ implies

$$[\![\eta(\Theta_1\circ\Theta_2)(x)]\!] = (\Theta_1\circ\Theta_2)([\![x]\!]) = \Theta_1([\![\eta(\Theta_2)(x)]\!]) = [\![\eta(\Theta_1)(\eta(\Theta_2)(x))]\!],$$

and so $\eta(\Theta_1\circ\Theta_2) = \eta(\Theta_1)\circ\eta(\Theta_2)$. Notice also that for any $\Theta\in\mathrm{Aut}(\mathcal{E};\mathrm{Id}_{\mathcal{L}^{\mathcal{R}}_{/0}},\mathbb{Z}_p^\times)$ and any $\varphi\in\mathcal{L}^{\mathcal{R}}(P,Q)$ one has $[\Theta(\varphi)] = [\varphi]$.

7.10. **Proposition.** *Let $\mathcal{G} = (S,\mathcal{F},\mathcal{L})$ be a p-local compact group and $\mathcal{R}$ be a collection which contains $\mathcal{H}^\bullet(\mathcal{F}^c)$. Let $\mathcal{E}$ denote the extension $(\mathcal{L}^{\mathcal{R}},\mathcal{L}^{\mathcal{R}}_{/0},\Phi,[-],[\![-]\!])$. Then*

(i) There exists a homomorphism

$$\mathrm{Aut}(\mathcal{E};Id_{\mathcal{L}^{\mathcal{R}}_{/0}},\mathbb{Z}_p^\times)\xrightarrow{\rho}\mathrm{SpAd}(\mathcal{G};\mathcal{R})$$

such that $\deg(\rho(\Theta)) = \eta(\Theta)\in\mathbb{Z}_p^\times$ (compare Definition 7.9).

(ii) Moreover, composition of ρ with the quotient by $\mathrm{Inn}_T(\mathcal{G})$ gives a surjective homomorphism

$$\mathrm{Aut}(\mathcal{E};Id_{\mathcal{L}^{\mathcal{R}}_{/0}},\mathbb{Z}_p^\times)\xrightarrow{\bar\rho}\mathrm{SpAd}(\mathcal{G};\mathcal{R})/\,\mathrm{Inn}_T(\mathcal{G})$$

whose kernel contains $\mathrm{Inn}(\mathcal{E})$, (compare Definition 6.9 and Lemma 6.12).

Proof. For any $\Theta \in \mathrm{Aut}(\mathcal{E}; \mathrm{Id}_{\mathcal{L}^{\mathcal{R}}_{/0}}, \mathbb{Z}_p^\times)$, fix the following elements:

(i) For any $\varphi \in \mathcal{L}^{\mathcal{R}}(P,Q)$ let $\tau_\varphi(\Theta)$ be the unique element of Q_0 such that

$$\Theta(\varphi) = [\![\tau_\varphi(\Theta)]\!] \circ \varphi.$$

(ii) For every $P \in \mathcal{R}$ set $\tau_P(\Theta) \overset{\mathrm{def}}{=} \tau_{\iota_P^S}(\Theta)$.

Notice that $\tau_P(\Theta) \in T$ for all $P \in \mathcal{R}$. Consider $\Theta \in \mathrm{Aut}(\mathcal{E}; \mathrm{Id}_{\mathcal{L}^{\mathcal{R}}_{/0}}, \mathbb{Z}_p^\times)$ such that $\eta(\Theta) = \zeta$. Set $\tau_P = \tau_P(\Theta)$ and $\tau_\varphi = \tau_\varphi(\Theta)$ for short, where $\tau_P(\Theta)$ and $\tau_\varphi(\Theta)$ are as in (i) and (ii) above. Notice that $\tau_S = 1$ since $\iota_S^S = \mathrm{id}_S$ in $\mathcal{L}$.

For any $x \in P$, where $P \in \mathcal{R}$, consider $\widehat{x} \in \mathrm{Aut}_{\mathcal{L}}(P)$. Then $[\Theta(\widehat{x})] = \overline{\Theta}([\widehat{x}]) = [\widehat{x}]$, so $\Theta(\widehat{x}) \in \widehat{P} \leq \mathrm{Aut}_{\mathcal{L}}(P)$. By identifying P with $\widehat{P}$ via δ_P, this shows that for every $P \in \mathcal{R}$, the functor Θ induces an automorphism $\theta_P \in \mathrm{Aut}(P)$ by the equation

$$\widehat{\theta_P(x)} = \Theta(\widehat{x}), \qquad (\forall x \in P).$$

Notice that $S \in \mathcal{R}$, and we set

$$\psi \overset{\mathrm{def}}{=} \theta_S.$$

Consider some $P \in \mathcal{R}$ and $x \in P$. By applying Θ to the equality $\iota_P^S \circ \delta_P(x) = \delta_S(x) \circ \iota_P^S$ we obtain $\widehat{\tau_P} \circ \widehat{\theta_P(x)} = \widehat{\psi(x)} \circ \widehat{\tau_P}$ in $\mathcal{L}$. Therefore

$$\psi(x) = (c_{\tau_P} \circ \theta_P)(x), \qquad (x \in P). \tag{7.11}$$

In particular it follows that

$$\tau_P \cdot P \cdot \tau_P^{-1} = \psi(P),$$

i.e, $\tau_P \in N_T(P, \psi(P))$.

Step 1. ψ is a normal Adams automorphism of degree ζ. Notice first that for any $x \in T \subseteq S$,

$$\widehat{\psi(x)} = \Theta(\widehat{x}) = \Theta([\![x]\!]) \overset{(6.2)}{=} [\![\eta(\Theta)_S(x)]\!] = [\![\zeta \cdot x]\!] = \widehat{\zeta \cdot x}.$$

Therefore $\psi|_T$ is multiplication by ζ. Also, the image of S in $\mathrm{Aut}_{\mathcal{L}_{/0}}(S)$ is exactly S/T and since $\overline{\Theta} = \mathrm{Id}$, it follows that ψ induces the identity on S/T. Hence ψ is a normal Adams automorphism, as claimed.

Step 2. ψ is fusion preserving. Since $\mathcal{R} \supseteq \mathcal{F}^\bullet$, it controls fusion. Therefore, it is enough to show that for any $P \in \mathcal{R}$ and any $f \in \mathrm{Hom}_{\mathcal{F}}(P,S)$ there exists $g \in \mathrm{Hom}_{\mathcal{F}}(\psi(P), S)$ such that $\psi \circ f = g \circ \psi|_P^{\psi(P)}$. Let $\varphi \in \mathcal{L}(P,S)$ be a lift for f. By axiom (C) of linking systems, for every $x \in P$ we have $\varphi \circ \widehat{x} = \widehat{f(x)} \circ \varphi$. By applying Θ,

$$\Theta(\varphi) \circ \widehat{\theta_P(x)} = \widehat{\psi(f(x))} \circ \Theta(\varphi). \tag{7.12}$$

Set $f' = \pi(\Theta(\varphi))$. By Axiom (C) and (7.11),

$$\Theta(\varphi) \circ \widehat{\theta_P(x)} = f'(\theta_P(x)) \circ \Theta(\varphi) = \widehat{f'(c_{\tau_P}{}^{-1}(\psi(x)))} \circ \Theta(\varphi). \tag{7.13}$$

Comparing the right hand sides of (7.12) and (7.13), and using the fact that $\Theta(\varphi)$ is an epimorphism in $\mathcal{L}$ by [JLL, Corollary 1.8],

$$\psi(f(x)) = f'(c_{\tau_P}{}^{-1}(\psi(x)))$$

for every $x \in P$. Set $g = f' \circ c_{\tau_P}{}^{-1}|_{\psi(P)}^P$. Then $g \in \mathrm{Hom}_{\mathcal{F}}(\psi(P), S)$ and $\psi \circ f = g \circ \psi|_P^{\psi(P)}$. This shows that ψ is fusion preserving and completes Step 2.

Define a functor $\Psi\colon \mathcal{L}^{\mathcal{R}} \to \mathcal{L}^{\mathcal{R}}$, on objects $P, Q \in \mathcal{R}$ and morphisms $\varphi \in \mathcal{L}^{\mathcal{R}}(P,Q)$, by

$$\Psi(P) \overset{\mathrm{def}}{=} \psi(P) \tag{7.14}$$

$$\Psi(\varphi) \overset{\mathrm{def}}{=} \widehat{\tau_Q} \circ \Theta(\varphi) \circ \widehat{\tau_P}^{-1}, \tag{7.15}$$

where τ_P and τ_Q are considered here as elements of $N_T(P, \psi(P))$ and $N_T(Q, \psi(Q))$ respectively. The functoriality of Ψ is clear from that of Θ. In fact, the morphisms $\widehat{\tau_P} \in \mathcal{L}(P, \psi(P))$ give a natural isomorphism

$$\widehat{\tau}\colon \Theta \to \Psi.$$

Step 3. Ψ covers ψ (see Definition 4.2). First, we show that $\pi \circ \Psi = \psi_* \circ \pi$. The functors on both sides agree on objects, by definition of Ψ and ψ_*, and since the projection π is the identity on objects. Let $\varphi \in \mathcal{L}^{\mathcal{R}}(P, Q)$ be a morphism. Consider the squares:

$$\begin{tikzcd} P \arrow[r, "\varphi"] \arrow[d, "\widehat{x}"'] & Q \arrow[d, "\widehat{\pi(\varphi)(x)}"] \\ P \arrow[r, "\varphi"'] & Q \end{tikzcd} \qquad \begin{tikzcd} P \arrow[r, "\Theta(\varphi)"] \arrow[d, "\widehat{\theta_P(x)}"'] & Q \arrow[d, "\theta_Q(\widehat{\pi(\varphi)(x)})"] \\ P \arrow[r, "\Theta(\varphi)"'] & Q \end{tikzcd}$$

The left square commutes by Axiom (C), and the right one is obtained from the left by applying Θ. Since $\Theta(\varphi)$ is an epimorphism in $\mathcal{L}$, Axiom (C) applied to the right square gives

$$\theta_Q(\pi(\varphi)(x)) = \pi\big(\Theta(\varphi)\big)\,(\theta_P(x)). \tag{7.16}$$

Therefore, for any $x \in P$

$$\pi\big(\Psi(\varphi)\big)(\psi(x)) = \pi\big(\widehat{\tau_Q} \circ \Theta(\varphi) \circ \widehat{\tau_P}^{-1}\big)(\psi(x)) = (c_{\tau_Q} \circ \pi(\Theta(\varphi)) \circ c_{\tau_P}^{-1})(\psi(x)) \overset{(7.11)}{=}$$
$$(c_{\tau_Q} \circ \pi(\Theta(\varphi)))(\theta_P(x)) \overset{(7.16)}{=} c_{\tau_Q}(\theta_Q(\pi(\varphi)(x))) \overset{(7.11)}{=} \psi(\pi(\varphi)(x))$$

Thus, $\pi(\Psi(\varphi)) \circ \psi|_P = \psi \circ \pi(\varphi)$ and consequently $(\pi \circ \Psi)(\varphi) = (\psi_* \circ \pi)(\varphi)$ as claimed.

