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Joint Hypothesis Tests for Multidimensional Inequality Indices

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Abstract

An inequality index over p dimensions of well-being is decomposable by attributes if it can expressed as a function of p unidimensional inequality indices and a measure of association between the various dimensions of well-being. We exploit this decomposition framework to derive joint hypothesis tests regarding the sources of multidimensional inequality, and present Monte Carlo evidence on their finite sample behavior.

Keywords: multidimensional inequality indices, large sample distributions, joint hypothesis tests, bootstrap.

JEL codes: $D63, C43$

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Economists often study inequality in well-being using indices that satisfy known sets of ethical axioms and/or statistical properties. The use of unidimensional inequality indices in the analysis of income distributions is well-established (e.g. Cowell and Flachaire, 2007). Recently, multidimensional indices that measure inequality in the joint distribution of several attributes (e.g. income and health) have also received attention, in recognition of well-being as a multidimensional concept (e.g. Justino et al., 2004; Zhong, 2009; Abu-Zaineh and Abul Naga, 2013).

There is an important class of multidimensional inequality indices that satisfy attribute decomposability (Abul Naga and Geoffard 2006; Kobus, 2012). This property allows a multidimensional index to be disaggregated as a function of unidimensional indices for individual attributes and a measure of association between attributes. Intergroup or temporal variations in the overall multidimensional inequality can therefore be traced to variations in its unidimensional inequality and association measures. The related statistical inference naturally entails joint hypothesis tests on particular subsets of those components. But the inferential framework has not been developed yet, and empirical results to date are presented without statistical tests.

This paper complements the existing literature on single hypothesis tests on unidimensional indices (e.g. Davidson and Flachaire, 2007) and the aggregated forms of multidimensional indices (Abul Naga, 2010), by developing a framework for testing joint hypotheses on unidimensional indices and an association measure that arise from decomposing a multidimensional index. Our Monte Carlo evidence suggests that, combined with bootstrapping, the proposed chi-squared tests provide reliable tools for drawing finite sample inferences.

1 Attribute decomposable inequality indices

Consider data on p attributes of well-being in a sample of n individuals. Individual i has resources $\boldsymbol{x}_i := (x_{i1}, ..., x_{ip})$, where $\boldsymbol{x}_i \in \mathbb{R}^p_{++}$. We gather the data in a matrix $\boldsymbol{X} := \begin{bmatrix} \boldsymbol{x}_1 \\ \vdots \\ \boldsymbol{x}_n \end{bmatrix} \in \mathbb{R}^{n\times p}_{++}$, and let $\bar{\boldsymbol{x}} := \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{x}_i = (\bar{x}_1, ..., \bar{x}_p)$ denote the vector of sample means.

Let $\hat{\imath} : \mathbb{R}^{n\times p}_{++} \to \mathbb{R}_+$ denote a multidimensional inequality index, $W : \mathbb{R}^{n\times p}_{++} \to \mathbb{R}$ be the social welfare function underlying the derivation of $\hat{\imath}$, and $\omega := W(\boldsymbol{X})$ be the level of welfare attained by $\boldsymbol{X}$. If W satisfies the standard axioms of anonymity, additivity across individuals, continuity, increasing monotonicity and equality preference, one may define an increasing and concave utility function $u(.)$ such that $W(\boldsymbol{X}) =$

$\frac{1}{n}\sum_{i=1}^n u(\boldsymbol{x}_i)$, and a scalar $\hat{\theta}(\boldsymbol{X})$ in the unit interval such that $u(\hat{\theta}\bar{\boldsymbol{x}}) = \frac{1}{n}\sum_{i=1}^n u(\boldsymbol{x}_i)$. The scalar $\hat{\theta}$ is an index of multidimensional *equality* in $\boldsymbol{X}$, and $\hat{\imath}(\boldsymbol{X}) := 1 - \hat{\theta}(\boldsymbol{X})$ is the corresponding inequality index.

