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SPATIAL SIGN CORRELATION

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ABSTRACT. A new robust correlation estimator based on the spatial sign covariance matrix (SSCM) is proposed. We derive its asymptotic distribution and influence function at elliptical distributions. Finite sample and robustness properties are studied and compared to other robust correlation estimators by means of numerical simulations.

1. INTRODUCTION

The problem studied in this article is robust and high-dimensional correlation estimation. By *robust* we mean insusceptible to outliers and erroneous observations, that is, we examine alternatives to the commonly used, but highly non-robust Pearson correlation. Over the last few decades, many robust multivariate scatter estimators — and consequently robust correlation matrix estimators — have been proposed, see Maronna et al. (2006) for a review. Much attention has been paid to affine equivariant estimators. If we denote by $\mathbb{X}_n = (\mathbf{X}_1, \dots, \mathbf{X}_n)^T$ the $n \times p$ data matrix containing the p -dimensional observations $\mathbf{X}_1, \dots, \mathbf{X}_n$ as rows, then the data set $\mathbb{Y}_n = \mathbb{X}_n A^T + \mathbf{1}_n \mathbf{b}^T$ is obtained by applying the affine linear transformation $\mathbf{x} \mapsto A\mathbf{x} + \mathbf{b}$ to each data point. An affine equivariant scatter estimator \hat{S}_n satisfies $\hat{S}_n(\mathbb{Y}_n) = A\hat{S}_n(\mathbb{X}_n)A^T$ for any $\mathbf{b} \in \mathbb{R}^p$ and any full rank square matrix A , i.e. it behaves as the covariance matrix under linear transformations of the data.

The second attribute *high-dimensional* means two things: being fast to compute, also in high-dimensions, and being defined also for sparse, high-dimensional data, i.e. in the $p > n$ situation. Both properties basically prohibit robust, affine equivariant estimators: they are usually hard to compute in high dimensions, and they are not defined in the $p > n$ setting — or coincide with a multiple of the sample covariance matrix (Tyler, 2010) and are thus not robust. In fact, both requirements suggest the use of pairwise correlation estimators. In a pairwise correlation estimate $\hat{P}_n \in \mathbb{R}^{p \times p}$ each entry $\hat{\rho}_{i,j}$ is computed only from the i th and the j th coordinate of the data, implying that the computing time increases quadratically with p .

The price one usually has to pay for dropping affine equivariance and resorting to pairwise correlation estimators is the loss of non-negative definiteness of the matrix estimate \hat{P}_n . For example, many nonparametric correlation matrix estimators (see Section 4) are based on an initial scatter matrix estimate which is non-negative definite, but not affine equivariant. The loss of non-negative definiteness occurs when a component-wise transformation is applied to render the entries consistent for the moment correlation. However, in applications where non-negative definiteness is important, one can “orthogonalize” the matrix estimate as suggested by Maronna and Zamar (2002), which involves an eigenvalue decomposition.

The new proposal is based on the spatial sign covariance matrix (SSCM). This is the covariance matrix of the projections of the centered observations onto the p -dimensional unit sphere. This scatter estimator is of frequent use in multivariate data analysis due to its robustness. Since every observation is basically trimmed to length 1, the impact of any contamination is bounded. It is known that within symmetric data models, the SSCM consistently estimates the eigenvectors of the covariance matrix, but not the eigenvalues. In fact, the connection between the eigenvalues of the population SSCM and the covariance matrix is an open problem. We solve this problem for

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the special case of two-dimensional elliptical distributions. This enables us to robustly estimate a two-dimensional covariance matrix (up to scale) based on the SSCM and hence devise a correlation estimator, which we call *spatial sign correlation*. We further derive the asymptotic distribution of the SSCM and the spatial sign correlation and compute the influence function of the latter.

The main advantage of the new estimator is its simplicity. It is very fast to compute, it requires neither an iterative algorithm nor any ranking or sorting of the data. It is furthermore distribution-free within the elliptical model, it behaves equally well for very heavy-tailed and very peaked distributions, which is true for hardly any other robust scatter estimator.*

The paper has two parts: In part 1, consisting of Sections 2 – 3, we develop the spatial sign correlation estimator and derive its asymptotics. Being aware that this estimator is one out of many that were introduced for similar purposes, the second part, consisting of Sections 4 and 5, gathers together analytic results about a variety of alternatives and compares them in an elaborate simulation study to provide some guidance within the ever increasing number of robust correlation estimates. All proofs are deferred to the appendix. We close this section by introducing some recurrent terms and notation.

In order to study the properties of the new estimator analytically we will assume the data to stem from the elliptical model. A continuous distribution F on \mathbb{R}^p is said to be *elliptical* if it has a Lebesgue-density f of the form

$$(1) \quad f(\mathbf{x}) = \det(V)^{-\frac{1}{2}} g((\mathbf{x} - \boldsymbol{\mu})^T V^{-1} (\mathbf{x} - \boldsymbol{\mu}))$$

for some $\boldsymbol{\mu} \in \mathbb{R}^p$ and symmetric, positive definite $p \times p$ matrix V . We call $\boldsymbol{\mu}$ the *location* or *symmetry center* and V the *shape matrix*, since it describes the shape of the elliptical contour lines of the density. The class of all continuous elliptical distributions F on \mathbb{R}^p having these parameters is denoted by $\mathcal{E}_p(\boldsymbol{\mu}, V)$. The shape matrix V is unique only up to scale, that is, $\mathcal{E}_p(\boldsymbol{\mu}, V) = \mathcal{E}_p(\boldsymbol{\mu}, cV)$ for any $c > 0$. For scale-free functions of V , such as correlations, which we consider here, this ambiguity is irrelevant. A common view on the *shape* of an elliptical distribution is to treat it as an equivalence class of positive definite random matrices being proportional to each other. We adopt this notion here: in the results of this exposition, V can be any representative from its equivalence class. For example, if second moments exist, one can always take the covariance matrix — or any suitably scaled multiple of it. However, the results are more general, the existence of second — or even first — moments is not required. Throughout the paper we let

$$(2) \quad V = U\Lambda U^T$$

denote an eigenvalue decomposition of V , where U is an orthogonal matrix containing the eigenvectors of V as columns and $\Lambda \text{diag}(\lambda_1, \dots, \lambda_p)$ is such that $0 < \lambda_p \leq \dots \leq \lambda_1$. We use $\|\cdot\|$ to denote the L_2 norm of a vector.

2. THE SPATIAL SIGN COVARIANCE MATRIX

We define the spatial sign covariance matrix of a multivariate distribution and derive its connection to the shape matrix V in case of a two-dimensional elliptical distribution. For $\mathbf{x} \in \mathbb{R}^p$ define the *spatial sign* $\mathbf{s}(\mathbf{x})$ of \mathbf{x} as $\mathbf{s}(\mathbf{x}) = \mathbf{x}/\|\mathbf{x}\|$ if $\mathbf{x} \neq \mathbf{0}$ and $\mathbf{s}(\mathbf{x}) = \mathbf{0}$ otherwise. Let \mathbf{X} be a p -dimensional random vector ($p \geq 2$) having distribution F . We call

$$\boldsymbol{\mu}(F) = \boldsymbol{\mu}(\mathbf{X}) = \arg \min_{\boldsymbol{\mu} \in \mathbb{R}^p} \mathbb{E} (\|\mathbf{X} - \boldsymbol{\mu}\| - \|\mathbf{X}\|)$$

the *spatial median* and, following the terminology of Visuri et al. (2000),

$$S(F) = S(\mathbf{X}) = \mathbb{E} (\mathbf{s}(\mathbf{X} - \boldsymbol{\mu}) \mathbf{s}(\mathbf{X} - \boldsymbol{\mu})^T)$$

the *spatial sign covariance matrix* (SSCM) of F (or \mathbf{X}). If there is no unique minimizing point of $\mathbb{E} (\|\mathbf{X} - \boldsymbol{\mu}\| - \|\mathbf{X}\|)$, then $\boldsymbol{\mu}(F)$ is the barycenter of the minimizing set. This may only

*For these statements to be true, the SSCM has to be based on an appropriate location estimator.

happen if F is concentrated on a line. For results on existence and uniqueness of the spatial median see Haldane (1948), Kemperman (1987), Milasevic et al. (1987) or Koltchinskii and Dudley (2000). If the first moments of F are finite, then the spatial median allows the more descriptive characterization as $\arg \min_{\boldsymbol{\mu} \in \mathbb{R}^p} \mathbb{E} \|\mathbf{X} - \boldsymbol{\mu}\|$. The spatial median always exists.