Next we show that for any $P, Q \in \mathcal{R}$ and $g \in N_S(P, Q)$ we have $\Psi(\widehat{g}) = \widehat{\psi(g)}$. First, notice that $\psi(g) \in N_S(\psi(P), \psi(Q))$. Next,

$$\begin{aligned}
\iota^S_{\psi(Q)} \circ \Psi(\widehat{g}) &= && \text{by definition of } \Psi \text{ (7.15)} \\
\iota^S_{\psi(Q)} \circ \widehat{\tau_Q} \circ \Theta(\widehat{g}) \circ \widehat{\tau_P}^{-1} &= && \text{by definition of } \tau_Q \\
\Theta(\iota^S_Q) \circ \Theta(\widehat{g}) \circ \widehat{\tau_P}^{-1} &= && \\
\Theta(\iota^S_Q \circ \widehat{g}) \circ \widehat{\tau_P}^{-1} &= && \\
\Theta(\widehat{g} \circ \iota^S_P) \circ \widehat{\tau_P}^{-1} &= && \\
\Theta(\widehat{g}) \circ \Theta(\iota^S_P) \circ \widehat{\tau_P}^{-1} &= && \text{by definition of } \theta_S \\
\widehat{\theta_S(g)} \circ \Theta(\iota^S_P) \circ \widehat{\tau_P}^{-1} &= && \text{by definition of } \tau_P \\
\widehat{\theta_S(g)} \circ \iota^S_{\psi(P)} &= && \\
\iota^S_{\psi(Q)} \circ \widehat{\psi(g)}. & &&
\end{aligned}$$

Since $\iota^S_{\psi(Q)}$ is a monomorphism in $\mathcal{L}$ it follows that $\Psi(\widehat{g}) = \widehat{\psi(g)}$. This shows that Ψ covers ψ and completes the proof of the claim.

Step 4. Proof of (i). By Lemma 7.8, Ψ extends to an unstable Adams operation $(\rho(\Theta), \psi)$. We retain the notation $\Psi = \rho(\Theta)$ for convenience. Then Ψ is special relative to $\mathcal{R}$ because for any $P \in \mathcal{R}$,

$$\psi(P) = c_{\tau_P}(\theta_P(P)) = c_{\tau_P}(P) = \tau_P P \tau_P^{-1},$$

where the first and second equality follow from (7.11), and for any $\varphi \in \mathcal{L}^{\mathcal{R}}(P, Q)$,

$$\Psi(\varphi) = \widehat{\tau_Q} \circ \Theta(\varphi) \circ \widehat{\tau_P}^{-1} = \widehat{\tau_Q} \circ [\![\tau_\varphi]\!] \circ \varphi \circ \widehat{\tau_P}^{-1},$$

where the second equality follows from the definition of τ_φ and (7.15). It has degree ζ because for any $x \in T$,

$$[\![\psi(x)]\!] = [\![\theta_S(x)]\!] = \Theta([\![x]\!]) = \eta(\Theta)([\![x]\!]) = \zeta \cdot [\![x]\!].$$

It remains to prove that ρ is a homomorphism. Choose $\Theta, \Theta' \in \mathrm{Aut}(\mathcal{E}; \mathrm{Id}_{\mathcal{L}/0}, \mathbb{Z}_p^\times)$ and set $\Theta'' = \Theta' \circ \Theta$. Let Ψ, Ψ', Ψ'' denote their images in $\mathrm{Ad}(\mathcal{G}; \mathcal{R})$ by ρ. We need to show that $\Psi'' = \Psi' \circ \Psi$. Keeping the notation above, if $P \in \mathcal{R}$ then it is clear that $\theta''_S = \theta'_S \circ \theta_S$, namely $\psi'' = \psi' \circ \psi$ and therefore

$$\Psi''(P) = \psi''(P) = \psi'(\psi(P)) = (\Psi' \circ \Psi)(P).$$

Hence Ψ'' and $\Psi' \circ \Psi$ agree on objects. Moreover, observe that by the definition of the elements $\tau_P, \tau'_P, \tau''_P$ of T

$$\iota^S_{\psi''(P)} \circ \widehat{\tau'_{\psi(P)}} \circ \Theta'(\widehat{\tau_P}) = \Theta'(\iota^S_{\psi(P)}) \circ \Theta'(\widehat{\tau_P}) = \Theta'(\iota^S_{\psi(P)} \circ \widehat{\tau_P}) = \Theta'(\Theta(\iota^S_P)) = \Theta''(\iota^S_P) = \iota^S_{\psi''(P)} \circ \widehat{\tau''_P}.$$

Since $\iota^S_{\psi''(P)}$ is a monomorphism in $\mathcal{L}$ it follows that

$$\widehat{\tau_{\psi(P)}} \circ \Theta'(\widehat{\tau_P}) = \widehat{\tau''_P}.$$

Now suppose that $P, Q \in \mathcal{R}$ and that $\varphi \in \mathcal{L}(P, Q)$. Then

$$\Psi'(\Psi(\varphi)) = \Psi'(\widehat{\tau_Q} \circ \Theta(\varphi) \circ \widehat{\tau_P}^{-1}) = \widehat{\tau'_{\psi(Q)}} \circ \Theta'(\widehat{\tau_Q} \circ \Theta(\varphi) \circ \widehat{\tau_P}^{-1}) \circ \widehat{\tau'_{\psi(P)}}^{-1} =$$
$$\widehat{\tau'_{\psi(Q)}} \circ \Theta'(\widehat{\tau_Q}) \circ \Theta'(\Theta(\varphi)) \circ \Theta'(\widehat{\tau_P}^{-1}) \circ \widehat{\tau'_{\psi(P)}}^{-1} = \widehat{\tau''_Q} \circ \Theta'(\Theta(\varphi)) \circ \widehat{\tau''_P}^{-1} = \Psi''(\varphi).$$

This shows that Ψ'' and $\Psi' \circ \Psi$ agree on morphisms and completes the proof of (i).

Step 5. Proof of (ii). Suppose that $(\Psi, \psi) \in \mathrm{SpAd}(\mathcal{G}; \mathcal{R})$ has degree ζ. For each $P \in \mathcal{R}$ and each $\varphi \in \mathrm{Mor}(\mathcal{L}^{\mathcal{R}})$, fix elements τ_P and τ_φ, as in Definition 7.1. Define a functor $\Theta \colon \mathcal{L}^{\mathcal{R}} \to \mathcal{L}^{\mathcal{R}}$ by

$$\Theta(P) = P, \qquad (P \in \mathcal{R})$$
$$\Theta(\varphi) = \widehat{\tau_Q}^{-1} \circ \Psi(\varphi) \circ \widehat{\tau_P}, \qquad (\varphi \in \mathcal{L}^{\mathcal{R}}(P, Q)).$$

The functoriality of Θ is clear, and it is a morphism of extensions by Propositions 6.15 and 6.16. Since (Ψ, ψ) is special relative to $\mathcal{R}$,

$$[\Theta(\varphi)] = [\widehat{\tau_Q}^{-1}] \circ [\widehat{\tau_Q} \circ \widehat{\tau_\varphi} \circ \varphi \circ \widehat{\tau_P}^{-1}] \circ [\widehat{\tau_P}] = [\varphi],$$

and so $\overline{\Theta} = \mathrm{Id}_{\mathcal{L}^{\mathcal{R}}_{/0}}$. Also, given $P \in \mathcal{R}$ and $g \in P_0$, since T is abelian we get

$$\Theta([\![g]\!]) = \Theta(\widehat{g}) = \widehat{\tau_P}^{-1} \circ \Psi(\widehat{g}) \circ \widehat{\tau_P} = \widehat{\tau_P}^{-1} \circ \widehat{\psi(g)} \circ \widehat{\tau_P} = [\![\psi(g)]\!] = [\![\zeta \cdot g]\!].$$

Hence $\eta(\Theta) = \zeta$ by Lemma 6.10. Thus, $\Theta \in \mathrm{Aut}(\mathcal{E}; \mathrm{Id}_{\mathcal{L}^{\mathcal{R}}_{/0}}; \mathbb{Z}_p^\times)$.

It remains to show that $\rho(\Theta)$ and (Ψ, ψ) differ by $c_{\widehat{\tau_S}} \in \mathrm{Inn}_T(\mathcal{G})$. Observe first that $\tau_P(\Theta)$ is the unique element satisfying $\Theta(\iota^S_P) = [\![\tau_P(\Theta)]\!] \circ \iota^S_P$. On the other hand,

$$\Theta(\iota^S_P) = \widehat{\tau_S}^{-1} \circ \Psi(\iota^S_P) \circ \widehat{\tau_P} = \widehat{\tau_S}^{-1} \circ \iota^S_{\psi(P)} \circ \widehat{\tau_P} = \tau_S^{-1}\tau_P \circ \iota^S_P.$$

By comparing the two expressions for $\Theta(\iota^S_P)$ and since ι^S_P is an epimorphism in $\mathcal{L}$ we deduce that

$$\tau_P(\Theta) = \tau_S^{-1}\tau_P.$$

By (7.11), (7.14) and (7.15) it follows that

$$\rho(\Theta)(P) = \tau_P(\Theta) \cdot P \cdot \tau_P(\Theta)^{-1} = \tau_S^{-1}\tau_P \cdot P \cdot \tau_P^{-1}\tau_S = \tau_S^{-1} \cdot \psi(P) \cdot \tau_S$$
$$\rho(\Theta)(\varphi) = \widehat{\tau_Q(\Theta)} \circ \Theta(\varphi) \circ \widehat{\tau_P(\Theta)}^{-1} = \widehat{\tau_S^{-1}} \circ \widehat{\tau_Q} \circ \Theta(\varphi) \circ \widehat{\tau_P}^{-1} \circ \widehat{\tau_S} = \widehat{\tau_S}^{-1} \circ \Psi(\varphi) \circ \widehat{\tau_S}.$$

Therefore $\Psi = c_{\widehat{\tau_S}^{-1}} \circ \rho(\Theta)$, and this shows that $\bar{\rho}$ is surjective.

Finally, consider some $\Theta \in \mathrm{Inn}(\mathcal{E})$. Then for every $P, Q \in \mathcal{R}$ and any $\varphi \in \mathcal{L}^{\mathcal{R}}(P, Q)$, there is an element $t_P \in P_0$, such that $\Theta(\varphi) = \widehat{t_Q}^{-1} \circ \varphi \circ \widehat{t_P}$. In particular $\tau_P(\Theta) = t_S^{-1}t_P$. Then for any $\varphi \in \mathcal{L}^{\mathcal{R}}(P, Q)$

$$\rho(\Theta)(\varphi) = \widehat{\tau_Q(\Theta)} \circ \Theta(\varphi) \circ \widehat{\tau_P(\Theta)}^{-1} = \widehat{t_s}^{-1} \circ \varphi \circ \widehat{t_s}.$$

This shows that $\rho(\Theta)|_{\mathcal{R}} = c_{\widehat{t_S}^{-1}}$ and by Lemma 7.8, $\rho(\Theta) = c_{\widehat{t_S}^{-1}}$. This completes the proof of Step 5 and hence of the proposition. □

We recall from (4.5) that if $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$ is a p-local compact group then every element of $\mathrm{Aut}_{\mathcal{L}}(S)$ induces a "conjugation automorphism" on $\mathcal{G}$ which we call "inner". We will write $\mathrm{Aut}_T(\mathcal{G})$ for the inner automorphisms of $\mathcal{G}$ induced by the elements of $\widehat{T} \leq \mathrm{Aut}_{\mathcal{L}}(S)$. Observe that conjugation by $t \in T$ induces an Adams automorphism of S of degree 1 and c_t is a special unstable Adams operation with $\tau_P = t$ for every $P \in \mathcal{F}^c$ and $\tau_\varphi = 1$ for any $\varphi \in \mathrm{Mor}(\mathcal{L})$.

Suppose that M is a $\mathbb{Z}_p$-module and that $x \in M$ has order p^m for some m, i.e $p^m x = 0$. Then for any $\zeta \in \mathbb{Z}_p^\times$ we claim that $\zeta \cdot x = x$ if and only if $\zeta \in \Gamma_m(p)$. To see this write $\zeta = u + p^m v$ for some $u \in \mathbb{Z} \subseteq \mathbb{Z}_p$ and $v \in \mathbb{Z}_p$. Then $\zeta \cdot x = u \cdot x$ so $\zeta \cdot x = x$ if and only if $u \cdot x = x$ which happens if and only if $u \in \mathrm{Ker}(\mathbb{Z}_p^\times \to \mathrm{Aut}(\mathbb{Z}/p^m)) = \Gamma_m(p)$.