Assume furthermore that W is scale-invariant, and consider for expositional simplicity the case of two attributes ($p = 2$) of well-being.[1] Then the utility function underlying the definition of W takes one of three possible forms (Aczél, 1988; ch. 4, Corollary 4):

$$u(x_{i1}, x_{i2}) \ : \ = x_{i1}^{\alpha} x_{i2}^{\beta} \qquad \alpha, \beta > 0, \qquad \alpha + \beta \leq 1 \tag{1}$$

$$u(x_{i1}, x_{i2}) \ : \ = -x_{i1}^{\alpha} x_{i2}^{\beta} \qquad \alpha, \beta < 0 \tag{2}$$

$$u(x_{i1}, x_{i2}) \ : \ = \alpha \ln x_{i1} + \beta \ln x_{i2} \qquad \alpha, \beta > 0 \tag{3}$$

A multidimensional Atkinson-Kolm-Sen (mAKS) inequality index arises from (1) and (2), while a multidimensional mean-logarithmic deviation (mMLD) inequality index arises from (3) (Tsui, 1995).

Abul Naga and Geoffard (2006) show that both mAKS and mMLD indices are decomposable by attributes. The mAKS *equality* index can be written as a function of three components, $\hat{\theta}_{mAKS}(\boldsymbol{X}) = \exp(\frac{\alpha}{(\alpha+\beta)} \ln \hat{\delta}_1 + \frac{\beta}{(\alpha+\beta)} \ln \hat{\delta}_2 + \frac{1}{(\alpha+\beta)} \ln \hat{\delta}_3)$ where

$$\hat{\delta}_1 = \left(\frac{1}{n}\sum_{i=1}^n x_{i1}^{\alpha}\right)^{1/\alpha} / \bar{x}_1 \tag{4}$$

$$\hat{\delta}_2 = \left(\frac{1}{n}\sum_{i=1}^n x_{i2}^{\beta}\right)^{1/\beta} / \bar{x}_2 \tag{5}$$

$$\hat{\delta}_3 = \frac{\frac{1}{n}\sum_{i=1}^n x_{i1}^{\alpha} x_{i2}^{\beta}}{\left(\frac{1}{n}\sum_{i=1}^n x_{i1}^{\alpha}\right)\left(\frac{1}{n}\sum_{i=1}^n x_{i2}^{\beta}\right)}. \tag{6}$$

$\hat{\theta}_{mAKS}(\boldsymbol{X})$ is the *aggregated form* and $\hat{\boldsymbol{\delta}} = (\hat{\delta}_1 \ \hat{\delta}_2 \ \hat{\delta}_3)'$ the *disaggregated form* of the mAKS index. $\hat{\delta}_1$ ($\hat{\delta}_2$) is the unidimensional AKS index of *equality* in the first (second) attribute, and $\hat{\delta}_3$ is a measure of association between the two attributes.[2] The

[1] Let $\boldsymbol{Y} \in \mathbb{R}_{++}^{n \times p}$ be another data matrix and Λ be a $p \times p$ positive-definite diagonal matrix. $W(.)$ is scale-invariant when $W(\boldsymbol{X}) = W(\boldsymbol{Y})$ if and only if $W(\boldsymbol{X}\boldsymbol{\Lambda}) = W(\boldsymbol{Y}\boldsymbol{\Lambda})$.

[2] As in the multidimensional case, $1 - \widehat{\delta}_1$ and $1 - \widehat{\delta}_2$ are the corresponding inequality indices.

mMLD *equality* index can be written as a function of two components, $\hat{\theta}_{mMLD}(\boldsymbol{X}) = \exp(\frac{\alpha}{(\alpha+\beta)} \ln \hat{\delta}_1 + \frac{\beta}{(\alpha+\beta)} \ln \hat{\delta}_2)$, where

$$\hat{\delta}_j = \exp\left[\frac{1}{n}\sum_{i=1}^{n} \ln(x_{ij}) - \ln \bar{x}_j\right] \qquad j = 1, 2. \tag{7}$$

$\hat{\delta}_1$ and $\hat{\delta}_2$ are unidimensional MLD *equality* indices. The mMLD index is *strongly decomposable* à la Kobus (2012), because $\hat{\theta}_{mMLD}$ is entirely characterized by the two equality indices pertaining to each attribute's marginal distribution. Since the mAKS and mMLD inequality indices are $\widehat{\iota}_{mAKS}(\boldsymbol{X}) := 1 - \hat{\theta}_{mAKS}(\boldsymbol{X})$ and $\widehat{\iota}_{mMLD}(\boldsymbol{X}) := 1 - \hat{\theta}_{mMLD}(\boldsymbol{X})$, they are also decomposable by attributes.