Let $\mathbb{X}_n = (\mathbf{X}_1, \dots, \mathbf{X}_n)^T$ be a data sample of size n , where the \mathbf{X}_i , $i = 1, \dots, n$, are i.i.d., each with distribution F . Define

$$\hat{S}_n(\mathbb{X}_n; \mathbf{t}) = \text{ave}_{i=1, \dots, n} \mathbf{s}(\mathbf{X}_i - \mathbf{t}) \mathbf{s}(\mathbf{X}_i - \mathbf{t})^T$$

where $\mathbf{t} \in \mathbb{R}^p$. Choosing $\mathbf{t} = \boldsymbol{\mu}(F)$, we call the estimator $\hat{S}_n(\mathbb{X}_n; \boldsymbol{\mu}(F))$ the *empirical SSCM with known location*. However, the location is usually unknown, and \mathbf{t} has to be replaced by a suitable location estimator $(\mathbf{T}_n)_{n \in \mathbb{N}}$, and we refer to $\hat{S}_n(\mathbb{X}_n; \mathbf{T}_n)$ as the *empirical SSCM with unknown location*. The canonical location functional in this case is the (*empirical*) *spatial median*

$$\hat{\boldsymbol{\mu}}_n = \hat{\boldsymbol{\mu}}_n(\mathbb{X}_n) = \min_{\boldsymbol{\mu} \in \mathbb{R}^p} \sum_{i=1}^n \|\mathbf{X}_i - \boldsymbol{\mu}\|.$$

Under regularity conditions (the data points do not lie on a line and none of them coincides with $\hat{\boldsymbol{\mu}}_n$, see Kemperman (1987), p. 228) the spatial signs w.r.t. the empirical spatial median are centered, i.e. $\text{ave}_{i=1}^n \mathbf{s}(\mathbf{X}_i - \hat{\boldsymbol{\mu}}_n) = \mathbf{0}$. Hence, the empirical spatial sign covariance matrix $\hat{S}_n(\mathbb{X}_n; \hat{\boldsymbol{\mu}}_n)$ is indeed the covariance matrix of the spatial signs — if the latter are taken w.r.t. the spatial median. Our first proposition draws a connection between the shape of an elliptical distribution and the corresponding spatial sign covariance matrix.

Proposition 1. *Let $F \in \mathcal{E}_p(\boldsymbol{\mu}, V)$ and $V = U\Lambda U^T$ denote an eigenvalue decomposition of V with $0 < \lambda_p \leq \dots \leq \lambda_1$. Then*

- (1) $\boldsymbol{\mu}(F) = \boldsymbol{\mu}$ and
- (2) $S(F) = U\Delta U^T$, where $\Delta = \text{diag}(\delta_1, \dots, \delta_p)$ is a diagonal matrix with $0 < \delta_p \leq \dots \leq \delta_1$.
- (3) If $p = 2$, then $\delta_j = \sqrt{\lambda_j} / (\sqrt{\lambda_1} + \sqrt{\lambda_2})$, $j = 1, 2$.

The proof is given in the appendix. Part (2) of Proposition 1 states that the SSCM $S(F)$ and the shape matrix V have the same eigenvectors and the same order of the corresponding eigenvalues. This has been known for some time, and the use of the SSCM has been proposed to robustify such multivariate analyses that are based on this information only, most notably principal component analysis, (Marden, 1999; Locantore et al., 1999; Croux et al., 2002; Gervini, 2008). Other such applications are direction-of-arrival estimation (Visuri et al., 2001), or testing sphericity in the elliptical model (Sirkiä et al., 2009). Part (3) enables us to reconstruct the whole shape matrix V from $S(F)$ in dimension $p = 2$. Thus the SSCM can be directly employed for applications that rely on the shape information, but do not require any knowledge about the overall scale, most notably correlations. This result seems to be quite recent. It appears in a similar form in Croux et al. (2010) and has also been used by Vogel et al. (2008), but neither of these articles provide a proof.

The next result concerns the asymptotic behavior of the empirical SSCM. It is formulated using the vec operator, which stacks the columns of a matrix from left to right underneath each other, and the Kronecker product \otimes (e.g. Magnus and Neudecker, 1999, Sec. 2). Both are connected by the identity $\text{vec}(ABC) = (C^T \otimes A) \text{vec } B$.

Proposition 2. *Let $\mathbf{X}, \mathbf{X}_1, \dots, \mathbf{X}_n$ be i.i.d. random vectors with distribution F satisfying $\mathbb{E} \|\mathbf{X} - \boldsymbol{\mu}\|^{-1} < \infty$ and $(\mathbf{T}_n)_{n \in \mathbb{N}}$ a sequence of random variables converging almost surely to $\boldsymbol{\mu}(F)$. Then, as $n \rightarrow \infty$, we have*

- (1) $\hat{S}_n(\mathbb{X}_n; \mathbf{T}_n) \xrightarrow{a.s.} S(F)$, and

(2) if furthermore $\sqrt{n}\|\mathbf{T}_n - \boldsymbol{\mu}\|$ converges in distribution, $\mathbb{E}\{\|\mathbf{X} - \boldsymbol{\mu}\|^{-3/2}\} < \infty$ and $(\mathbf{X} - \boldsymbol{\mu}) \stackrel{\mathcal{L}}{=} -(\mathbf{X} - \boldsymbol{\mu})$, then $\hat{S}_n(\mathbb{X}_n; \mathbf{T}_n)$ is asymptotically normal, i.e. there is a non-negative definite $p^2 \times p^2$ matrix $W_{\hat{S}}$ such that

$$\sqrt{n} \operatorname{vec} \{ \hat{S}_n(\mathbb{X}_n; \mathbf{T}_n) - S(F) \} \xrightarrow{\mathcal{L}} N_{p^2}(\mathbf{0}, W_S).$$

(3) If additionally $F \in \mathcal{E}_2(\boldsymbol{\mu}, V)$, then

$$W_S = \frac{-\lambda_1 \lambda_2 + \frac{1}{2} \sqrt{\lambda_1 \lambda_2} (\lambda_1 + \lambda_2)}{(\lambda_1 - \lambda_2)^2} (U \otimes U) W_0 (U \otimes U)^T$$

with

$$W_0 = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}.$$

Parts (1), (2) are proved in Dürre et al. (2014), where also alternative assumptions to guarantee strong consistency and asymptotic normality are given, The mentioned regularity conditions are rather mild, they are fulfilled for elliptical distributions with bounded density.