Proof of Proposition 7.2. Suppose that $(\Psi, \psi) \in \mathrm{SpAd}(\mathcal{G}; \mathcal{R})$ and let $\zeta = \deg(\psi)$. By Propositions 6.15 and 6.16, $\Psi|_{\mathcal{R}} \in \mathrm{Aut}(\mathcal{E})$. By Proposition 7.3, $\zeta \cdot [\mathcal{L}^{\mathcal{R}}] = [\mathcal{L}^{\mathcal{R}}]$. By the remark above, $\zeta \in \Gamma_m(p)$. Conversely, if $\zeta \in \Gamma_m(p)$ the same remark shows that $\zeta \cdot [\mathcal{L}^{\mathcal{R}}] = [\mathcal{L}^{\mathcal{R}}]$ and Proposition 6.25 implies that there exists $\Theta \in \mathrm{Aut}(\mathcal{E})$ such that $\overline{\Theta} = \mathrm{Id}_{\mathcal{L}^{\mathcal{R}}_{/0}}$, and $\eta(\Theta) = \zeta$. By Proposition 7.10, $\rho(\Theta)$ is a special unstable Adams operation relative to $\mathcal{R}$ of degree ζ. This shows that the degree homomorphism is onto $\Gamma_m(p)$.

The kernel of deg is the group of special unstable Adams operations relative to $\mathcal{R}$ of degree 1. Its preimage under ρ is $\mathrm{Aut}(\mathcal{E}; \mathrm{Id}_{\mathcal{L}^{\mathcal{R}}_{/0}}, \mathrm{Id}_\Phi)$. The exactness of the sequence at $\mathrm{SpAd}(\mathcal{G}; \mathcal{R})/\mathrm{Inn}_T(\mathcal{G})$ now follows from Proposition 7.10(ii) and Proposition 6.26. □

We end this section with an analysis of the group $\mathrm{SpAd}(\mathcal{G}; \mathcal{R})$ as a subgroup of $\mathrm{Ad}(\mathcal{G})$. The next two lemmas will be needed.

7.17. **Lemma.** *Let $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$ be a p-local finite group and $\mathcal{R} \subseteq \mathcal{F}^c$ a collection. Then $\mathrm{SpAd}(\mathcal{G}; \mathcal{R})$ is a normal subgroup of $\mathrm{Ad}(\mathcal{G})$.*

Proof. Suppose (Ψ, ψ) and (Ψ', ψ') are special unstable Adams operations relative to $\mathcal{R}$. Fix structure elements $\{\tau_P\}_{P \in \mathcal{R}}$ and $\{\tau_\varphi\}_{\varphi \in \mathrm{Mor}(\mathcal{L}^{\mathcal{R}})}$ for (Ψ, ψ), and $\{\tau'_P\}_{P \in \mathcal{R}}$ and $\{\tau'_\varphi\}_{\varphi \in \mathrm{Mor}(\mathcal{L}^{\mathcal{R}})}$ for (Ψ', ψ'). Set $(\Psi'', \psi'') = (\Psi \circ \Psi', \psi \circ \psi')$, $\tau''_P = \tau_P \psi(\tau'_P)$ for every $P \in \mathcal{R}$, and $\tau_\varphi \psi(\tau'_\varphi)$ for any $\varphi \in \mathrm{Mor}(\mathcal{L}^{\mathcal{R}})$. Notice that $\psi''(P) = \tau''_p P \tau''^{-1}_P$ and that $\tau''_\varphi \in \psi''(Q)_0$ for any $\varphi \in \mathcal{L}^{\mathcal{R}}(P, Q)$. Using the fact that $\Psi(\widehat{g}) = \widehat{\psi(g)}$ it is straightforward to check that (Ψ'', ψ'') is a special unstable Adams operation with structure elements $\{\tau''_P\}_{P \in \mathcal{R}}$ and $\{\tau''_\varphi\}_{\varphi \in \mathrm{Mor}(\mathcal{L}^{\mathcal{R}})}$. The details are straightforward and left to the reader. In addition Ψ^{-1} is a special unstable Adams operation with structure elements $\{\psi^{-1}(\tau_P^{-1})\}_{P \in \mathcal{R}}$ and $\{\psi^{-1}(\tau_\varphi^{-1})\}_{\varphi \in \mathrm{Mor}(\mathcal{L}^{\mathcal{R}})}$. The verification uses the commutativity of T and the fact that $\psi^{-1}(\tau_\varphi^{-1}) \in Q_0$ for $\varphi \in \mathcal{L}^{\mathcal{R}}(P, Q)$. It follows that $\mathrm{SpAd}(\mathcal{G}; \mathcal{R})$ is a subgroup of $\mathrm{Ad}(\mathcal{G})$.

Let (Ψ, ψ) be a special unstable Adams operation relative to $\mathcal{R}$. For any $(\Theta, \theta) \in \mathrm{Ad}(\mathcal{G})$ consider the Adams operation $\Psi' \stackrel{\mathrm{def}}{=} \Theta^{-1} \circ \Psi \circ \Theta$. Set $\tau'_P \stackrel{\mathrm{def}}{=} \theta^{-1}(\tau_{\theta(P)})$ and $\tau'_\varphi \stackrel{\mathrm{def}}{=} \theta^{-1}(\tau_{\Theta(\varphi)})$. For any $P \in \mathcal{L}^{\mathcal{R}}$ and any $\varphi \in \mathcal{L}^{\mathcal{R}}(P, Q)$,

$$
\begin{aligned}
\Psi'(P) &= \theta^{-1}(\psi(\theta(P)) = \theta^{-1}(\tau_{\theta(P)} \cdot \theta(P) \cdot \tau^{-1}_{\theta(P)}) = \tau'_P \cdot P \cdot \tau'^{-1}_P, \\
\Psi'(\varphi) &= \Theta^{-1}(\widehat{\tau_{\theta(Q)}} \circ \widehat{\tau_{\Theta(\varphi)}} \circ \Theta(\varphi) \circ \widehat{\tau_{\theta(P)}}^{-1}) = \widehat{\tau'_Q} \circ \widehat{\tau'_\varphi} \circ \varphi \circ \widehat{\tau'_P}.
\end{aligned}
$$

Therefore the sets $\{\tau'_P\}_{P \in \mathcal{R}}$ and $\{\tau'_\varphi\}_{\varphi \in \mathrm{Mor}(\mathcal{L}^{\mathcal{R}})}$ give Ψ' the structure of a special Adams operation, and so $\mathrm{SpAd}(\mathcal{G}; \mathcal{R}) \trianglelefteq \mathrm{Ad}(\mathcal{G})$. □

7.18. **Lemma.** *Let $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$ be a p-local compact group and let $\mathcal{R} \subseteq \mathcal{F}^c$ be any collection with finitely many $\mathcal{F}$-conjugacy classes. Then the the order of the class $[\mathcal{L}^{\mathcal{R}}] \in H^2(\mathcal{L}^{\mathcal{R}}_{/0}, \Phi)$ is a power of p.*

Proof. The category $\mathcal{L}^{\mathcal{R}}_{/0}$ can be replaced with a finite subcategory $\mathcal{M}$ such that $H^*(\mathcal{L}^{\mathcal{R}}_{/0}, \Phi) \cong H^*(\mathcal{M}, \Phi)$. Since $\mathcal{M}$ is finite the cobar construction has the property that for any $n \geq 0$ the group $C^n(\mathcal{M}, \Phi)$ is a product of finitely many discrete p-tori and it is therefore a p-torsion group. □

7.19. **Proposition.** *Let $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$ be a p-local compact group and $\mathcal{R} \subseteq \mathcal{F}^c$ be a collection which contains finitely many $\mathcal{F}$-conjugacy classes. Then* $\mathrm{SpAd}(\mathcal{G}; \mathcal{R})$ *had finite index in* $\mathrm{Ad}(\mathcal{G})$ *and the factor group is solvable of class at most* 3.

Proof. We have seen in Lemma 7.17 that $\mathrm{SpAd}(\mathcal{G}; \mathcal{R}) \trianglelefteq \mathrm{Ad}(\mathcal{G})$. By Lemma 7.18 and Proposition 7.2 there is a morphism of exact sequences

$$\begin{tikzcd} 1 \arrow[r] & \mathrm{SpAd}^{\deg=1}(\mathcal{G};\mathcal{R}) \arrow[r] \arrow[d, hook] & \mathrm{SpAd}(\mathcal{G};\mathcal{R}) \arrow[r, "\deg"] \arrow[d, hook] & \Gamma_m(p) \arrow[r] \arrow[d, hook] & 1 \\ 1 \arrow[r] & \mathrm{Ad}^{\deg=1}(\mathcal{G}) \arrow[r] & \mathrm{Ad}(\mathcal{G}) \arrow[r, "\deg"'] & \mathbb{Z}_p^\times . \end{tikzcd}$$

The cokernel of the last column is a finite abelian group. Thus, by the snake lemma it remains to show that the quotient group in the first column is a solvable finite group of class at most 2.

Consider the following commutative diagram with exact rows

$$\begin{tikzcd} 1 \arrow[r] & \mathrm{SpAd}^{\mathrm{id}_S}(\mathcal{G};\mathcal{R}) \arrow[r] \arrow[d, hook] & \mathrm{SpAd}^{\deg=1}(\mathcal{G};\mathcal{R}) \arrow[r, "(\Psi,\psi)\mapsto\psi"] \arrow[d, hook] & \mathrm{Ad}^{\deg=1}(S) \arrow[d, equal] \\ 1 \arrow[r] & \mathrm{Ad}^{\mathrm{id}_S}(\mathcal{G}) \arrow[r] & \mathrm{Ad}^{\deg=1}(\mathcal{G}) \arrow[r, "(\Psi,\psi)\mapsto\psi"] & \mathrm{Ad}^{\deg=1}(S) \end{tikzcd}$$

where the superscript id_S means operations whose underlying Adams automorphism is the identity on S. Let $U \leq V$ be the images of the right horizontal maps. Notice that $\mathrm{Aut}_T(S) \leq U$ because for every $t \in T$, $c_{\widehat{t}} \in \mathrm{SpAd}^{\deg=1}(\mathcal{G};\mathcal{R})$. By [JLL, Proposition 2.8] $\mathrm{Ad}^{\deg=1}(S)/\mathrm{Aut}_T(S) \cong H^1(S/T;T)$. Since this is a finite abelian group it follows that V/U is a finite abelian group. It remains to show that the quotient group in the left column in this diagram is a finite abelian group. This follows from the next two claims.

Claim 1: $\mathrm{Ad}^{\mathrm{id}_S}(\mathcal{G})$ is an abelian discrete p-toral group.

Proof: Recall from [BLO3, Lemma 3.2] that $\mathcal{H}^\bullet(\mathcal{F})$ has finitely many S-conjugacy classes and hence finitely many T-conjugacy classes. Let $\mathcal{Q} = \{Q_1, \dots, Q_r\}$ be a set of representatives and for any $P \in \mathcal{H}^\bullet(\mathcal{F})$ we choose once and for all some $t_P \in T$ such that $t_P P t_P^{-1} \in \mathcal{Q}$. If $(\Psi, \mathrm{id}_S) \in \mathrm{Ad}(\mathcal{G})$, then for $\varphi \in \mathcal{L}(P,Q)$ we have $\pi(\Psi(\varphi)) = \pi(\varphi)$, so $\Psi(\varphi) = \varphi \circ \widehat{z(\varphi)}$ for a unique $z(\varphi) \in Z(P)$. This gives a function

$$z\colon \mathrm{Ad}^{\mathrm{id}_S}(\mathcal{G}) \xrightarrow{\Psi \mapsto (z(\varphi))} \prod_{Q,Q' \in \mathcal{Q}} \prod_{\mathcal{L}(Q,Q')} Z(Q).$$

This is a homomorphism because if $\Psi, \Psi' \in \mathrm{Ad}^{\mathrm{id}_S}(\mathcal{G})$, then for any $\varphi \in \mathcal{L}(P,Q)$

$$(\Psi' \circ \Psi)(\varphi) = \Psi'(\varphi \circ \widehat{z(\Psi)(\varphi)}) = \Psi'(\varphi) \circ \Psi'(\widehat{z(\Psi)(\varphi)}) = \varphi \circ \widehat{z(\Psi')(\varphi)} \circ \widehat{z(\Psi)(\varphi)},$$

so $z(\Psi' \circ \Psi) = z(\Psi') \cdot z_(\Psi)$.