2 Large sample distribution

Let $\hat{\boldsymbol{\delta}}(\boldsymbol{X}) = (\hat{\delta}_1, ..., \hat{\delta}_k)'$ denote the vector of equality indices and measure of association in the sample.[3] For estimation and inference, it is convenient to define $\hat{\boldsymbol{\delta}}(\boldsymbol{X})$ in relation to m sample moments of $\boldsymbol{X}$,

$$\boldsymbol{s} := \left[\begin{array}{ccc} \frac{1}{n}\sum_{i=1}^{n} g_1(\boldsymbol{x}_i) & \cdots & \frac{1}{n}\sum_{i=1}^{n} g_m(\boldsymbol{x}_i) \end{array}\right] \tag{8}$$

via a function $\boldsymbol{F} : \mathbb{R}^m \longrightarrow \mathbb{R}^k$ such that $\hat{\boldsymbol{\delta}} = \boldsymbol{F}(\boldsymbol{s})$. One can then define the population indices $\boldsymbol{\delta}_o$ in relation to m population moments,

$$\boldsymbol{\sigma}_o := \left[\begin{array}{ccc} E[g_1(\boldsymbol{x}_i)] & \cdots & E[g_m(\boldsymbol{x}_i)] \end{array}\right] \tag{9}$$

such that $\boldsymbol{\delta}_o = \boldsymbol{F}(\boldsymbol{\sigma}_o)$. For instance, in the context of the mAKS index we have

$$\boldsymbol{s} := \left[\begin{array}{ccccc} \bar{x}_1 & \bar{x}_2 & \frac{1}{n}\sum_{i=1}^{n} x_{i1}^{\alpha} x_{i2}^{\beta} & \frac{1}{n}\sum_{i=1}^{n} x_{i1}^{\alpha} & \frac{1}{n}\sum_{i=1}^{n} x_{i2}^{\beta} \end{array}\right] \tag{10}$$

$$\boldsymbol{\sigma}_o := \left[\begin{array}{ccccc} E(x_1) & E(x_2) & E(x_1^{\alpha} x_2^{\beta}) & E(x_1^{\alpha}) & E(x_2^{\beta}) \end{array}\right] \tag{11}$$

and in the context of the mMLD index,

$$\boldsymbol{s} := \left[\begin{array}{cccc} \bar{x}_1 & \bar{x}_2 & \frac{1}{n}\sum_{i=1}^{n} \ln(x_{i1}) & \frac{1}{n}\sum_{i=1}^{n} \ln(x_{i2}) \end{array}\right] \tag{12}$$

[3] In general, $k = p + 1$ as the decomposition produces p equality indices together with an association measure. However, $k = p$ when the multidimensional index is strongly decomposable (as is the case in the context of $\hat{\theta}_{mMLD}$).

$$\boldsymbol{\sigma}_o := \left[\begin{array}{cccc} E(x_1) & E(x_2) & E(\ln x_1) & E(\ln x_2) \end{array} \right]. \tag{13}$$

Let $\bar{g}_j$ denote the jth component of $\boldsymbol{s}$, i.e. $\bar{g}_j = \sum_{i=1}^{n} g_j(\boldsymbol{x}_i)/n$, and $\boldsymbol{Z}$ be an $n \times m$ matrix where

$$\boldsymbol{Z} := \left[\begin{array}{ccc} g_1(\boldsymbol{x}_1) - \bar{g}_1 & \cdots & g_m(\boldsymbol{x}_1) - \bar{g}_m \\ \vdots & & \vdots \\ g_1(\boldsymbol{x}_n) - \bar{g}_1 & \cdots & g_m(\boldsymbol{x}_n) - \bar{g}_m \end{array} \right]. \tag{14}$$

Define also the $k \times m$ Jacobian matrix $\boldsymbol{J} := \partial \boldsymbol{F} / \partial \boldsymbol{\sigma}$,

$$\boldsymbol{J} := \left(\begin{array}{ccc} \partial \boldsymbol{F}_1/\partial\sigma_1 & \cdots & \partial \boldsymbol{F}_1/\partial\sigma_m \\ & \vdots & \\ \partial \boldsymbol{F}_k/\partial\sigma_1 & \cdots & \partial \boldsymbol{F}_k/\partial\sigma_m \end{array} \right). \tag{15}$$

where σ_j refers to the jth component of $\boldsymbol{\sigma}_o$.