3. A SPATIAL SIGN BASED CORRELATION ESTIMATOR

In the following let $\mathbf{X}_i = (X_i, Y_i)^T$, $i = 1, \dots, n$, be an i.i.d. sample from $F \in \mathcal{E}_2(\boldsymbol{\mu}, V)$. Denoting the entries of V by v_{ij} , we want to estimate the parameter

$$\rho = v_{12} / \sqrt{v_{11} v_{22}}.$$

We call ρ the *generalized correlation coefficient* of the elliptical distribution F , since it coincides with the correlation coefficient if second moments are finite. In a slight abuse of notation, we will refer to ρ simply as the correlation (coefficient) of F in the following. Propositions 1 and 2 from the previous section give rise to an estimator of ρ constructed as follows: compute the SSCM $\hat{S}_n = \hat{S}_n(\mathbb{X}_n; \hat{\boldsymbol{\mu}}_n)$, perform an eigenvalue decomposition $\hat{S}_n = \hat{U}_n \hat{\Delta}_n \hat{U}_n^T$ with $\hat{\Delta}_n = \operatorname{diag}(\hat{\delta}_1, \hat{\delta}_2)$ and compute the matrix $\hat{V}_n = \hat{U}_n \hat{\Lambda}_n \hat{U}_n^T$ with $\hat{\Lambda}_n = \operatorname{diag}(\hat{\lambda}_1, \hat{\lambda}_2)$ and $\hat{\lambda}_1 = \hat{\delta}_1 / \hat{\delta}_2$, $\hat{\lambda}_2 = \hat{\delta}_2 / \hat{\delta}_1$.[†] Finally compute the correlation coefficient from the matrix \hat{V}_n , i.e. let $\hat{\rho}_n = \hat{v}_{12} / \sqrt{\hat{v}_{11} \hat{v}_{22}}$. In dimension two, the eigenvalue decomposition can be computed explicitly with justifiable effort, and we obtain the following explicit expression for the thus defined estimator:

$$\hat{\rho}_n = \frac{c \hat{s}_{12} b}{\sqrt{(\hat{s}_{12}^2 + b^2)^2 + (\hat{s}_{12} c b)^2}},$$

where

$$(3) \quad c = \frac{2d-1}{d(1-d)}, \quad d = \frac{1}{2} + \sqrt{(\hat{s}_{11} - \frac{1}{2})^2 + \hat{s}_{12}^2}, \quad b = d - \hat{s}_{11}$$

and \hat{s}_{ij} denote the entries of \hat{S}_n . We call $\hat{\rho}_n$ the *spatial sign correlation coefficient*. This must not be confused with the correlation of the spatial signs of the observations. This would be $\hat{\rho}_{\text{SSCM}} = \hat{s}_{12} / \sqrt{\hat{s}_{11} \hat{s}_{22}}$. Also note that knowing $\hat{\rho}_{\text{SSCM}}$ alone is not sufficient for computing $\hat{\rho}_n$. Despite the rather lengthy definition of $\hat{\rho}_n$, its asymptotic variance has a surprisingly simple form.

Proposition 3. *Let $F \in \mathcal{E}_2(\boldsymbol{\mu}, V)$ have a bounded density at $\boldsymbol{\mu}$. Then, as $n \rightarrow \infty$,*

(1) $\hat{\rho}_n \xrightarrow{a.s.} \rho$, and

[†]The overall scaling of \hat{V}_n is, of course, irrelevant for the correlation, and its eigenvalues $\hat{\lambda}_1$ and $\hat{\lambda}_2$ may as well be chosen differently. Their ratio has to satisfy $\hat{\lambda}_1 / \hat{\lambda}_2 = (\hat{\delta}_1 / \hat{\delta}_2)^2$.

(2) $\sqrt{n}(\hat{\rho}_n - \rho) \xrightarrow{\mathcal{L}} N\left(0, (1 - \rho^2)^2 + \frac{1}{2}(a + a^{-1})(1 - \rho^2)^{3/2}\right)$, where $a = \sqrt{v_{11}/v_{22}}$ is the root of the ratio of the diagonal elements of V .

Proposition 3 (2) gives the asymptotic variance $ASV(\hat{\rho}_n)$ as a function of the true correlation ρ and the ratio of the diagonal elements of the shape matrix V . The elliptical generator g , cf. (1), does not enter, which may be phrased as “ $\hat{\rho}_n$ is asymptotically distribution-free within the elliptical model”. It is furthermore consistent and asymptotically normal without any moment condition.

For fixed ρ , the asymptotic variance $ASV(\hat{\rho}_n)$ is minimal for equal marginal variances, but can get arbitrarily large for heteroscedastic data. It is therefore advisable to apply this estimator to standardized data, i.e. the components should be divided beforehand by a scale measure to yield equally dispersed margins. Margin-wise standardization generally should be administered with caution in multivariate data analysis, since it changes the shape, e.g., the direction of the eigenvectors, and will alter the results of, e.g., a principal component analysis. The inefficiency of the spatial sign covariance matrix at strongly “shaped”, i.e. non-spherical, distributions has led to criticism regarding its use for robust principal component analysis, where a strong “shapedness” is the working assumption, cf. e.g. Remark 5.1 in Bali et al. (2011). We define shapedness as deviation from sphericity (and measure it for instance by the condition number of V). There are two sources that contribute to the shapedness: collinearity and heteroscedasticity. The formula in Proposition 3 (2) nicely visualizes the individual influences of these two sources of shapedness on the asymptotic variance of $\hat{\rho}_n$. Since we are interested in correlation — a function of the shape that is invariant with respect to margin-wise scale changes —, we can avoid the inefficiency due to the heteroscedasticity by margin-wise standardisation.

Technical conditions that ensure the asymptotic equivalence of such a two-step procedure to the spatial sign correlation estimation at spherical distributions are yet to be established, but by heuristic arguments we can work for all practical purposes with an asymptotic variance of

$$ASV(\hat{\rho}_n) = (1 - \rho^2)^2 + (1 - \rho^2)^{3/2}.$$

In light of robustness, we recommend to use a highly robust scale estimator for standardization, such as the MAD or the Q_n (see also next Section). Both have a breakdown point of 1/2, a property which they share with the spatial sign covariance matrix (Croux et al., 2010). The thus obtained two-stage correlation estimator is highly robust, but we refrain from considering breakdown points of correlation estimators, see the discussion and the rejoinder of Davies and Gather (2005).

In the next section we will compare several correlation estimators with respect to their efficiency at the normal model. As a first glimpse in this direction, we recall the asymptotic variance of the Pearson correlation $\hat{\rho}_{\text{Pea}}$ at elliptical distributions

$$ASV(\hat{\rho}_{\text{Pea}}) = \left(1 + \frac{\kappa}{3}\right) (1 - \rho^2)^2,$$

where κ is the excess kurtosis of the components of F . The asymptotic relative efficiency of $\hat{\rho}_n$ with respect to $\hat{\rho}_{\text{Pea}}$,

$$ARE(\hat{\rho}_n, \hat{\rho}_{\text{Pea}}) = \frac{ASV(\hat{\rho}_{\text{Pea}})}{ASV(\hat{\rho}_n)} = \frac{1 + \kappa/3}{1 + \frac{1}{2}(a + a^{-1})(1 - \rho^2)^{-1/2}},$$

is depicted in Figure 1.

At normality, the maximum 1/2 is attained for $a = 1$ and $\rho = 0$. If we fix $a = 1$, the asymptotic relative efficiency declines with increasing $|\rho|$, even tending to 0 for $|\rho| \rightarrow 1$. But it declines very slowly, for $|\rho| < 0.7$ it stays above 0.4. Under heavy-tailed distributions, however, the spatial sign correlation can be more efficient than the Pearson correlation. Specifically, $ARE(\hat{\rho}_n, \hat{\rho}_{\text{Pea}}) \geq 1$ if

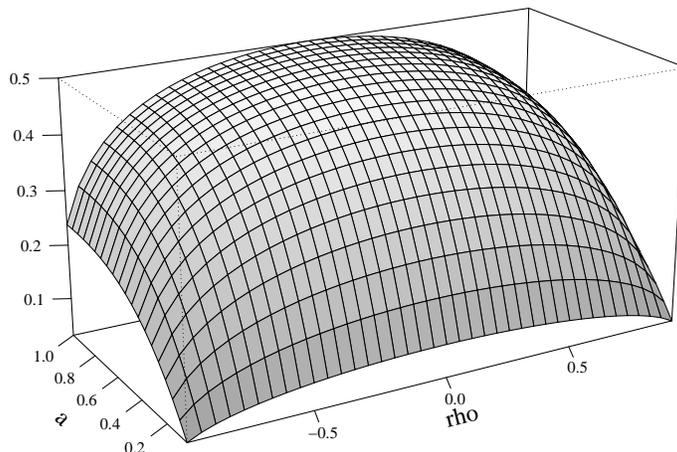


FIGURE 1. The asymptotic relative efficiency of $\hat{\rho}$ with respect to the empirical correlation under normality as a function of ρ and $a = \sqrt{v_{11}/v_{22}}$.