We claim that z is injective. Choose some $\Psi \in \ker(z)$ and suppose that $z(\Psi)(\varphi) = 1$ for all $\varphi \in \mathcal{L}^{\mathcal{Q}}$. This means that Ψ is the identity on $\mathcal{L}^{\mathcal{Q}}$. If $P, P' \in \mathcal{H}^\bullet(\mathcal{F}^c)$, then with the notation above, there are unique $Q, Q' \in \mathcal{Q}$ such that $\mathcal{L}(P,P') = \widehat{t_{P'}}^{-1} \circ \mathcal{L}(Q,Q') \circ \widehat{t_P}$. Since $\Psi(\widehat{t_P}) = \widehat{\mathrm{id}_S(t_P)} = \widehat{t_P}$ and $\Psi(\widehat{t_{P'}}) = \widehat{t_{P'}}$, and since Ψ is the identity on $\mathcal{L}(Q,Q')$, it follows that Ψ is the identity on $\mathcal{L}^{\mathcal{H}^\bullet(\mathcal{F}^c)}$. By the uniqueness part in Lemma 7.8 it follows that $\Psi = \mathrm{Id}$.

Since z is injective and the codomain of z is a finite product of discrete p-tori, hence a discrete p-torus itself, it follows that its image is an abelian discrete p-toral group.

Claim 2: $\mathrm{SpAd}^{\mathrm{id}_S}(\mathcal{G};\mathcal{R})$ has finite index in $\mathrm{Ad}^{\mathrm{id}_S}(\mathcal{G})$.

Proof: By the the previous claim, $\mathrm{Ad}^{\mathrm{id}_S}(\mathcal{G})$ is an abelian discrete p-toral. Thus by the structure theorem of abelian discrete p-toral groups, it is isomorphic to $D_0 \times A$, where D_0 is a discrete p-torus and A is a finite abelian p-group. Therefore, $n \cdot D$ has finite index in D for any $n \geq 1$. We will show that there exists some n such that $\Psi^n \in \mathrm{SpAd}^{\mathrm{id}_S}(\mathcal{G};\mathcal{R})$ for any $\Psi \in \mathrm{Ad}^{\mathrm{id}_S}(\mathcal{G})$, and this will complete the proof.

Any $(\Psi, \mathrm{id}_S) \in \mathrm{Ad}(\mathcal{G})$ induces, by Propositions 6.15 and 6.16, a functor $\overline{\Psi} \in \mathrm{Aut}(\mathcal{L}_{/0})$ which is the identity on objects and therefore induces a permutation on $\mathcal{L}_{/0}(P,Q)$ for all $P,Q \in \mathcal{L}$. We therefore obtain a homomorphism

$$\sigma\colon \mathrm{Ad}^{\{\mathrm{id}_S\}}(\mathcal{G}) \to \prod_{P,Q \in \mathcal{L}^{\mathcal{R}}} \mathrm{Sym}(\mathcal{L}_{/0}(P,Q))$$

where $\mathrm{Sym}(\Omega)$ is the symmetric group of a set Ω. Since the sets $\mathcal{L}_{/0}(P,Q)$ are finite and since $\mathcal{R}$ has finitely many T-conjugacy classes, we obtain a well defined

$$r = \max\{|\mathcal{L}_{/0}(P,Q)| : P,Q \in \mathcal{H}^\bullet(\mathcal{F}^c)\}.$$

Then $n = r!$ annihilates any element in the codomain of σ, and in particular $\Psi^n \in \mathrm{Ker}(\sigma)$ for any $\Psi \in \mathrm{Ad}^{\mathrm{id}_S}(\mathcal{G})$. It remains to show that $\mathrm{Ker}(\sigma) \leq \mathrm{SpAd}^{\mathrm{id}_S}(\mathcal{G};\mathcal{R})$. Suppose that $(\Psi, \mathrm{id}_S) \in \mathrm{Ker}(\sigma)$. Then $\Psi(P) = P$ for any $P \in \mathcal{R}$ and also $[\Psi(\varphi)] = [\varphi]$ for any $\varphi \in \mathcal{L}^{\mathcal{R}}(P,Q)$, namely $\Psi(\varphi) = \widehat{\tau_\varphi} \circ \varphi$ for a unique $\tau_\varphi \in Q_0$. This shows that (Ψ, ψ) has the structure of a special unstable Adams operation with structure $\tau_P = 1$ and the elements τ_φ above. This completes the proof of the claim and the proposition follows. □

8. Not all Adams operations are special

specials

In this section we find examples of weakly connected p-local compact groups $\mathcal{G}$ that afford unstable Adams operations which are not special relative to the collection $\mathcal{R}$ of the $\mathcal{F}$-centric $\mathcal{F}$-radical subgroups. The idea is to find such $\mathcal{G}$ which fulfils the conditions of Lemma 8.1 below.

The collection $\mathcal{R}$ is a very natural one to look at since $|\mathcal{L}^{\mathcal{R}}| \to |\mathcal{L}|$ is a mod-p equivalence. To see this observe that $\mathcal{R} \subseteq \mathcal{H}^\bullet(\mathcal{F}^c)$ by [BLO3, Corollary 3.5] and that $|\mathcal{L}^{\mathcal{H}^\bullet(\mathcal{F}^c)}| \to |\mathcal{L}|$ is a homotopy equivalence by [JLL, Proposition 1.12] which provides a natural transformation from the identity on $\mathcal{L}$ to the functor $\mathcal{L} \xrightarrow{P \mapsto P^\bullet} \mathcal{L}$. Since $\mathcal{H}^\bullet(\mathcal{F}^c)$ has finitely many $\mathcal{F}$-conjugacy classes by [BLO3, Lemma 3.2], there results a finite filtration of $\mathcal{L}^{\mathcal{H}^\bullet(\mathcal{F}^c)}$ which together with the Λ-functors machinery in [BLO3, Section 5], can be used to prove that $|\mathcal{L}^{\mathcal{R}}| \to |\mathcal{L}^{\mathcal{H}^\bullet(\mathcal{F}^c)}|$ induces an isomorphism in $H^*(-, \mathbb{Z}_{(p)})$.

8.1. **Lemma.** *Let $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$ be a p-local compact group and $\mathcal{R}$ be a collection of $\mathcal{F}$-centric subgroup. Let $(\Psi, \psi) \in \mathrm{Ad}(\mathcal{G})$ and assume that*

(i) $\deg(\psi) \in \mathbb{Z}_p^\times \setminus \Gamma_1(p)$ *and that*
(ii) there exists $P \in \mathcal{R}$ such that P is not a semi-direct product of P_0 with P/P_0.

Then $(\Psi, \psi) \notin \mathrm{SpAd}(\mathcal{G}; \mathcal{R})$.

Proof. Assume by contradiction that $(\Psi, \psi) \in \mathrm{SpAd}(\mathcal{G}; \mathcal{R})$ and set $\zeta = \deg(\psi)$. We claim that $[\mathcal{L}^{\mathcal{R}}]$ is the trivial element in $H^2(\mathcal{L}^{\mathcal{R}}_{/0}, \Phi)$. To see this, let L be the $\mathbb{Z}_p$-submodule of $H^2(\mathcal{L}^{\mathcal{R}}_{/0}, \Phi)$ generated by $[\mathcal{L}^{\mathcal{R}}]$. By Proposition 7.3, ζ acts as the identity on L. If L is infinite then $L \cong \mathbb{Z}_p$ and therefore $\zeta = 1 \in \Gamma_1(p)$ which is a contradiction. Therefore $L \cong \mathbb{Z}/p^m$ and $\zeta \in \Gamma_m(p)$ which by hypothesis (i) implies that $m = 0$, namely $L = 0$.

Lemma 6.22 now implies that there exists a functor $s\colon \mathcal{L}^{\mathcal{R}}_{/0} \to \mathcal{L}^{\mathcal{R}}$ which is a right inverse to the projection $\mathcal{L}^{\mathcal{R}} \to \mathcal{L}^{\mathcal{R}}_{/0}$. Consider any $P \in \mathcal{R}$. Notice that $\mathrm{Aut}_{\mathcal{L}}(P)$ contains $\widehat{P}$ as a copy of P and similarly $\mathrm{Aut}_{\mathcal{L}_{/0}}(P)$ contains a copy of P/P_0. The functor s gives a section $s\colon P/P_0 \to P$ for the projection $P \to P/P_0$. This is a contradiction to hypothesis (ii). □

It turns out that compact Lie groups provide examples of p-local compact group $\mathcal{G}$ which satisfy the conditions of the lemma. First, let us recall from [BLO3, Section 9] how compact Lie groups give rise to p-local compact groups.

The poset of all discrete p-toral subgroups of a compact Lie group G contains a maximal element S. Every discrete p-toral $P \leq G$ is conjugate to a subgroup of S and in particular all maximal discrete p-toral subgroups of G are conjugate. The fusion system $\mathcal{F} = \mathcal{F}_S(G)$ over S has by definition $\mathrm{Hom}_{\mathcal{F}}(P,Q) = \mathrm{Hom}_G(P,Q)$ namely the homomorphisms $P \to Q$ induced by conjugation by elements of g. This fusion system is saturated by [BLO3, Lemma 9.5], it admits an associated centric linking system $\mathcal{L} = \mathcal{L}^c_S(G)$ that is unique up to isomorphism, and $|\mathcal{L}^c_S(G)|^\wedge_p \simeq BG^\wedge_p$ by [BLO3, Theorem 9.10].

Recall that a closed subgroup $Q \le G$ is called p-toral if it is an extension of a torus by a finite p-group. Let P be a discrete p-toral subgroup of G. Then $\overline{P}$ (the closure of P) is a a p-toral subgroup of G. In fact, $\overline{P_0}$ is the maximal torus of $\overline{P}$.

A discrete p-toral $P \le G$ is called *snugly embedded* if $\overline{P} = \overline{P_0} \cdot P$ and every p-power torsion element in $(\overline{P})_0$ belongs to P_0.

8.2. **Lemma.** *Let G be a compact Lie group and $P \le G$ a snugly embedded discrete p-toral subgroup. If P is the semidirect product of P_0 with P/P_0 then $\overline{P}$ is the semidirect product of $\overline{P}_0$ with P/P_0.*

Proof. Suppose $\pi \le P$ is a complement of P_0. Since P is snugly embedded, $\overline{P} = \overline{P_0}\cdot P = (\overline{P})_0\cdot P = (\overline{P})_0\cdot\pi$. Also, P_0 contains all p-torsion in $(\overline{P})_0$ so $(\overline{P})_0 \cap \pi \le P_0 \cap \pi = 1$. □

Fix a compact Lie group G. A subgroup P is called *p-stubborn*, if P is p-toral and if $N_G(P)/P$ is finite and $O_p(N_G(P)/P) = 1$, where $O_p(K)$ denotes the largest normal p-subgroup of a finite group K [JMO]. Let us now recall from [O] the structure of the p-stubborn subgroups of the classical groups $U(n)$ and $SU(n)$ when $p > 2$.