Under appropriate assumptions, the Continuous Mapping Theorem ensures that $\hat{\boldsymbol{\delta}} = \boldsymbol{F}(\boldsymbol{s})$ is a consistent estimator of $\boldsymbol{\delta}_o = \boldsymbol{F}(\boldsymbol{\sigma}_o)$. Specifically, consider the following assumptions:

[$A1$] The observations $\boldsymbol{x}_1, ..., \boldsymbol{x}_n$ are independently and identically distributed with cross-moment vector $\boldsymbol{\sigma}_o$ where $\boldsymbol{\sigma}_o$ is a finite m-dimensional vector.

[$A2$] $\text{plim}\left(\frac{1}{n}\boldsymbol{Z}'\boldsymbol{Z}\right) = \boldsymbol{V}_o$, where $\boldsymbol{V}_o$ is a finite $m \times m$ positive definite matrix.

[$A3$] The vector function $\boldsymbol{F}(\boldsymbol{\sigma})$ does not involve n and is continuously differentiable at the point $\boldsymbol{\sigma}_o$.

[$A4$] The Jacobian matrix $\boldsymbol{J}$ has full rank $k \leq m$ at $\boldsymbol{\sigma}_o$.

Proposition 1 *Under Assumptions* [$A1 - A4$], $\hat{\boldsymbol{\delta}}(\boldsymbol{X}) = \boldsymbol{F}(\boldsymbol{s})$ *converges in distribution to a* $k-$variate *normal vector such that*:

$$n^{1/2}[\boldsymbol{F}(\boldsymbol{s}) - \boldsymbol{F}(\boldsymbol{\sigma}_o)] \longrightarrow \mathcal{N}(\boldsymbol{0}, \boldsymbol{J}_o \boldsymbol{V}_o \boldsymbol{J}_o') \tag{16}$$

where $\boldsymbol{J}_o$ *is the Jacobian matrix given by*

$$\boldsymbol{J}_o := \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{\sigma}} \mid_{\boldsymbol{\sigma}=\boldsymbol{\sigma}_o}$$

Proof From [$A1$], the Law of Large Numbers implies that $\text{plim}(\boldsymbol{s}) = \boldsymbol{\sigma}_o$ and the Central Limit Theorem ensures that $n^{1/2}(\boldsymbol{s} - \boldsymbol{\sigma}_o)$ converges to a normal distribution, $\mathcal{N}(\boldsymbol{0}, \boldsymbol{V}_o)$. Given [$A3$], the delta method applies and $n^{1/2}[\boldsymbol{F}(\boldsymbol{s}) - \boldsymbol{F}(\boldsymbol{\sigma}_o)]$ converges to a multivariate normal distribution $\mathcal{N}(\boldsymbol{0}, \boldsymbol{J}_o \boldsymbol{V}_o \boldsymbol{J}_o')$, which is non-degenerate due to [$A2$] and [$A4$]. $\square$

2.1 Covariance matrices

The formula $\boldsymbol{J}_o\boldsymbol{V}_o\boldsymbol{J}_o'$ of the asymptotic covariance matrix (16) is generally applicable to all attribute decomposable indices, though the specific components of $\boldsymbol{V}_o$ and $\boldsymbol{J}_o$ would vary from index to index depending on the underlying moments and the functional form of $\boldsymbol{F}$. In empirical work, one can construct a consistent estimator of $\boldsymbol{J}_o\boldsymbol{V}_o\boldsymbol{J}_o'$ using the matrix $\hat{\boldsymbol{J}}\hat{\boldsymbol{V}}\hat{\boldsymbol{J}}'$ where

$$\hat{\boldsymbol{V}} \ : \ = \boldsymbol{Z}'\boldsymbol{Z}/n \tag{17}$$

$$\hat{\boldsymbol{J}} \ : \ = \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{\sigma}} \,|_{\boldsymbol{\sigma}=\boldsymbol{s}} \ . \tag{18}$$