$\kappa \geq (3/2)(a + a^{-1})/\sqrt{1 - \rho^2}$. For instance, with the kurtosis of the t_ν distribution being $6/(\nu - 4)$, the spatial sign correlation is more efficient at the bivariate spherical t_ν distribution for $\nu < 6$.

In the remainder of this section, we examine the influence function of the spatial sign correlation. The influence function is based on the notion that estimators are statistical functionals working on distributions. The specific estimate computed from the data set \mathbb{X}_n is then the functional applied to the corresponding empirical distribution. We use \hat{S} and $\hat{\rho}$ to denote the statistical functionals corresponding to the SSCM and the spatial sign correlation, respectively. The influence function $IF(\mathbf{x}, \hat{\rho}, F)$ describes the effect of an infinitesimal small contamination at point \mathbf{x} on the functional $\hat{\rho}$ if the latter is evaluated at distribution F . It is an important tool describing the robustness properties of estimators. For a precise definition, interpretation and further details, see, e.g., Hampel et al. (1986) or Maronna et al. (2006).

Croux et al. (2010) give the influence function of the off-diagonal element of the SSCM for $p = 2$. Calculation of the diagonal elements is straightforward, and we obtain for $F \in \mathcal{E}_2(\boldsymbol{\mu}, V)$:

$$IF(\mathbf{x}, \hat{S}, F) = \mathbf{x}\mathbf{x}^T / (\mathbf{x}^T \mathbf{x}) - S(F).$$

Applying the chain rule and using the derivatives calculated in the proof of Proposition 5 in the appendix, we arrive at the influence function of the spatial sign correlation.

Proposition 4. *Let $F \in \mathcal{E}_2(\boldsymbol{\mu}, V)$. Then $IF(\mathbf{x}, \hat{\rho}, F) =$*

$$\frac{-\{(a^2+1)\rho\sqrt{1-\rho^2}+2a\rho(1-\rho^2)\}(a^2x_2^2+x_1^2)-\{(a^4+6a^2+1)(\rho^2-1)+2a(a^2+1)\sqrt{1-\rho^2}(\rho^2-2)\}x_1x_2}{\{2a^2\sqrt{1-\rho^2}+a(a^2+1)\}(x_2^2+x_1^2)},$$

where $\mathbf{x} = (x_1, x_2)^T$ and a and ρ are as in Proposition 3.

The influence function for $a = 1$ and $\rho = 0$ is illustrated in Figure 2 on the right. It has a discontinuity at the origin and is bounded. Its extreme values ± 2 are attained on the diagonals. Furthermore, $IF(\mathbf{x}, \hat{\rho}, F)$ is bounded in \mathbf{x} for any fixed values a and ρ , but it may get arbitrarily large as a varies. A robustness index that is derived from the influence function is the gross-error sensitivity (GES), defined as

$$GES(\hat{\rho}, F) = \sup_{\mathbf{x} \in \mathbb{R}^2} |IF(\mathbf{x}, \hat{\rho}, F)|.$$

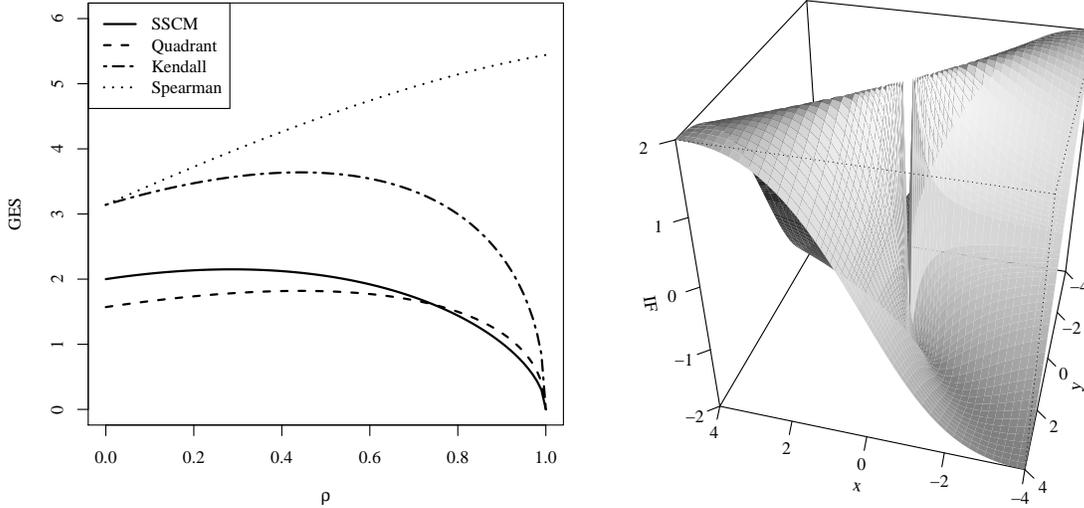


FIGURE 2. GES for the spatial correlation compared to other popular nonparametric correlation estimators under equal marginal variances (left) and influence function of the spatial correlation for $\rho = 0$ and $a = 1$ (right).

For $a = 1$, we obtain

$$GES(\hat{\rho}, F) = \frac{\{(\rho^2 - 1)(-\rho^4 + 8\rho^2 + 4\sqrt{1 - \rho^2}(\rho^2 - 2) - 8)\}^{1/2} + |\rho|(\sqrt{1 - \rho^2} - \rho^2 + 1)}{\sqrt{1 - \rho^2} + 1}.$$

which is depicted in Figure 2 (left). Croux and Dehon (2010) compute the gross-error sensitivities of several nonparametric correlation measures at bivariate normal distributions. Figure 2 (left) corresponds to their Figure 2, complemented by the GES curve of the spatial sign correlation. The GES is small for any ρ , indicating a good robustness against small amounts of outliers. We refrain from stating the GES for arbitrary a and ρ explicitly since the formula is rather lengthy.

4. ANALYTICAL COMPARISON OF ROBUST CORRELATION ESTIMATORS

There are many proposals for robust correlation estimators in the literature. In the second part of this exposition, consisting of Sections 4 and 5, we compare the spatial sign correlation $\hat{\rho}_n$ to a number of prominent alternatives, without claiming or attempting any completeness or ranking. In Section 4, we gather the basic analytic results, particularly the asymptotic efficiencies at the normal model, and in Section 5 we compare the finite-sample and robustness properties numerically.

It is important to note that — in general — the estimators mentioned are known to be Fisher-consistent for the correlation only under normality, which often — as in the case of the spatial sign correlation — can be relaxed to ellipticity. To put it differently, each of the various correlation estimators[‡] $\hat{\theta}_n$ estimates some parameter θ of the bivariate population distribution, which may serve as a measure of monotone dependence, but does in general not coincide with the moment correlation ρ . The exact functional connection between θ and ρ is usually hard to assess for arbitrary distributions, but is known for important subclasses, such as the normal model. If no such function is mentioned in the examples below, it is the identity.