First, consider the regular representation of $\mathbb{Z}/p$ on $\mathbb{C}^p$ with the standard basis $e_0, \dots, e_{p-1}$. This representation sends a generator of $\mathbb{Z}/p$ to the permutation matrix in $U(p)$:

$$B = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & & \dots & & \dots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix} \qquad B\colon e_i \mapsto e_{i-1}$$

Since the regular representation contains one copy of every irreducible representation of $\mathbb{Z}/p$, it is clear that B is conjugate in $U(p)$ to the matrix

$$A = \mathrm{diag}(1, \zeta, \zeta^2, \dots, \zeta^{p-1}) = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \zeta & 0\dots & 0 & \\ 0 & 0 & \zeta^2 & \dots & 0 \\ \dots & & & & \dots \\ 0 & 0 & 0 & \dots & \zeta^{p-1} \end{pmatrix} \qquad A\colon e_i \mapsto \zeta^i e_i$$

where ζ is a p-th root of unity. It is easy to check that

$$[A, B] = ABA^{-1}B^{-1} = \zeta I_p.$$

Thus, A and $[A, B]$ belong to the standard maximal torus of $U(p)$ (namely the unitary diagonal matrices).

Consider the natural action of $U(p^k)$ on $\mathbb{C}^{p^k} \cong \mathbb{C}^p \otimes \dots \otimes \mathbb{C}^p$. For every $i = 0, \dots, k-1$ we let A_i and B_i denote the matrices that correspond to the action of A and B on the ith factor of the tensor product and the identity on the other factor. From this description it is clear that

$$[A_i, A_j] = [B_i, B_j] = I, \qquad [A_i, B_j] = I \ (i \ne j), \qquad [A_i, B_i] = \zeta I. \tag{8.3}$$

In matrix notation, $A_i = I^{\otimes i-1} \otimes A \otimes I^{\otimes k-i}$ and $B_i = I^{\otimes i-1} \otimes B \otimes I^{\otimes k-i}$ where I denotes the $p \times p$ identity matrix and we use the Kronecker tensor product of matrices.

For any $k \ge 0$ define $\Gamma^U_{p^k} \le U(p^k)$ by

$$\Gamma^U_{p^k} \overset{\text{def}}{=} \Big\langle A_0, \dots, A_{k-1},\ B_0, \dots, B_{k-1},\ u \cdot I \ :\ u \in U(1) \Big\rangle,$$

where I denotes the identity matrix in $U(p^k)$. It is clear from (8.3) that the identity component of $\Gamma^U_{p^k}$ is isomorphic to $U(1)$ and that the factor group is isomorphic to

$$\mathbb{Z}/p^{2k} = \langle \overline{A_0}, \dots, \overline{A_{k-1}}, \overline{B_0}, \dots, \overline{B_{k-1}} \rangle \tag{8.4}$$

where $\overline{A_i}$ and $\overline{B_i}$ are the images of A_i and B_i in the quotient. Thus, $\Gamma^U_{p^k}$ are p-toral groups. Notice that

$$\Gamma^U_1 = U(1) \qquad \text{and} \qquad \Gamma^U_p = \big\langle A,\ B,\ u\cdot I\ :\ u \in U(1)\big\rangle.$$

Next, fix some $k \geq 1$ and recall that $\Sigma_{p^k} \leq U(p^k)$ via permutation matrices. Let $E_{p^k} = (\mathbb{Z}/p)^k$ act on itself by left translation. This gives a monomorphism $E_{p^k} \to \Sigma_{p^k}$ and identifies E_{p^k} as a subgroup of $U(p^k)$. Given any $H \leq U(m)$ the wreath product $H \wr E_{p^k}$ is naturally a subgroup of $U(mp^k)$.

Now fix some $n \geq 1$ and let p be an odd prime. Write $n = p^{m_1} + \cdots + p^{m_r}$. Identifying the product $U(p^{m_1}) \times \cdots \times U(p^{m_r})$ as a subgroup of $U(n)$ in the standard way, consider the subgroups $Q_1 \times \cdots \times Q_r$, where for each $1 \leq i \leq r$, $Q_i \leq U(P^{m_i})$ has the form

$$\Gamma^U_{p^k} \wr E_{q_1} \wr \cdots \wr E_{q_t},$$

and where $t \geq 0$ and $q_1, \ldots, q_t$ are all p-powers such that $p^{m_i} = p^k q_1 \ldots q_t$. By [O, Theorems 6 and 8], these groups give a complete set of representatives for the conjugacy classes of p-stubborn subgroups of $U(n)$ when $p > 2$. By [O, Theorem 10] the assignment $P \mapsto P \cap SU(n)$ gives a bijection between the p-stubborn subgroups of $U(n)$ and $SU(n)$.

8.5. **Lemma.** *Let G be a connected compact Lie group and $S \leq G$ a maximal discrete p-toral group. Then $\mathcal{F}_S(G)$ is weakly connected, namely $T = S_0$ is self centralising in S.*

Proof. Let $T \leq S$ be the maximal discrete p-torus in S. Then $\overline{T}$ is the maximal torus in $\overline{S}$, and hence it is the maximal torus of G. Since G is connected, its maximal torus is self-centralising and in particular $C_S(T) = C_S(\overline{T}) = S \cap C_G(\overline{T}) = S \cap \overline{T} = T$, where the last equality holds since S is snugly embedded in G. □

Recall that a space X is called p-good in the sense of Bousfield and Kan [BK72] if the natural map $X \to X^\wedge_p$ induces isomorphism in $H_*(-;\mathbb{Z}/p)$. Also recall that $H^*_{\mathbb{Q}_p}(X) \overset{\text{def}}{=} H^*(X;\mathbb{Z}_p) \otimes \mathbb{Q}$.

8.6. **Lemma.** *Let X be a CW-complex. If X is p-good then $H^*_{\mathbb{Q}_p}(X) \cong H^*_{\mathbb{Q}_p}(X^\wedge_p)$ via the natural map. If $H_*(X;\mathbb{Z})$ is finitely generated in every degree then $H^*_{\mathbb{Q}_p}(X) \cong H^*(X) \otimes \mathbb{Q}_p \cong H^*(X;\mathbb{Q}) \otimes \mathbb{Z}_p$.*

Proof. If X is p-good then $X \to X^\wedge_p$ induces an isomorphism in $H_*(-;\mathbb{Z}/p)$ and hence in $H^*(-;\mathbb{Z}/p^n)$ for all $n \geq 1$. Since $\mathbb{Z}_p = \varprojlim_n \mathbb{Z}/p^n$ a standard spectral sequence argument gives $H^*(X;\mathbb{Z}_p) \cong H^*(X^\wedge_p;\mathbb{Z}_p)$ and hence $H^*_{\mathbb{Q}_p}(X) \cong H^*_{\mathbb{Q}_p}(X^\wedge_p)$.

Suppose $H_*(X)$ is finitely generated in every degree. If A is a torsion-free abelian group, [Sp, Theorem 10 in Chap. 5.5] implies that $H^*(X;A) \cong H^*(X) \otimes A$. Therefore

$$H^*_{\mathbb{Q}_p}(X) = H^*(X;\mathbb{Z}_p) \otimes \mathbb{Q} \cong H^*(X) \otimes \mathbb{Z}_p \otimes \mathbb{Q} = H^*(X) \otimes \mathbb{Q}_p = H^*(X) \otimes \mathbb{Q} \otimes \mathbb{Z}_p = H^*(X;\mathbb{Q}) \otimes \mathbb{Z}_p.$$

□

The next proposition is the main result of this section.

8.7. **Proposition.** *Let $p \geq 3$ be a prime and set $G = \mathrm{PSU}(2p)$. Let $S \leq G$ be a maximal discrete p-toral group and let $\mathcal{G} = (S, \mathcal{F}_S(G), \mathcal{L}^c_S(G))$ be the associated p-local compact group. Let $\mathcal{R}$ denote the collection of all centric radical subgroups of S. Then $\mathrm{SpAd}(\mathcal{G};\mathcal{R}) \lneq \mathrm{Ad}(\mathcal{G})$. In fact, the following composition is not surjective*

$$\mathrm{SpAd}(\mathcal{G};\mathcal{R}) \xrightarrow{incl} \mathrm{Ad}(\mathcal{G}) \xrightarrow{(\Psi,\psi)\mapsto|\Psi|^\wedge_p} \mathrm{Ad}^{\mathrm{geom}}(B\mathcal{G}).$$

Proof. Write $\mathcal{G} = (S, \mathcal{F}, \mathcal{L})$ for short. Let W be the Weyl group of G. Since $p > 2$, Dirichlet's theorem on the existence of infinitely many primes in arithmetic progressions implies that there are infinitely many primes k such that $k \not\equiv 1 \mod p$. In particular there exists such k that, in addition, satisfies $(k, |W|) = 1$. By [JMO, Theorem 2] there exist unstable Adams operations $f\colon BG \to BG$ of degree k. That means that $f^*\colon H^{2m}(BG;\mathbb{Q}) \to H^{2m}(BG;\mathbb{Q})$ is multiplication by k^m for every $m > 0$. Lemma 8.6 and the functoriality of $H^*(-)$ imply that $f^\wedge_p$ induces multiplication by k^m on $H^{2m}(BG;\mathbb{Q}) \otimes \mathbb{Z}_p \cong H^{2m}_{\mathbb{Q}_p}(BG) \cong H^*_{\mathbb{Q}_p}(BG^\wedge_p)$. Since $BG^\wedge_p \simeq B\mathcal{G}$, we obtain a self equivalence h of $B\mathcal{G}$ which induces multiplication by k^m

on $H^*_{\mathbb{Q}_p}(B\mathcal{G})$. Since $\mathcal{G}$ is weakly connected by Lemma 8.5 it follows from [BLO7, Proposition 3.2] that $h \in \mathrm{Ad}^{\mathrm{geom}}(B\mathcal{G})$. Theorem 1.3 applies to show that h is homotopic to $|\Psi|^\wedge_p$ for some $(\Psi, \psi) \in \mathrm{Ad}(\mathcal{G})$.

Set $\zeta = \deg(\psi)$. Thus, $\psi|_T$ is multiplication by ζ. Commutativity of

$$\begin{tikzcd} BT \arrow[r] \arrow[d, "B\psi|_T"'] & B\mathcal{G} \arrow[d, "|\Psi|^\wedge_p"] \\ BT \arrow[r] & B\mathcal{G} \end{tikzcd}$$

and the fact that $H^*_{\mathbb{Q}_p}(B\mathcal{G}) \cong H^*_{\mathbb{Q}_p}(BT)^{W(\mathcal{G})}$, [BLO7, Theorem A], imply that $|\Psi|^\wedge_p$ induces multiplication by ζ^m on $H^{2m}_{\mathbb{Q}_p}(B\mathcal{G})$ for all $m > 0$. On the other and, $|\Psi|^\wedge_p \simeq h$, so $\zeta^m = k^m$ for all $m > 0$.

Now, $G = \mathrm{PSU}(2p)$ and since $Z(\mathrm{SU}(2p))$ is a finite group, BG and $BSU(2p)$ are rationally equivalent. It is well known that

$$H^*(BSU(n)) \cong \mathbb{Z}[c_2, c_3, \dots, c_n], \qquad c_i \in H^{2i}(BSU(n)).$$

It follows from Lemma 8.6 that $H^*_{\mathbb{Q}_p}(B\mathcal{G})$ is a polynomial algebra over $\mathbb{Q}_p$ generated by $c_2, \dots, c_p, \dots, c_{2p}$. In particular $H^{2p}(B\mathcal{G})$ is a non-trivial vector space over $\mathbb{Q}_p$ on which $|\Psi|^\wedge_p$ induces multiplication by ζ^p, and this map is the same as multiplication by k^p. This implies that ζ and k differ by a pth root of unity in $\mathbb{Q}_p$ and [Ro, Sec. 6.7, Prop. 1,2] implies that $\zeta = k$ since $\mathbb{Q}_p$ contains no pth roots of unity if $p > 2$.