Constructing $\hat{\boldsymbol{J}}$ however requires an analytic derivation of the Jacobian $\boldsymbol{J} := \partial \boldsymbol{F}/\partial \boldsymbol{\sigma}$, prior to its evaluation at $\boldsymbol{\sigma} = \boldsymbol{s}$. Our purpose here is to derive the Jacobian matrices of the disaggregated forms of the mAKS and mMLD families, in the context of two attributes of well-being. For the mAKS family (4–6), $\boldsymbol{J}$ is the following rank three matrix

$$\boldsymbol{J} = \begin{bmatrix} -\frac{\sigma_4^{1/\alpha}}{\sigma_1^2} & 0 & 0 & \frac{1}{\alpha}\frac{\sigma_4^{(1/\alpha-1)}}{\sigma_1} & 0 \\ 0 & -\frac{\sigma_5^{1/\beta}}{\sigma_2^2} & 0 & 0 & \frac{1}{\beta}\frac{\sigma_5^{(1/\beta-1)}}{\sigma_2} \\ 0 & 0 & \frac{1}{\sigma_4\sigma_5} & -\frac{\sigma_3}{\sigma_4^2\sigma_5} & -\frac{\sigma_3}{\sigma_4\sigma_5^2} \end{bmatrix}. \tag{19}$$

For the mMLD family (7) the Jacobian $\boldsymbol{J}$ is the following rank two matrix

$$\boldsymbol{J} = \begin{bmatrix} -\frac{\exp(\sigma_3 - \ln\sigma_1)}{\sigma_1} & 0 & \exp(\sigma_3 - \ln\sigma_1) & 0 \\ 0 & -\frac{\exp(\sigma_4 - \ln\sigma_2)}{\sigma_2} & 0 & \exp(\sigma_4 - \ln\sigma_2) \end{bmatrix}. \tag{20}$$

2.2 Test statistics

Proposition 1 provides a basis for carrying out Wald-type joint hypothesis tests on the vector of k indices $\boldsymbol{\delta}_o$. Let $\boldsymbol{\eta}$ be an $l \times 1$ vector and let $\boldsymbol{A}$ denote a matrix that satisfies the following assumption:

[$A5$] $\boldsymbol{A}$ is a $l \times k$ non-random matrix of rank $l \leq k$.

Consider a test of $l \leq k$ hypotheses of the form $H_0 : \boldsymbol{A\delta}_o = \boldsymbol{\eta}$. The corresponding test statistic is of the form

$$\varsigma = \left(\boldsymbol{A}\hat{\boldsymbol{\delta}} - \boldsymbol{\eta}\right)' \left[\boldsymbol{A}\left(\frac{1}{n}\hat{\boldsymbol{J}}\hat{\boldsymbol{V}}\hat{\boldsymbol{J}}'\right)\boldsymbol{A}'\right]^{-1}\left(\boldsymbol{A}\hat{\boldsymbol{\delta}} - \boldsymbol{\eta}\right). \tag{21}$$

Next consider a joint hypothesis involving two different population vectors, for example as in an analysis of urban-rural differences in inequality structures. Let superscript a (b) denote parameters and estimates specific to population a (b). For a test of the form $H_0 : \boldsymbol{A}(\boldsymbol{\delta}_o^a - \boldsymbol{\delta}_o^b) = \boldsymbol{\eta}$, consider the test statistic

$$\begin{aligned}\tau \;=\; & \left(\boldsymbol{A}\left(\hat{\boldsymbol{\delta}}^a - \hat{\boldsymbol{\delta}}^b\right) - \boldsymbol{\eta}\right)' \left[\boldsymbol{A}\left(\frac{1}{n^a}\hat{\boldsymbol{J}}^a\widehat{\boldsymbol{V}^a}\hat{\boldsymbol{J}}^{a\prime} + \frac{1}{n^b}\hat{\boldsymbol{J}}^b\widehat{\boldsymbol{V}^b}\hat{\boldsymbol{J}}^{b\prime}\right)\boldsymbol{A}'\right]^{-1} \cdot \\ & \cdot \left(\boldsymbol{A}\left(\hat{\boldsymbol{\delta}}^a - \hat{\boldsymbol{\delta}}^b\right) - \boldsymbol{\eta}\right).\end{aligned} \tag{22}$$

On the basis of [$A5$] and Proposition 1, we readily obtain:

Corollary 2 *Under Assumptions* [$A1 - A5$]*, each of the test statistics* ς *and* τ *is asymptotically distributed as a* $\chi^2(l)$ *variate.*

3 Monte Carlo experiments

Our Monte Carlo study considers a multidimensional inequality index defined over household expenditure (EXP) and nutritional health (HLT), and a joint hypothesis test comparing urban and rural populations. We base our Monte Carlo populations on 1066 women (374 from urban areas, 692 from rural areas) from The Egyptian Integrated Household Survey (EIHS) 1997-1999: each urban (rural) observation i, $\boldsymbol{x}_i := (EXP_i, HLT_i)$, is independently drawn from a discrete uniform distribution whose support points are data points in the urban (rural) EIHS sample.