Let the data be denoted by $\mathbf{X}_i = (X_i, Y_i)^T$, $i = 1, \dots, n$, independent and normally distributed. Relative efficiencies reported below are with respect to the sample correlation, which is denoted by $\hat{\rho}_{\text{Pea}}$. The estimators we will consider can roughly be divided into three groups: We call

[‡]In this sense, “correlation” is understood as monotone dependence.

the first group *nonparametric estimators* since they depend on signs and ranks. Besides the spatial sign correlation, these are the Gaussian rank correlation, Spearman's ρ , Kendall's τ , and the quadrant correlation. The second group are the *Gnanadesikan-Kettenring-type estimators*, where we consider the τ -scale and the Q_n as scale estimators. We label the third group *affine equivariant estimators*, i.e. estimators that are derived from an affine equivariant two-dimensional scatter estimator. Here we consider Tyler's M-estimator, raw and reweighted MCD, the Stahel-Donoho-estimator and the S-estimator with Tukey's biweight-function. The estimators in detail:

4.1. Nonparametric estimators. The Gaussian rank correlation is defined as the sample correlation of the normal scores of the data, i.e.

$$\hat{\rho}_{\text{GRK}} = \frac{1}{c_n} \sum_{i=1}^n \Phi^{-1} \left(\frac{R(X_i)}{n+1} \right) \Phi^{-1} \left(\frac{R(Y_i)}{n+1} \right),$$

where $c_n = \sum_{i=1}^n \Phi^{-1} \left(\frac{i}{n+1} \right)^2$, $R(X_i)$ is the rank of X_i among X_1, \dots, X_n and Φ^{-1} is the quantile function of the standard normal distribution. The influence function of the Gaussian rank correlation is unbounded, but in finite samples it is much more robust than the Pearson correlation (Boudt et al., 2012). Since the Gaussian rank correlation corresponds to the Pearson correlation of transformed data, the pairwise estimation of multidimensional correlation matrices leads always to a non-negative definite estimate.

Another rank based estimator is Spearman's ρ , which is the sample correlation of the ranks $R(X_1), \dots, R(X_n)$ and $R(Y_1), \dots, R(Y_n)$. To obtain a consistent estimator for ρ , one has to apply the transformation $\hat{\rho}_{\text{Sp.c}} = 2 \sin(\pi \hat{\rho}_{\text{Sp}}/6)$, which goes back to Pearson (1907). Another popular nonparametric estimator is Kendall's τ , which is defined as

$$\hat{\rho}_{\text{Ken}} = \frac{2}{n(n-1)} \sum_{i>j} \mathbf{s}((X_i - X_j)(Y_i - Y_j)),$$

where $\mathbf{s}(\cdot)$ is the sign function defined at the beginning of Section 2, here applied to a univariate argument. It also requires a consistency transformation, which is valid under ellipticity (e.g. Möttönen et al., 1999): $\hat{\rho}_{\text{Ken.c}} = \sin(\pi \hat{\rho}_{\text{Ken}}/2)$. A highly robust, non-parametric procedure based on signs is the quadrant correlation, which can be expressed as

$$\hat{\rho}_{\text{Q}} = \frac{1}{n} \sum_{i=1}^n \mathbf{s}((X_i - \text{med}(X))(Y_i - \text{med}(Y))),$$

where $\text{med}(X)$ denotes the median of X_1, \dots, X_n . The same transformation $\hat{\rho}_{\text{Q.c}} = \sin(\pi \hat{\rho}_{\text{Q}}/2)$ renders this estimator consistent for ρ under elliptical distributions. All three nonparametric estimators $\hat{\rho}_{\text{Sp.c}}$, $\hat{\rho}_{\text{Ken.c}}$, $\hat{\rho}_{\text{Q.c}}$ have a bounded influence function and are therefore called B-robust. Their influence functions, asymptotic variances and gross-error sensitivities can be found in Croux and Dehon (2010).

4.2. GK estimators. Gnanadesikan and Kettenring (1972) introduced an estimation principle based on robust variance estimation,

$$\hat{\rho} = \frac{\hat{\sigma}^2(X/\alpha + Y/\beta) - \hat{\sigma}^2(X/\alpha - Y/\beta)}{\hat{\sigma}^2(X/\alpha + Y/\beta) + \hat{\sigma}^2(X/\alpha - Y/\beta)},$$

where $\hat{\sigma}$ can be any robust scale measure and $\alpha = \hat{\sigma}(X)$, $\beta = \hat{\sigma}(Y)$. Such an estimator can be seen to be Fisher-consistent for ρ , regardless of the choice of the scale measure $\hat{\sigma}$, if $X + Y$, $X - Y$ as well as X and Y have the same distribution up to location and scale, which is fulfilled for elliptical distributions. According to Ma and Genton (2001), the correlation estimator has the same asymptotic efficiency as the underlying variance estimator. There is also a relationship

between the influence functions, which guarantees that the B-robustness translates from the variance to the correlation estimator, see Genton and Ma (1999). In the recent literature, there are two proposals for the variance estimation. Maronna and Zamar (2002) favor the so-called τ -scale:

$$\hat{\sigma}_\tau = \frac{\sigma_0^2}{n} \sum_{i=1}^n d_{c_2} \left(\frac{X_i - \hat{\mu}(X)}{\sigma_0} \right), \quad \text{where} \quad \hat{\mu}(X) = \frac{\sum_{i=1}^n w_i X_i}{\sum_{i=1}^n w_i},$$

$w_i = W_{c_1}\{(X_i - \text{med}(X))/\sigma_0\}$, $\sigma_0 = \text{med}\{|X_i - \text{med} X| : i = 1, \dots, n\}$, $d_c(x) = \min(x^2, c^2)$ and $W_c(x) = (1 - (x/c)^2)^2 \mathbf{1}_{\{|x| \leq c\}}$. They use $c_1 = 4.5$ and $c_2 = 3$ to get an efficiency of approximately 0.8 under normality distribution. Ma and Genton (2001) use the Q_n , which is defined as

$$Q_n(X) = d \cdot \{|x_i - x_j| : i < j\}_{(k)},$$

where $k = \binom{[n/2]+1}{2}$ and d is a consistency factor equaling $1/(\sqrt{2}\Phi^{-1}(5/8))$ for the normal distribution. This estimator has an efficiency of 0.82, see Rousseeuw and Croux (1993). The influence function of the resulting covariance estimator is bounded and can also be found in Ma and Genton (2001).

4.3. Affine equivariant estimators. One can estimate the correlation by means of any affine equivariant, bivariate scatter estimator \hat{V}_n using the relation $\hat{\rho} = \hat{v}_{1,2}/\sqrt{\hat{v}_{1,1}\hat{v}_{2,2}}$. Taskinen et al. (2006) derive the influence function of the correlation estimator from the influence function of \hat{V}_n under elliptical distributions. Furthermore, the asymptotic variance of $\hat{\rho}$ is of the form $(1 - \rho^2)^2 \cdot ASV(\hat{v}_{1,2})$, where $ASV(\hat{v}_{1,2})$ is the asymptotic variance of $\hat{v}_{1,2}$ under the corresponding spherical distribution. We consider four examples of robust affine equivariant scatter estimators.

Tyler (1987a) proposed an M-estimator for the shape matrix V , being a suitably scaled solution of

$$\frac{2}{n} \sum_{i=1}^n \frac{(\mathbf{X}_i - \hat{\boldsymbol{\mu}}_n)(\mathbf{X}_i - \hat{\boldsymbol{\mu}}_n)^T}{(\mathbf{X}_i - \hat{\boldsymbol{\mu}}_n)^T \hat{V}^{-1}(\mathbf{X}_i - \hat{\boldsymbol{\mu}}_n)} = \hat{V}_n,$$

where $\hat{\boldsymbol{\mu}}_n$ is a suitable multivariate location estimate. In the simulations in Section 5 we take the spatial median. The Tyler estimator can be regarded as an affine equivariant version of the SSCM and is also distribution-free within the elliptical model. The corresponding correlation estimate in two dimensions has an efficiency of 0.5 (e.g. Taskinen et al., 2006). A highly robust, affine equivariant scatter estimator is the minimum covariance determinant estimator (MCD) proposed by Rousseeuw (1985). For a given trimming constant α , it is defined as the sample covariance matrix of the subset of the observations that yields the smallest determinant of the estimated matrix among all subsets of size $\lfloor (1 - \alpha) \cdot n \rfloor$. Choosing $\alpha = 0.5$ results in an asymptotic breakdown point of 0.5. Since the asymptotic efficiency, especially in small dimensions, is rather low, the raw MCD is usually followed by a reweighting step. We will call this two-step estimated the weighted MCD. For both, the raw and the weighted MCD, influence functions, consistency factors and asymptotic efficiencies can be found in Croux and Haesbroeck (1999).