In order to complete the proof we apply Lemma 8.1 to show that (Ψ, ψ) is not special relative to $\mathcal{R}$. Since $k \not\equiv 1 \mod p$, it follows that $k \notin \Gamma_1(p)$, and so Condition (i) of Lemma 8.1 holds. The remainder of the proof is dedicated to showing that Condition (ii) of the lemma also holds.

Let $\tilde{Q} \le U(2p)$ be the subgroup

$$\tilde{Q} = \Gamma^U_p \times \Gamma^U_p$$

that, by the discussion above, is p-stubborn in $U(2p)$. As we explained above, $\tilde{Q}$ is generated by the matrices

$$A^{(1)} = \begin{pmatrix} A & 0 \\ 0 & I \end{pmatrix}, \qquad B^{(1)} = \begin{pmatrix} B & 0 \\ 0 & I \end{pmatrix},$$
$$A^{(2)} = \begin{pmatrix} I & 0 \\ 0 & A \end{pmatrix}, \qquad B^{(2)} = \begin{pmatrix} I & 0 \\ 0 & B \end{pmatrix},$$
$$\begin{pmatrix} uI & 0 \\ 0 & vI \end{pmatrix}, \qquad (u, v \in U(1)).$$

Throughout we will write I for the identity matrix in $U(p)$ and we will also write $\mathrm{diag}(uI, vI)$ for the matrices of the last type, and $\mathrm{diag}(A, I)$ for $A^{(1)}$ etc. It is clear from (8.3) that

$$[A^{(1)}, A^{(2)}] = [A^{(1)}, B^{(2)}] = [B^{(1)}, B^{(2)}] = 1,$$
$$[A^{(1)}, B^{(1)}] = \mathrm{diag}(\zeta I, I), \qquad [A^{(2)}, B^{(2)}] = \mathrm{diag}(I, \zeta I).$$

Also, $A^{(1)}, A^{(2)}, B^{(1)}, B^{(2)}$ commute with all the matrices $\mathrm{diag}(uI, vI)$. Set

$$Q = \tilde{Q} \cap SU(2p).$$

By [O, Theorem 10], Q is p-stubborn in $\mathrm{SU}(2p)$ and by [O, Lemma 7] it is also centric in $\mathrm{SU}(2p)$, namely $C_{\mathrm{SU}(2p)}(Q) = Z(Q)$. Notice that Q contains $A^{(1)}, A^{(2)}, B^{(1)}, B^{(2)}$ because $\det(B) = 1$ since it is the signature of the odd cycle $(1, 2, \dots, p)$. Therefore Q is generated by $A^{(1)}, A^{(2)}, B^{(1)}, B^{(2)}$ and by the subgroup consisting of the matrices

$$\begin{pmatrix} uI & 0 \\ 0 & vI \end{pmatrix}, \qquad u, v \in U(1),\ u^p v^p = 1. \tag{8.8}$$

This subgroup is easily seen to be isomorphic to $U(1) \times \mathbb{Z}/p$. The maximal torus of Q is therefore the set of matrices

$$Q_0 = \{\mathrm{diag}(uI, vI) : v = u^{-1}\} \cong U(1).$$

Clearly Q contains $Z(SU(2p))$ which is the set of matrices $u \cdot I$ where $u^{2p} = 1$.

Let R be the image of Q in $\mathrm{PSU}(2p)$. It is generated by the images of $A^{(1)}, A^{(2)}, B^{(1)}, B^{(2)}$ which we denote by adding "bars" - $\bar{A}^{(i)}$ and $\bar{B}^{(i)}$ - and by the images of the matrices in (8.8) which we denote by $\bar{U}$. The maximal torus of R is the image of Q_0, namely these matrices U with $v = u^{-1}$.

By [JMO, Proposition 1.6(i)], R is p-stubborn in G and hence, by [O, Lemma 7(ii)] it is also centric in $G = \mathrm{PSU}(2p)$, namely $C_G(R) = Z(R)$. We claim that the projection $R \to R/R_0$ does not have a section. Assume to the contrary that such a section $s\colon R/R_0 \to R$ exists. Consider the following elements in R

$$X = \bar{A}^{(1)}\bar{B}^{(2)}, \qquad Y = \bar{B}^{(1)}\bar{A}^{(2)}.$$

A straightforward calculation using (8.3) gives

$$[X, Y] = [\bar{A}^{(1)}\bar{B}^{(2)}, \bar{B}^{(1)}\bar{A}^{(2)}] = [\bar{A}^{(1)}, \bar{B}^{(1)}] \cdot [\bar{B}^{(2)}, \bar{A}^{(2)}] = \overline{\mathrm{diag}(\zeta I, I)} \cdot \overline{\mathrm{diag}(I, \zeta^{-1} I)} = \overline{\mathrm{diag}(\zeta I, \zeta^{-1} I)}.$$

Let x, y denote the images of X, Y in R/R_0. These are clearly non-trivial elements, and since $s\colon R/R_0 \to R$ is a section, there are $\bar{U}, \bar{V} \in R_0$ such that

$$s(x) = X \cdot \bar{U} \qquad \text{and} \qquad s(y) = Y \cdot \bar{V}.$$

Say, $U = \mathrm{diag}(uI, u^{-1}I)$ and $V = \mathrm{diag}(vI, v^{-1}I)$ where $u, v \in U(1)$. Then U, V commute with the matrices $A^{(i)}$ and $B^{(i)}$, and since s is a homomorphism and R/R_0 is abelian, (8.4), it follows that

$$1 = [s(x), s(y)] = [X\bar{U}, Y\bar{V}] = [X, Y] = \overline{\mathrm{diag}(\zeta I, \zeta^{-1} I)}.$$

This means that $\mathrm{diag}(\zeta I, \zeta^{-1} I) \in Z(\mathrm{SU}(2p))$ so it is diagonal, namely $\zeta = \zeta^{-1}$. Therefore $\zeta^2 = 1$ in $\mathbb{Z}_p$ which implies that $\zeta = \pm 1$. However, we have seen that $\zeta = k$, and k was chosen such that $k > 0$ and $k \neq 1 \mod p$, which is absurd. We conclude that $R \to R/R_0$ does not have a section.

Let P be a maximal discrete p-toral subgroup of R. Up to conjugation in G we may assume that $P \leq S$. Since R is centric, it is p-centric (namely $Z(R)$ is the maximal p-toral subgroup of $C_G(R)$) and [BLO3, Lemma 9.6(c)] shows that P is $\mathcal{F}$-centric. Also, P is snugly embedded in R, hence in G, and by [BLO3, Lemma 9.4] there is an isomorphism of groups

$$\mathrm{Out}_G(P) = \mathrm{Rep}_G(P, P) \xrightarrow{\cong} \mathrm{Rep}_G(R, R) = \mathrm{Out}_G(R).$$

Since R is centric in G, it follows that $\mathrm{Out}_G(R) = N_G(R)/R$ and since R is p-stubborn, $\mathrm{Out}_G(R)$ is finite and

$$O_p(\mathrm{Out}_{\mathcal{F}}(P)) = O_p(\mathrm{Out}_G(P)) \cong O_p(\mathrm{Out}_G(R)) = O_p(N_G(R)/R) = 1$$

Therefore P is $\mathcal{F}$-radical. We deduce that $P \in \mathcal{R}$. Since $\overline{P} = R$, Lemma 8.2 shows that P cannot have a complement for P_0 in P. This shows that P satisfies the condition (ii) in Lemma 8.1 and finishes the proof of this proposition. □

Appendix A. Extensions of categories - Proofs of results

Proof of Lemma 6.20. Let $c \in \mathcal{C}(X_0, X_1)$ be a morphism. By the definition of z_σ we have $\llbracket z_\sigma(1_{X_1}, c) \rrbracket \circ \sigma(1_{x_1} \circ c) = \sigma(1_{X_1}) \circ \sigma(c) = \sigma(c)$ and since $\Phi(X_1)$ acts freely on $\mathcal{D}(X_0, X_1)$ it follows that $z_\sigma(1_{X_1}, c) = 1$. Similarly $z_\sigma(c, 1_{X_0}) = 1$.

Now we show that z_σ is a 2-cocycle. We need to show that $\delta(z_\sigma)(c_2, c_1, c_0) = 1$ for any 3-chain $X_0 \xrightarrow{c_0} X_1 \xrightarrow{c_1} X_2 \xrightarrow{c_2} X_3$. Consider the following diagram in $\mathcal{D}$

$$\begin{tikzcd}
& X_0 \arrow[rr, equal] \arrow[d, "\sigma(c_1\circ c_0)"] \arrow[ddl, "\sigma(c_2\circ c_1\circ c_0)"'] & & X_0 \arrow[d, "\sigma(c_0)"'] \arrow[ddr, "\sigma(c_2\circ c_1\circ c_0)"] & \\
& X_2 \arrow[r, "\llbracket z_\sigma(c_1,c_0)\rrbracket"'] \arrow[d, "\sigma(c_2)"'] & X_2 \arrow[d, "\sigma(c_2)"] & X_1 \arrow[l, "\sigma(c_1)"] \arrow[d, "\sigma(c_2\circ c_1)"] & \\
X_3 \arrow[r, "\llbracket z_\sigma(c_2,c_1\circ c_0)\rrbracket"'] & X_3 \arrow[r, "\llbracket c_{2*}(z_\sigma(c_1,c_0))\rrbracket"'] & X_3 & X_3 \arrow[l, "\llbracket z_\sigma(c_2,c_1)\rrbracket"] & X_3 \arrow[l, "\llbracket z_\sigma(c_2\circ c_1,c_0)\rrbracket"]
\end{tikzcd}$$

The rectangle, the bottom-right square and the two triangles commute by the definition of z_σ. The bottom-left square commutes by (6.2). The diagram is therefore commutative. Since $\Phi(X_3)$ acts freely on $\mathcal{D}(X_0, X_3)$ it follows that the two morphisms $X_3 \to X_3$ defined at the bottom of the diagram are equal because their composition with $\sigma(c_2 \circ c_1 \circ c_0)$ give the same morphism in $\mathcal{D}(X_0, X_2)$. This, in turn, is equivalent to the 2-cocycle condition for z_σ.

Now suppose that σ' is another regular section. Since $[\sigma(c)] = c = [\sigma'(c)]$ for any $c \in \mathcal{C}(X_0, X_1)$, there exists a unique $u(c) \in \Phi(X_1)$ such that $\sigma'(c) = [\![u(c)]\!] \circ \sigma(c)$. This gives a 1-cochain $u \in C^1(\mathcal{C}, \Phi)$. By the defining relation of z_σ and $z_{\sigma'}$ we get for any 2-cochain $X_0 \xrightarrow{c_0} X_1 \xrightarrow{c_1} X_2$ in $\mathcal{C}$

$$[\![z_{\sigma'}(c_1, c_0)]\!] \circ [\![u(c_1 \circ c_0)]\!] \circ \sigma(c_1 \circ c_0) \overset{(u)}{=} [\![z_{\sigma'}(c_1, c_0)]\!] \circ \sigma'(c_1 \circ c_0) \overset{(z_{\sigma'})}{=} \sigma'(c_1) \circ \sigma'(c_0) \overset{(u)}{=}$$
$$[\![u(c_1)]\!] \circ \sigma(c_1) \circ [\![u(c_0)]\!] \circ \sigma(c_0) \overset{(6.2)}{=} [\![u(c_1)]\!] \circ [\![c_{1*}(u(c_0))]\!] \circ \sigma(c_1) \circ \sigma(c_0) \overset{(z_\sigma)}{=}$$
$$[\![u(c_1)]\!] \circ [\![c_{1*}(u(c_0))]\!] \circ [\![z_\sigma(c_1, c_0)]\!] \circ \sigma(c_1 \circ c_0).$$

Since $\Phi(X_2)$ acts freely on $\mathcal{D}(X_0, X_2)$ and since $[\![-]\!] \colon \Phi(X_2) \to \mathrm{Aut}_{\mathcal{D}}(X_2)$ is injective, we deduce that $z_{\sigma'}(c_1, c_0) = z_\sigma(c_1, c_0) \cdot u(c_1) \cdot u(c_1 \circ c_0)^{-1} \cdot c_{1*}(u(c_0))$. Since this holds for all 2-cochains, $z_{\sigma'} = z_\sigma \cdot \delta(u)$, namely z_σ and $z_{\sigma'}$ are cohomologous. □

Proof of Lemma 6.22. Suppose first that s exists. Then it provides a regular section $s \colon \mathcal{C}_1 \to \mathcal{D}_1$ and the functoriality of s readily implies that $z_s = 1$, hence $[\mathcal{D}] = 0$.