Table 1 summarizes the population inequality structures for our experiments. The mAKS indices vary with parameters α and β, where a larger absolute value of α (β) reflects greater aversion to inequality in EXP (HLT). We explore several alternative configurations of α and β based on the existing empirical applications (Zhong, 2009; Abu-Zaineh and Abul Naga, 2013).

We consider two different joint hypotheses of the form (22). H_0^{ALL} states that the urban-rural difference in inequality structures is as shown in Table 1. For instance, in the context of the mAKS index with $(\alpha, \beta) = (-0.5, -0.5)$, A is an identity matrix, and $\eta = (0.38 - 0.50 \;\; 0.03 - 0.02 \;\; 1.00 - 1.00)' = (-0.12 \;\; 0.01 \;\; 0.00)'$. The resulting test statistic is asymptotically distributed as a $\chi^2(3)$ variate in the context of mAKS indices and a $\chi^2(2)$ variate in the mMLD family. The hypothesis H_0^{-EXP} states that the urban-rural difference of the HLT component and that of the association component are as shown in Table 1: it is only relevant to the mAKS indices and leads to an asymptotic $\chi^2(2)$ statistic.

We use three different types of p-values to test each hypothesis. These are constructed from (*i*) an asymptotic test (ASYM), (*ii*) a bootstrap test (BOOT), and (*iii*) a fast-double bootstrap test (FDB). BOOT utilizes critical values generated from a non-parametric bootstrapping scheme as described in Horowitz (2001, p.3181), and offers asymptotic refinement over ASYM.[4] Davidson and MacKinnon (2007) suggest that the FDB test sometimes achieves better finite sample accuracy than BOOT, as approximation errors from two stages of bootstrapping offset each other.

Table 2 reports the empirical rejection frequency of each test at the nominal rejection probability of 0.05. Each empirical rejection frequency is calculated using 1000 Monte Carlo samples of $n = 1066$ observations, comprising 374 observations from the urban population and 692 from the rural population. The BOOT and FDB procedures use $B = 999$ bootstrapping repetitions. Our findings can be summarized as follows:

1. ASYM is correctly sized for the mMLD index, but is over-sized for mAKS indices except when $|\alpha| < 1$. The log-transformation of expenditure and height data involved in the calculation of the mMLD index may explain the good finite sample behavior of ASYM.

2. BOOT however does allow a correctly sized test on mAKS indices in most cases. Even where it does not, it achieves substantial accuracy gains over ASYM. FDB does only as well as BOOT, and sometimes worse.

3. The size accuracy of ASYM deteriorates as $|\alpha|$ and $|\beta|$ increase. It is more sensitive to changes in α than to changes in β. Comparing H_0^{ALL} and H_0^{-EXP}, the influence of α on the size inaccuracy of ASYM (that operates through the association measure) is also substantial, even when the null excludes the expenditure data.

[4] This scheme generates a bootstrap sample by randomly drawing with replacement from the estimation sample. A bootstrap sample's test statistic is computed by comparing the bootstrap sample's inequality structure against the estimation sample's inequality structure, not against the population inequality structure. The quantiles of the distribution of bootstrap test statistics computed in this manner provide critical values for BOOT. As Horowitz (2001) summarizes, for an asymptotically chi-squared distributed test statistic, the approximation error of BOOT is $O(n^{-2})$ whereas that of ASYM is $O(n^{-1})$.

As Schluter and van Garderen (2009) recommend in the unidimensional context, it thus seems prudent to select the smallest values of $|\alpha|$ and $|\beta|$ from a set of acceptable configurations. Existing empirical applications of mAKS indices (Zhong, 2009; Abu-Zaineh and Abul Naga, 2013) focus on $|\alpha| \leq 2$ and $|\beta| \leq 4$: for such configurations, BOOT addresses the potential size inaccuracy of ASYM adequately. In the online appendix, we report results based on other conventional nominal rejection probabilities (0.10 and 0.01), and samples of $n = 356$ ($\approx 1/3 \times 1066$) and $n = 3198$ ($= 3 \times 1066$) observations.[5] These results convey similar major lessons as Table 2.