Stahel (1981) and Donoho (1982) proposed another covariance estimator with an asymptotic breakdown point of 0.5. It is defined as

$$\hat{V}_n = \left(\sum_{i=1}^n w_i \right)^{-1} \sum_{i=1}^n w_i (\mathbf{X}_i - \hat{\boldsymbol{\mu}}_n)(\mathbf{X}_i - \hat{\boldsymbol{\mu}}_n)^T \quad \text{where} \quad \hat{\boldsymbol{\mu}}_n = \left(\sum_{i=1}^n w_i \right)^{-1} \sum_{i=1}^n w_i \mathbf{X}_i,$$

$w_i = \min\{1, (c/r_i)^2\}$ and c is often chosen as the 0.95-quantile of the χ_2^2 -distribution. The value

$$r_i = \max_{a:|a|=1} \frac{a^T \mathbf{X}_i - \text{med}(a^T \mathbb{X}_n)}{\text{MAD}(a^T \mathbb{X}_n)},$$

	$\rho = 0$	$\rho = 0.5$		$\rho = 0$	$\rho = 0.5$
Pearson		1	GK- Q_n		0.823
Spatial sign	0.5	0.464	GK- τ		0.8
Gaussian rank		1	Tyler		0.5
Spearman	0.912	0.867	rMCD		0.033
Kendall	0.912	0.892	wMCD		0.401
Quadrant	0.405	0.342	S		0.377

TABLE 1. Asymptotic efficiency of correlation estimators for $p = 2$ under normality

is a measure of the outlyingness of X_i among all observations. Any other high-breakdown point location and scale estimators can be used instead of the median and median absolute deviation (MAD, Hampel, 1974). The influence function, asymptotic distribution and gross error sensitivity of the Stahel-Donoho estimator can be found in Gervini (2002), but an explicit value of its asymptotic efficiency does not seem to be available in the literature.

Davies (1987) proposed a multivariate generalization of S-estimators, being defined as

$$(\hat{\boldsymbol{\mu}}_n, \hat{V}_n) = \arg \min_{\boldsymbol{\mu}, V} \det(\hat{V}_n) \quad \text{subject to} \quad \text{ave}_{i=1}^n w(\hat{d}_i) = b,$$

where $\hat{d}_i = \{(\mathbf{X}_i - \hat{\boldsymbol{\mu}}_n)^T \hat{V}_n^{-1} (\mathbf{X}_i - \hat{\boldsymbol{\mu}}_n)\}^{1/2}$ and w is a suitable, smooth and bounded, weight function, usually the Tukey-biweight:

$$w_c(y) = \min \left(\frac{y^2}{2} - \frac{y^4}{2c^2} + \frac{y^6}{6c^4}, \frac{c^2}{6} \right).$$

Letting $b = E\{w_c(\|V^{-1/2}(\mathbf{X} - \boldsymbol{\mu})\|)\}$ yields Fisher-consistency at the elliptical population distribution F , and if c is chosen such that $rc^2/6 = E\{w_c(\|V^{-1/2}(\mathbf{X} - \boldsymbol{\mu})\|)\}$, the S-estimator achieves an asymptotic breakdown point of $0 < r \leq 1/2$. We consider the common standard choices $r = 1/2$ and consistency for Σ at the normal model. Asymptotics can be found in Davies (1987), the influence function was calculated by Lopuhaä (1989), and efficiencies under normal distribution were calculated for instance in Croux and Haesbroeck (1999).

Table 1 lists the asymptotic relative efficiencies of the mentioned correlation estimators with respect to the Pearson correlation under normality. Specific tuning constants, parameters, weight functions, etc., are chosen as described above. The efficiency of the nonparametric estimators is declining with $|\rho|$, but the loss is rather small for moderate values. As we can see, the spatial correlation can well compete with highly robust estimators in terms of efficiency.

5. NUMERICAL COMPARISON

We compared the correlation estimators in four different situations: under normality, under ellipticity, in outlier scenarios and at non-elliptical distributions. We used the statistical software R, including the packages ICSNP (Tyler's M-estimator), MNM (elliptical power exponential distribution), mvtnorm (multivariate normal and elliptical t -distributions), pcaPP (spatial median), rrcov (Stahel-Donoho and S-estimator) and robustbase (τ -scale, MCD and Q_n). In all scenarios, the estimators were transformed to be Fisher-consistent for the normal distribution. For some estimators, consistency-transformations for other distributions are known as well, but it is unrealistic in practice to know the kind of distribution in advance. Furthermore, the marginal variances are always chosen equal.

5.1. Results under normality. First we examine bias and variance under normality. We choose $\rho = 0.5$, let the sample size n vary from 5 to 100, and generate 100,000 samples for each sample size. In Figure 3 (left), we see that all correlation estimators are biased towards zero in small samples.

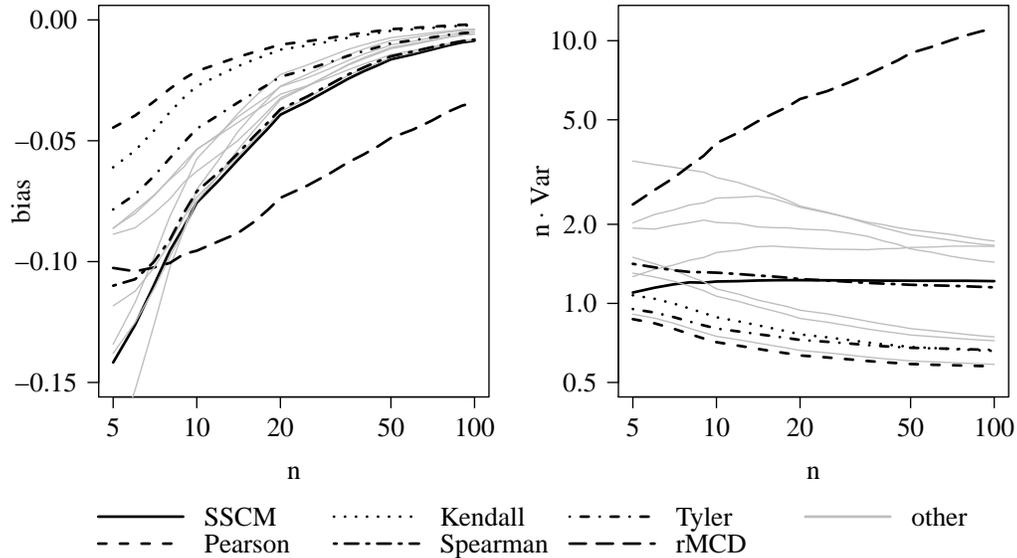


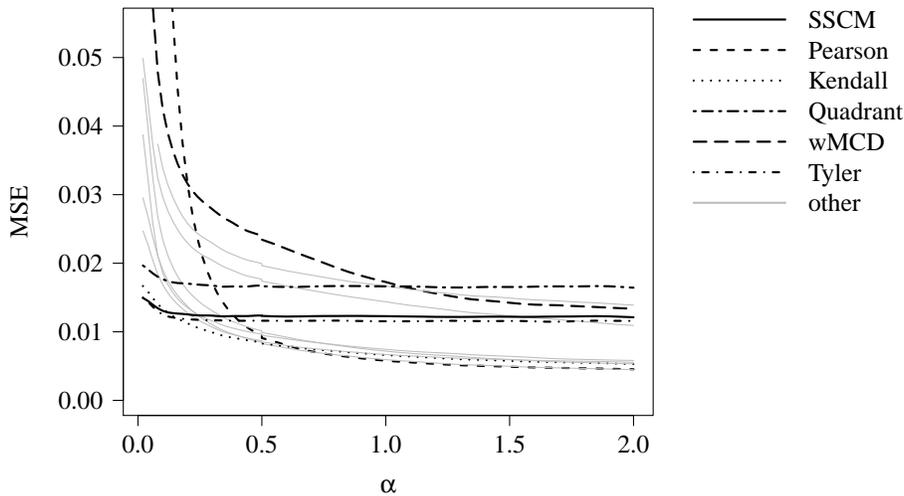
FIGURE 3. Simulated finite sample bias (left) and n -variance (right) under normality, $\rho = 0.5$ and different sample sizes n .