Conversely, suppose that $[\mathcal{D}] = 0$. Choose a regular section $\sigma \colon \mathcal{C}_1 \to \mathcal{D}_1$. Then $z_\sigma = \delta(u)$ where δ is the differential in the cobar construction and $u \in C^1(\mathcal{C}, \Phi)$. Thus, given a 2-chain $C_0 \xrightarrow{c_1} C_1 \xrightarrow{c_2} C_2$ in $\mathcal{C}$,

$$z_\sigma(c_2, c_1) = u(c_2) \cdot u(c_2 \circ c_1)^{-1} \cdot \Phi(c_2)(u(c_1)). \tag{A.1}$$

Define a functor $s \colon \mathcal{C} \to \mathcal{D}$ as follows. On objects $s(C) = C$ for all $C \in \mathcal{C}$. On morphisms $c \in \mathcal{C}(C_1, C_2)$,

$$s(c) = [\![u(c)]\!] \circ \sigma(c).$$

Then s is a functor because it respects units and compositions. First, since z_σ is regular $1 = z_\sigma(1_X, 1_X) = u(1_X) \cdot u(1_X \circ 1_X)^{-1} \cdot (1_X)_*(u(1_X)) = u(1_X)$, and therefore $u(1_X) = 1$. Hence $s(1_X) = 1_X$ because σ is regular. Next, s respects composition because

$$s(c_2 \circ c_1) = [\![u(c_2 \circ c_1)]\!] \circ \sigma(c_2 \circ c_1) = [\![u(c_2 \circ c_1)]\!] \circ [\![z_\sigma(c_2, c_1)]\!] \circ \sigma(c_2) \circ \sigma(c_1) \overset{=}{(\text{A.1})}$$
$$[\![u(c_2)]\!] \circ [\![\Phi(c_2)(u(c_1))]\!] \circ \sigma(c_2) \circ \sigma(c_1) \overset{=}{(6.2)} [\![u(c_2)]\!] \circ \sigma(c_2) \circ [\![u(c_1)]\!] \circ \sigma(c_1) = s(c_2) \circ s(c_1).$$

Clearly s is a right inverse to $\mathcal{D} \xrightarrow{[-]} \mathcal{C}$. □

A.2. **Remark.** Given a functor $\Phi \colon \mathcal{C} \to \mathbf{Ab}$, Thomason's construction [Th] gives rise to an extension $\mathcal{D}$. Inspection of this construction shows that is comes equipped with a section, namely a functor $s \colon \mathcal{C} \to \mathcal{D}$ which is a right inverse to the projection $\mathcal{D} \xrightarrow{[-]} \mathcal{C}$. It is not hard to see that $[\mathrm{Tr}_{\mathcal{C}}(\Phi)] = 0$ and conversely, if $\mathcal{D}$ is an extension with $[\mathcal{D}] = 0$ then $\mathcal{D}$ is isomorphic as an extension to $\mathrm{Tr}_{\mathcal{C}}(\Phi)$.

Proof of Lemma 6.24. To see that Γ is well defined we need to show that if $\mathcal{E}$ and $\mathcal{E}'$ are equivalent extensions of $\mathcal{C}$ by Φ then they give rise to the same element of $H^2(\mathcal{C}, \Phi)$. Fix regular sections $\sigma \colon \mathcal{C}_1 \to \mathcal{D}_1$ and $\sigma' \colon \mathcal{C}_1 \to \mathcal{D}'_1$ and let z_σ and $z_{\sigma'}$ be their associated 2-cocycles. Also fix an equivalence $\Psi \colon \mathcal{E} \to \mathcal{E}'$, i.e. $\overline{\Psi} = \mathrm{Id}_{\mathcal{C}}$ and $\eta(\Psi) = \mathrm{Id}_\Phi$. Notice that $\Psi(\sigma) \colon \mathcal{C}_1 \to \mathcal{D}'_1$ is a regular section for $\mathcal{E}'$ because $\overline{\Psi} = \mathrm{Id}_{\mathcal{C}}$. Since $\eta(\Psi) = \mathrm{Id}_\Phi$, by applying Ψ to the defining relation of z_σ and using (6.11) we get

$$[\![z_\sigma(c_1, c_0)]\!] \circ \Psi(\sigma(c_1 \circ c_0)) = \Psi(\sigma(c_1)) \circ \Psi(\sigma(c_0)).$$

Therefore $z_\sigma = z_{\Psi(\sigma)}$ which is cohomologous to $z_{\sigma'}$ by Lemma 6.20.

Now we construct $\Sigma \colon H^2(\mathcal{C}, \Phi) \to \mathrm{Ext}(\mathcal{C}, \Phi)$. Given $\zeta \in H^2(\mathcal{C}, \Phi)$ choose a regular 2-cocycle $z \in \zeta$ (this is possible by Lemma 6.17). Define a category $\mathcal{D}_z$ as follows. First, $\mathrm{Obj}(\mathcal{D}_z) = \mathrm{Obj}(C)$ and $\mathcal{D}_z(X, Y) = \Phi(Y) \times \mathcal{C}(X, Y)$. Composition of morphisms $(g_0, c_0) \in \mathcal{D}_z(X_0, X_1)$ and $(g_1, c_1) \in \mathcal{D}_z(X_1, X_2)$ is given by

$$(g_1, c_1) \circ (g_0, c_0) = (g_1 \cdot \Phi(c_1)(g_0) \cdot z(c_1, c_0), c_1 \circ c_0).$$

It is a standard calculation to show that composition defined in this way is unital and associative, hence making D_z a small category. The functor $\pi_z \colon \mathcal{D}_z \to \mathcal{C}$ is the identity on objects and the obvious projection on morphisms. The assignment $g \mapsto (g, 1_X)$ gives the maps $\delta_X\colon \Phi(X) \to \mathrm{Aut}_{\mathcal{D}_z}(X)$. It is easy to check that $\mathcal{D}_z$ is an extension of $\mathcal{C}$ by Φ.

Suppose that $z' \in \zeta$ is another regular 2-cocycle and let $D_{z'}$ be the associated extension. We claim that $\mathcal{D}_z$ and $\mathcal{D}_{z'}$ are equivalent extensions. Since z and z' are cohomologous there is $u \in C^1(\mathcal{C}, \Phi)$ such that $z = z' \cdot \delta(u)$. Regularity of z and z' implies that $u(1_X) = 1$ for all $X \in \mathcal{C}$ (by looking at the 2-chain $X \xrightarrow{\mathrm{id}} X \xrightarrow{\mathrm{id}} X$). Define $\Psi\colon \mathcal{D}_z \to \mathcal{D}_{z'}$ as the identity on objects, and on morphisms

$$\Psi\colon (g, c) \mapsto (u(c) \cdot g, c).$$

Then Ψ respects identity morphisms since $u(1_X) = 1$. One easily checks it respects compositions because $z(c_1, c_0) = z'(c_1, c_0) \cdot u(c_1) \cdot u(c_1 \circ c_0)^{-1} \cdot c_{1*}(u(c_0))$. Finally, $\overline{\Psi} = \mathrm{Id}_{\mathcal{C}}$ as is evident from the definitions and $\eta(\Psi) = \mathrm{Id}$. Therefore $\mathcal{D}_z$ and $\mathcal{D}_{z'}$ define equivalent extensions. This shows that a map $\Sigma\colon H^2(\mathcal{C}, \Phi) \to \mathrm{Ext}(\mathcal{C}, \Phi)$ is well defined. Notice that $\sigma\colon \mathcal{C}_1 \to (\mathcal{D}_z)_1$ defined by $c \mapsto (1, c)$ where $c \in \mathcal{C}(X_0, X_1)$ and 1 denotes the identity element of $\Phi(X_1)$ gives a regular section such that $z_\sigma = z$. This shows that $\Gamma \circ \Sigma = \mathrm{Id}$. If $\mathcal{E} = (\mathcal{D}, \mathcal{C}, \Phi, \pi, \delta)$ is an extension and $\sigma\colon \mathcal{C}_1 \to \mathcal{D}_1$ is a section then there is an equivalence $\Psi\colon \mathcal{D}_{z_\sigma} \to \mathcal{D}$ defined as the identity on objects and $\Psi\colon (g, c) \mapsto [\![g]\!] \circ \sigma(c)$ on morphisms. This shows that $\Sigma \circ \Gamma = \mathrm{Id}$. □

Proof of Proposition 6.25. Fix regular sections $\sigma\colon \mathcal{C}_1 \to \mathcal{D}_1$ and $\sigma'\colon \mathcal{C}'_1 \to \mathcal{D}'_1$ and let z_σ and $z_{z'_\sigma}$ be the associated 2-cocycles (6.19).

(i) $\implies$ (ii). Let $\Psi\colon \mathcal{E} \to \mathcal{E}'$ be a morphism of extensions such that $\overline{\Psi} = \psi$ and $\eta(\Psi) = \eta$. Then for any $c \in \mathcal{C}(X_0, X_1)$ we have $[\Psi(\sigma(c))] = \psi(c) = [\sigma'(\psi(c))]$ so $\Psi(\sigma(c)) = [\![g]\!] \circ \sigma'(\psi(c))$ for a unique $g \in \Phi'(\psi(c))$. Therefore there exists $u \in C^1(\mathcal{C}, \Phi' \circ \psi)$ such that

$$\Psi(\sigma(c)) = [\![u(c)]\!] \circ \sigma'(\psi(c)), \qquad (c \in \mathcal{C}_1). \tag{A.3}$$

Now, given a 2-chain $X_0 \xrightarrow{c_0} X_1 \xrightarrow{c_1} X_2$ in $\mathcal{C}$ we have

$$\begin{aligned}
&[\![\eta_{X_2}(z_\sigma(c_1, c_0))]\!] \circ [\![u(c_1 \circ c_0)]\!] \circ \sigma'(\psi(c_1 \circ c_0)) \overset{(A.3)}{=} [\![\eta_{X_2}(z_\sigma(c_1, c_0))]\!] \circ \Psi(\sigma(c_1 \circ c_0)) \overset{(6.11)}{=} \\
&\Psi\Big([\![z_\sigma(c_1, c_0)]\!] \cdot \sigma(c_1 \circ c_0)\Big) \overset{(6.19)}{=} \Psi(\sigma(c_1)) \circ \Psi(\sigma(c_0)) \overset{(A.3)}{=} \\
&[\![u(c_1)]\!] \circ \sigma'(\psi(c_1)) \circ [\![u(c_0)]\!] \circ \sigma'(\psi(c_0)) \overset{(6.2)}{=} [\![u(c_1)]\!] \circ [\![\Phi'(\psi(c_1))(u(c_0))]\!] \circ \sigma'(\psi(c_1)) \circ \sigma'(\psi(c_0)) \overset{(z_{\sigma'})}{=} \\
&[\![u(c_1)]\!] \circ [\![\Phi'(\psi(c_1))(u(c_0))]\!] \circ [\![z_{\sigma'}(\psi(c_1), \psi(c_0))]\!] \circ \sigma'(\psi(c_1 \circ c_0)).
\end{aligned}$$

Since $\Phi'(\psi(X_2))$ acts freely on $\mathcal{D}'(X_0, X_2)$ we deduce that

$$\eta_{X_2}(z_\sigma(c_1, c_0)) = z_{\sigma'}(\psi(c_1), \psi(c_0)) \cdot u(c_1) \cdot u(c_1 \circ c_0)^{-1} \cdot \Phi'(\psi(c_1))(u(c_0)).$$

This shows that $\eta_*(z_\sigma) = \psi^*(z_{\sigma'}) \cdot \delta(u)$, hence $\eta_*([\mathcal{D}]) = \psi^*([\mathcal{D}'])$.