Overall, we may conclude from our Monte Carlo experiments that, combined with nonparametric bootstrapping, our proposed chi-square tests provide reliable tools for finite sample inference.

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Table 1: Monte Carlo population characteristics and inequality indices

A. Population characteristics

	Urban population			Rural population		
	EXP	HLT	Height	EXP	HLT	Height
Mean	908.91	121.03	157.82	807.18	130.75	157.13
Std. Dev.	621.56	21.98	6.32	579.98	20.14	5.43
C.V.	0.68	0.18	0.04	0.72	0.15	0.04

B. Population inequality structures

	Urban population			Rural population		
	ι_{EXP}	ι_{HLT}	δ_{ASC}	ι_{EXP}	ι_{HLT}	δ_{ASC}
mMLD	0.22	0.02	N/A	0.26	0.01	N/A
mAKS: (-0.5,-0.5)	0.38	0.03	1.00	0.50	0.02	1.00
mAKS: (-1.0,-1.0)	0.58	0.04	0.98	0.75	0.03	1.00
mAKS: (-2.0,-2.0)	0.85	0.06	0.83	0.93	0.05	0.92
mAKS: (-0.5,-1.0)	0.38	0.04	1.00	0.50	0.03	1.00
mAKS: (-1.0,-2.0)	0.58	0.06	0.95	0.75	0.05	0.99
mAKS: (-2.0,-4.0)	0.85	0.10	0.68	0.93	0.10	0.77
mAKS: (-1.0,-0.5)	0.58	0.03	0.99	0.75	0.02	1.00
mAKS: (-2.0,-1.0)	0.85	0.04	0.91	0.93	0.03	0.96
mAKS: (-4.0,-2.0)	0.95	0.06	0.64	0.98	0.05	0.82

Based on adult female respondents of the Egyptian Integrated Household Survey 1997-1999. EXP is monthly non-durable household expenditure (in the Egyptian pounds) divided by the square-root of household size. HLT is BMI-adjusted height, a measure of nutritional health. Height is raw height in centimeters. Std. Dev. and C.V. are the standard deviation and coefficient of variation respectively. ι_{EXP} (ι_{HLT}) is the unidimensional index of inequality in EXP (HLT). δ_{ASC} is the mAKS measure of association between EXP and HLT. mAKS: (α,β) denotes mAKS indices where α and β are the inequality aversion parameters on EXP and HLT respectively.

Table 2: Empirical rejection frequency at the nominal significance level of 5%

	H_0^{ALL}			H_0^{-EXP}		
	ASYM	BOOT	FDB	ASYM	BOOT	FDB
mMLD	0.05	0.04	0.05	N/A	N/A	N/A
mAKS: (-0.5,-0.5)	0.06	0.05	0.05	0.05	0.05	0.05
mAKS: (-1.0,-1.0)	0.09	0.04	0.04	0.07	0.04	0.05
mAKS: (-2.0,-2.0)	0.19	0.06	0.07	0.13	0.06	0.07
mAKS: (-0.5,-1.0)	0.06	0.05	0.05	0.06	0.04	0.05
mAKS: (-1.0,-2.0)	0.08	0.04	0.05	0.07	0.04	0.04
mAKS: (-2.0,-4.0)	0.19	0.05	0.06	0.13	0.04	0.05
mAKS: (-1.0,-0.5)	0.09	0.05	0.05	0.07	0.05	0.05
mAKS: (-2.0,-1.0)	0.20	0.07	0.08	0.14	0.07	0.07
mAKS: (-4.0,-2.0)	0.42	0.10	0.13	0.32	0.07	0.10

The null hypothesis H_0^{ALL} states the true urban-rural difference in each of ι_{EXP}, ι_{HLT}, and δ_{ASC}. H_0^{-EXP} states the true urban-rural difference in each of ι_{HLT} and ι_{ASC}. ASYM, BOOT, FDB are rejection frequencies based on the asymptotic, bootstrap and fast-double bootstrap p-values respectively. Each rejection frequency is calculated using 1000 samples of 374 urban observations and 692 rural observations.