Next to the Pearson correlation, Kendall’s τ and Spearman’s ρ (the adequately transformed estimates) are least biased. The bias of the raw MCD still remains heavy even for $n = 100$. The spatial sign correlation behaves very well in terms of finite-sample variance. On the right-hand side of Figure 3, the variance times n (the “ n -stabilized variance”) is plotted against n , which is, contrary to most other estimators, nearly a horizontal line. This indicates that asymptotic tests and confidence intervals based on the spatial sign correlation provide good approximations also in small samples.

5.2. Results under elliptical distributions. Furthermore, the behavior under different elliptical tails is investigated. We consider two subclasses of elliptical distributions that generate varying tails: the t_ν -family and the power exponential family (e.g. Bilodeau and Brenner, 1999, pp. 208, 209). The results for the t_ν distribution are summarized in Table 5.2, where the mean squared error (MSE) of the various correlation estimators based on 100,000 samples are given for $\rho = 0$ and $\rho = 0.5$ and different degrees of freedom ν . Keep in mind that formally the correlation does not exist for one and two degrees of freedom, and we estimate the corresponding parameter ρ of the shape matrix instead, see the remark at the beginning of Section 3. The MSE of the spatial sign correlation remains constant with respect to ν , which applies only to the quadrant correlation and Tyler’s M-estimator among the other methods. For one degree of freedom and $\rho = 0.5$, the spatial sign correlation, together with Kendall’s τ and Tyler’s M-estimator, is most efficient. For one degree of freedom and $\rho = 0$, Spearman’s ρ yields the smallest MSE by far. But this is due to its (asymptotic) bias towards zero. Contrary to Kendall’s τ , the consistency transformation applied to Spearman’s ρ under normality is not valid under ellipticity in general.

The MSEs, again based on 100,000 repetitions, for the power exponential distribution are displayed in Figure 4. The sample size is $n = 100$, the true correlation $\rho = 0.5$, and the power parameter α ranges from 0.02 to 2 in 56 (non-equidistant) steps. Letting $\alpha = 1$ corresponds to the normal distribution and $\alpha = 0.5$ yields the elliptical Laplace distribution. With decreasing α , the distribution gets heavier tailed and more peaked in the origin, but all moments exist for any $\alpha > 0$ and the density remains bounded. As before, the MSE of the spatial correlation does not depend on the “tailedness parameter” α , which is in line with the asymptotic result. Only

ρ ν	0				0.5			
	1	2	5	10	1	2	5	10
Pearson	0.356	0.112	0.021	0.013	0.283	0.077	0.012	0.007
Spatial sign	0.020	0.020	0.019	0.020	0.012	0.012	0.012	0.012
Quadrant	0.024	0.024	0.024	0.017	0.017	0.016	0.016	0.016
Kendall	0.019	0.016	0.013	0.012	0.012	0.010	0.008	0.007
Spearman	0.016	0.014	0.012	0.012	0.015	0.011	0.008	0.007
Gaussian rank	0.021	0.017	0.013	0.012	0.019	0.013	0.008	0.007
GK- Q_n	0.021	0.017	0.015	0.014	0.012	0.010	0.009	0.008
GK- τ	0.024	0.019	0.015	0.014	0.014	0.011	0.009	0.008
Tyler	0.020	0.020	0.020	0.020	0.012	0.012	0.012	0.012
rMCD	0.076	0.099	0.132	0.149	0.047	0.062	0.085	0.098
wMCD	0.037	0.035	0.034	0.032	0.022	0.021	0.020	0.019
S	0.033	0.030	0.029	0.029	0.019	0.017	0.017	0.017
Stahel-Donoho	0.031	0.028	0.026	0.025	0.018	0.016	0.015	0.015

TABLE 2. MSE under t_ν distributions with different degrees of freedom and $n = 100$.FIGURE 4. MSE of correlation estimators under the power exponential distribution with different α , $\rho = 0.5$ and $n = 100$.

for very small α , we observe a slight incline. The power exponential distribution with a small α is particularly challenging for robust scatter estimators, since it possesses heavy tails and a probability mass concentration at the origin. Robust estimators downweight or reject outlying observations, which are in this case no contaminations, but carry — in contrast to the bulk of the data in the center — the main information about the shape. In fact, the MSE of the raw MCD is above the displayed region in Figure 4. The spatial sign covariance matrix can cope well with such peaked distributions. For $\alpha < 0.1$, we find it, together with Tyler’s estimator, to have the smallest MSE. However, it is crucial to use an appropriate location estimator that also works well with peaks at the center, see e.g. the discussion in Section 3.2 of Dürre et al. (2014). Altogether Kendall’s τ appears to perform best over the whole range of α .

5.3. Results under contamination. To assess the robustness properties, we consider two scenarios: a single outlier of varying size, and an increasing amount of outliers stemming from a contamination distribution. In the first situation, we start from a bivariate normal sample with

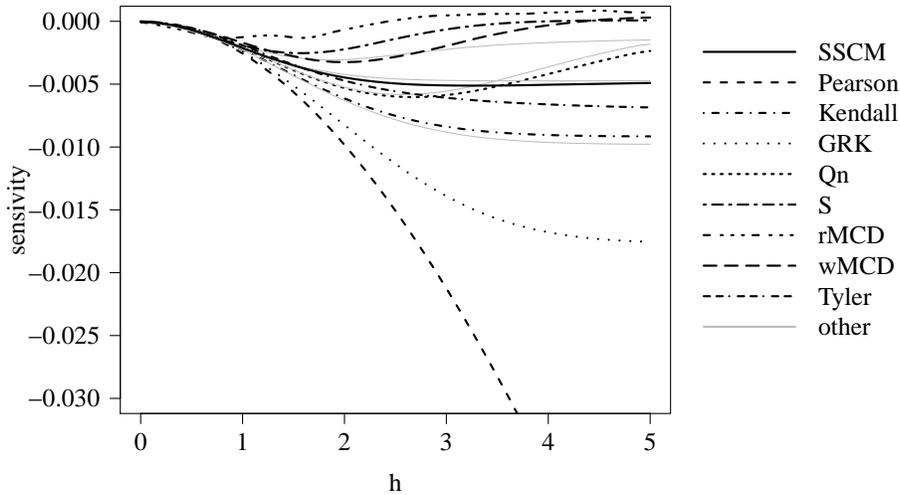


FIGURE 5. Bias of correlation estimators under normality with $\rho = 0.5$, $n = 100$ and one additive outlier of size h in the x -direction.

$\rho = 0.5$ and $n = 100$ and shift the first observation to the right by a distance h ranging from 0 to 5. This yields a high leverage point, suggesting a smaller correlation. We measure the influence of this one outlier in the x -direction by the difference of the estimate before and after the manipulation. The result is a sensitivity curve along the x -direction (divided by the factor $n = 100$), plotted in Figure 5. The influence of the additive outlier is very small for the spatial sign correlation and also for most other robust estimators. An exception is the Gaussian rank correlation, which is known to have an unbounded influence function. Several highly robust estimators (in particular, the S-estimator and the MCD) completely disregard outliers that are sufficiently far away from the bulk of the data, and their sensitivity curves tend back to 0 as h further increases.

In the second setting, we start as usual with normally distributed data, $\rho = 0.5$, marginal variances 1 and $n = 100$. Then we replace, one after another, the “good” observations by outliers, which stem from a normal distribution with marginal variances 4 and correlation $\rho = -0.5$. In Figure 6, the bias of the estimators (average of 50,000 repetitions) is plotted against the contamination fraction. Here the picture is somewhat reversed to the efficiency results under normality: the rather efficient rank-based estimators like Spearman’s ρ and Kendall’s τ are substantially biased, and the rather inefficient and highly robust estimators (MCD, S, Stahel-Donoho) perform better. As before, the spatial sign correlation takes a place in the middle.