(ii) $\implies$ (i). Assume that $\eta_*([\mathcal{D}]) = \psi^*([\mathcal{D}'])$. We will construct a morphism $\Psi\colon \mathcal{E} \to \mathcal{E}'$. By the hypothesis, there exists $u \in C^1(\mathcal{C}, \Psi' \circ \psi)$ such that $\eta_*(z_\sigma) = \psi^*(z_{\sigma'}) \cdot \delta(u)$. Notice that since z_σ and $z_{\sigma'}$ are regular 2-cocycles, $u(1_X) = 1$ for every $X \in \mathcal{C}$ (to see this, evaluate $\eta_*(z_\sigma)$ and $\psi^*(z_{\sigma'})$ on the 2-chain $X \xrightarrow{\mathrm{id}} X \xrightarrow{\mathrm{id}} X$).

Define $\Psi\colon \mathcal{D} \to \mathcal{D}'$ a follows. On objects $\Psi\colon X \mapsto \psi(X)$. Every morphism in $\mathcal{D}(X, Y)$ has the form $[\![g]\!] \circ \sigma(c)$ for unique $c \in \mathcal{C}(X, Y)$ and $g \in \Phi(Y)$. Define

$$\Psi\colon [\![g]\!] \circ \sigma(c) \mapsto [\![\eta_Y(g)]\!] \circ [\![u(c)]\!] \circ \sigma'(\psi(c)). \tag{A.4}$$

Then Ψ respects identities since $u(1_X) = 1$ and $\sigma'(1_X) = 1_X$. It respects composition as one verifies directly for $X_0 \xrightarrow{[\![g_0]\!] \circ \sigma(c_0)} X_1 \xrightarrow{[\![g_1]\!] \circ \sigma(c_1)} X_2$. On one hand,

$$\begin{aligned}
&\Psi([\![g_1]\!] \circ c_1) \circ \Psi([\![g_0]\!] \circ c_0) = [\![\eta_{X_2}(g_1) \cdot u(c_1)]\!] \circ \sigma'(\psi(c_1)) \circ [\![\eta_{X_1}(g_0) \cdot u(c_0)]\!] \circ \sigma'(\psi(c_0)) \overset{(6.2)}{=} \\
&[\![\eta_{X_2}(g_1) \cdot u(c_1) \cdot \Phi'(\psi(c_1))(\eta_{X_1}(g_0)) \cdot \Phi'(\psi(c_1))(u(c_0))]\!] \circ \sigma'(\psi(c_1)) \circ \sigma'(\psi(c_0)) \overset{\text{nat. of } \eta}{=} \\
&[\![\eta_{X_2}(g_1) \cdot u(c_1) \cdot \eta_{X_2}(\Phi(c_1)(g_0)) \cdot \Phi'(\psi(c_1))(u(c_0))]\!] \circ \sigma'(\psi(c_1)) \circ \sigma'(\psi(c_0)) \overset{(z_{\sigma'})}{=} \\
&[\![\eta_{X_2}(g_1) \cdot u(c_1) \cdot \eta_{X_2}(\Phi(c_1)(g_0)) \cdot \Phi'(\psi(c_1))(u(c_0)) \cdot z_{\sigma'}(\psi(c_1), \psi(c_0))]\!] \circ \sigma'(\psi(c_1 \circ c_0))
\end{aligned}$$

On the other hand

$$\Psi(\llbracket g_1\rrbracket \circ c_1 \circ \llbracket g_0\rrbracket \circ c_0) \overset{(6.2),(z_\sigma)}{=} \Psi(\llbracket g_1 \cdot \Phi(c_1)(g_0) \cdot z_\sigma(c_1,c_0)\rrbracket \circ \sigma(c_1 \circ c_0)) \overset{(6.11),(A.4)}{=}$$
$$\llbracket \eta_{X_2}(g_1) \cdot \eta_{X_2}(\Phi(c_1)(g_0)) \cdot \eta_{X_2}(z_\sigma(c_1,c_0))\rrbracket \circ \llbracket u(c_1 \circ c_0)\rrbracket \circ \sigma'(\psi(c_1 \circ c_0))$$

The right hand sides of the two equations are equal since $\eta_*(z_\sigma) = \psi^*(z_{\sigma'}) \cdot \delta(u)$. We have just shown that $\Psi\colon \mathcal{D} \to \mathcal{D}'$ is a functor. By its construction $\pi' \circ \Psi = \psi \circ \pi$ and $\eta(\Psi) = \eta$ because for any $g \in \Phi(X)$ we have $\Psi(\llbracket g\rrbracket \circ \sigma(1_X)) = \llbracket \eta_X(g) \cdot u(1_X)\rrbracket \circ \sigma'(\psi(1_X)) = \llbracket \eta_X(g)\rrbracket$ and use (6.11). $\square$

Proof of Proposition 6.26. For any 1-cocycle $z \in C^1(\mathcal{C};\Phi)$ define $\alpha_z\colon \mathcal{D} \to \mathcal{D}$ as the identity on objects and for any $d \in \mathcal{D}(X,Y)$

$$\alpha_z\colon d \mapsto \llbracket z([d])\rrbracket \circ d.$$

Since z is a 1-cocycle, $z(1_X) = 1$ for any $X \in \mathcal{C}$ and therefore α_z respects identity morphisms. It respects composition since z is a 1-cocycle:

$$\alpha_z(d_1) \circ \alpha_z(d_0) = \llbracket z([d_1])\rrbracket \circ d_1 \circ \llbracket z([d_0])\rrbracket \circ d_0 \overset{(6.2)}{=} \llbracket z([d_1]) \cdot [d_1]_*(z([d_0]))\rrbracket \circ d_1 \circ d_0 \overset{\text{1-cocycle}}{=}$$
$$\llbracket z([d_1 \circ d_0])\rrbracket \circ d_1 \circ d_0 = \alpha_z(d_1 \circ d_0).$$

It easily follows from (6.2) that given $u \in C^0(\mathcal{C},\Phi)$ we have $\alpha_{\delta(u)} = \tau_u$. This shows that Γ is well defined. The inverse of Γ sends every self equivalence Ψ to the 1-cochain $z \in C^1(\mathcal{C},\Phi)$ which is defined by the relation $\Psi(d) = \llbracket z([d]\rrbracket \circ d$ which follows from the fact that $[\Psi(d)] = \overline{\Psi}([d]) = [d]$. It is left as an exercise to check that z is a 1-cocycle and that $\alpha_z = \Psi$. $\square$

Appendix B. The unstable Adams operations in [JLL] are special

Fix a p-local compact group $\mathcal{G} = (S,\mathcal{F},\mathcal{L})$ and set $\mathcal{R} = \mathcal{H}^\bullet(\mathcal{F}^c)$. Let $\mathcal{P}$ be a set of representatives for the S-conjugacy classes in $\mathcal{R}$. By [BLO3, Lemma 3.2] the set $\mathcal{P}$ is finite. For any $P,Q \in \mathcal{P}$ let $\mathcal{M}_{P,Q}$ be a set of representatives for the orbits of $N_S(Q)$ on $\mathcal{L}(P,Q)$. These sets are finite by [BLO3, Lemma 2.5]. Also we remark that since every $R \in \mathcal{R}$ is $\mathcal{F}$-centric, $N_S(R)/R_0$ is finite.

In [JLL] we showed that there exists some m such that for any $\zeta \in \Gamma_m(p)$ we can construct an unstable Adams operation (Ψ,ψ) of degree ζ. This operation has the following properties

(1) $\psi(P) = P$ for any $P \in \mathcal{P}$.
(2) If $P \in \mathcal{P}$ then $\psi|_{N_S(P)}$ is an automorphism of $N_S(P)$ which induces the identity on $N_S(P)/P_0$.
(3) $\Psi(\varphi) = \varphi$ for every $\varphi \in \mathcal{M}_{P,Q}$ where $P,Q \in \mathcal{P}$.

In the rest of this section we show that any (Ψ,ψ) which satisfies these conditions must be special relative to $\mathcal{H}^\bullet(\mathcal{F}^c)$. As usual we will write T for the identity component of S.

For any $R \in \mathcal{H}^\bullet(\mathcal{F}^c)$ we fix once and for all $g_R \in S$ such that $R = g_R P g_R^{-1}$ where $P \in \mathcal{P}$ (clearly P is unique). For any $R,R' \in \mathcal{H}^\bullet(\mathcal{F}^c)$ and any $\varphi \in \mathcal{L}(R,R')$ we consider $P = g_R^{-1} R g_R$ and $P' = g_{R'}^{-1} R' g_{R'}$; these are the representatives in $\mathcal{P}$ for the S-conjugacy classes of R and R' respectively. There is a unique $\mu \in \mathcal{M}_{P,P'}$ and a unique $n_\varphi \in N_S(P')$ such that

$$\widehat{g_{R'}^{-1}} \circ \varphi \circ \widehat{g_R} = \widehat{n_\varphi} \circ \mu. \tag{B.1}$$

Since ψ induces the identity on S/T, there is a unique $\tau_R \in T$ such that

$$\psi(g_R) = \tau_R \cdot g_R.$$

Similarly, with the notation above ψ induces the identity on $N_S(P')/P'_0$ and therefore there exists a unique $t_\varphi \in P'_0$ such that

$$\psi(n_\varphi) = t_\varphi \cdot n_\varphi.$$

Notice that $g_{R'} \in N_S(P',R')$ and therefore conjugation by g_R carries P'_0 onto R'_0. Set

$$\tau_\varphi = g_{R'} \cdot t_\varphi \cdot g_{R'}^{-1} \in R'_0.$$

We now claim that τ_R and τ_φ chosen above render (Ψ,ψ) a special unstable Adams operation relative to $H^\bullet(\mathcal{F}^c)$, see Definition 7.1.

First, for any $R \in H^\bullet(\mathcal{F}^c)$,

$$\psi(R) = \psi(g_R P g_R^{-1}) = \psi(g_R) P \psi(g_R)^{-1} = \tau_R g_R P g_R^{-1} \tau_R^{-1} = \tau_R R \tau_R^{-1}.$$

Next, if $R, R' \in \mathcal{H}^\bullet(\mathcal{F}^c)$ and $\varphi \in \mathcal{L}(R, R')$ then (B.1) implies

$$\Psi(\varphi) = \Psi(\widehat{g_{R'}} \circ \widehat{n_\varphi} \circ \mu \circ \widehat{g_R^{-1}}) = \widehat{\psi(g_{R'})} \circ \widehat{\psi(n_\varphi)} \circ \Psi(\mu) \circ \widehat{\psi(g_R^{-1})} =$$

$$\widehat{\tau_{R'}} \circ \widehat{g_{R'}} \circ \widehat{t_\varphi} \circ \widehat{n_\varphi} \circ \mu \circ \widehat{g_R^{-1}} \circ \widehat{\tau_R^{-1}} = \widehat{\tau_{R'}} \circ \widehat{\tau_\varphi} \circ \widehat{g_{R'}} \circ \widehat{n_\varphi} \circ \mu \circ \widehat{g_R^{-1}} \circ \widehat{\tau_R^{-1}} = \widehat{\tau_{R'}} \circ \widehat{\tau_\varphi} \circ \varphi \circ \widehat{\tau_R^{-1}}$$

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