5.4. Results under non-ellipticity. The robust correlation estimators are designed to estimate Pearson’s moment correlation at the normal model, and the questions remains, what happens in data models that exhibit none of the basic geometric characteristics of the normal distribution, such as symmetry or unimodality. Is the estimate at least somewhere near the actual moment correlation of the population distribution? An in-depth answer, alone for spatial sign correlation, is beyond the scope of the paper, but we want to get a rough impression in a simulated example. We consider a unimodal, but heavily skewed distribution. Let $X = \alpha Z_1 + Z_2$ and $Y = Z_1 + \alpha Z_2$, where α is a scalar parameter and Z_1 and Z_2 are two independent, exponentially distributed random variables (with parameter $\lambda = 1$). By letting α vary between 0 and 1, one can generate any (positive) correlation ρ between X and Y . The explicit formula is

$$\alpha = (1 - \sqrt{1 - \rho^2})/\rho$$

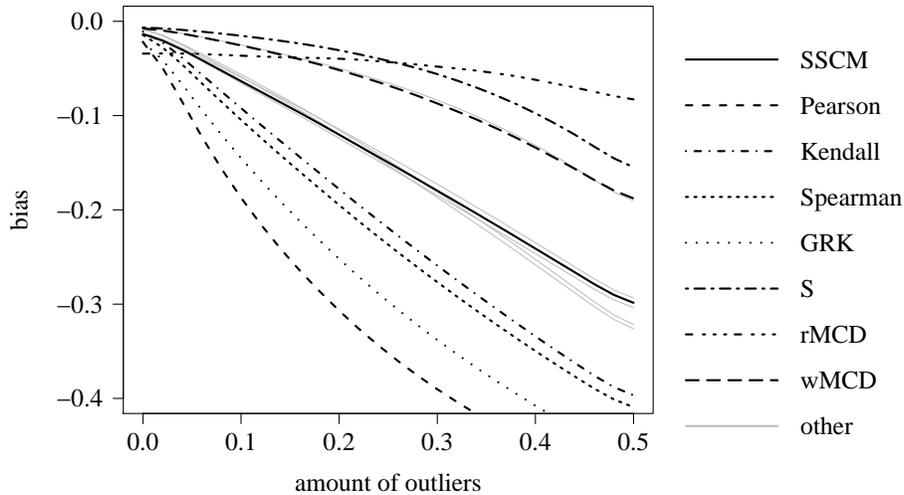


FIGURE 6. Bias of correlation estimators under normality with $\rho = 0.5$, $n = 100$ and with a different amount of outliers with correlation $\rho = -0.5$.

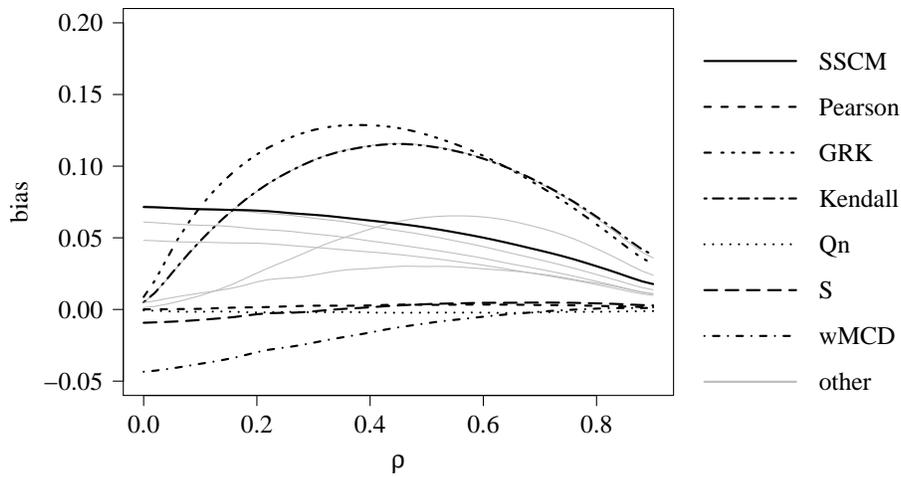


FIGURE 7. Bias of correlation estimators at a bivariate exponential distribution for $n = 100$ and different ρ .

In Figure 7, the bias of the estimators (based 50000 repetitions) is plotted against the correlation ρ . The sample size is $n = 100$. It is not surprising that most estimators, particularly the nonparametric ones including the spatial sign correlation, are substantially biased. Besides the sample correlation, we find the Gnanadesikan-Kettenring-estimator based on the Q_n (but not on the τ -scale) and the S -estimator to be nearly unbiased.

APPENDIX A. PROOFS

Proof of Proposition 1. The proofs of parts (1) and (2) are fairly straightforward employing the definitions of $\boldsymbol{\mu}(\mathbf{X})$ and $S(\mathbf{X})$. The key is the orthogonal equivariance of the spatial median and the orthogonal invariance of the spatial sign. A proof of a more general version of part (2) can also be found in Visuri (2001). It remains to show part (3). We only consider the non-trivial case $\lambda_1 \neq \lambda_2$. Since $\mathbf{X} \sim F \in \mathcal{E}_p(\boldsymbol{\mu}, V)$, there exists a spherical random variable \mathbf{Y} such that

$\mathbf{X} = U\Lambda^{\frac{1}{2}}\mathbf{Y} + \boldsymbol{\mu}$, with U and Λ as in (2). We thus have

$$S(\mathbf{X}) = \mathbb{E} \left\{ \frac{(\mathbf{X} - \boldsymbol{\mu})(\mathbf{X} - \boldsymbol{\mu})^T}{(\mathbf{X} - \boldsymbol{\mu})^T(\mathbf{X} - \boldsymbol{\mu})} \right\} = U \mathbb{E} \left\{ \frac{\Lambda^{1/2}\mathbf{Y}\mathbf{Y}^T\Lambda^{1/2}}{\mathbf{Y}^T\Lambda\mathbf{Y}} \right\} U^T.$$

It remains to evaluate the diagonal elements δ_1 and δ_2 of the expectation on the right-hand side. (Since spherical distributions have symmetrically distributed margins, the off-diagonal elements are zero.) The spatial sign is distribution-free within the elliptical model, i.e. $\mathbf{s}(\mathbf{X}) = \mathbf{s}(\tilde{\mathbf{X}})$ in distribution for any two elliptical random vectors \mathbf{X} and $\tilde{\mathbf{X}}$ with the same shape matrix V . The distribution of $\mathbf{s}(\mathbf{X})$ for elliptical \mathbf{X} is also known as the *angular central Gaussian distribution*, cf. Tyler (1987b). Hence we can choose any spherical distribution for \mathbf{Y} , e.g. the uniform distribution on the unit circle with density

$$(4) \quad f(\mathbf{y}) = \frac{1}{\pi} \mathbb{1}_{[0,1]}(\mathbf{y}^T \mathbf{y}),$$

which yields with $\mathbf{y} = (y_1, y_2)$

$$\delta_1 = \frac{1}{\pi} \int_0^1 \int_{-\sqrt{1-y_1^2}}^{\sqrt{1-y_1^2}} \frac{\lambda_1 y_1^2}{\lambda_1 y_1^2 + \lambda_2 y_2^2} dy_2 dy_1.$$

Substituting spherical coordinates $y_1 = r \cos(\alpha)$, $y_2 = r \sin(\alpha)$, we obtain

$$\delta_1 = \frac{1}{\pi} \int_0^1 \int_0^{2\pi} \frac{\lambda_1 r^3 \cos(\alpha)^2}{\lambda_1 r^2 \cos(\alpha)^2 + \lambda_2 r^2 \sin(\alpha)^2} d\alpha dr.$$

Using the identities $\cos(\alpha) = (e^{i\alpha} + e^{-i\alpha})/2$ and $\sin(\alpha) = (e^{i\alpha} - e^{-i\alpha})/(2i)$, we further substitute $z = e^{i\alpha}$ and get

$$(5) \quad \delta_1 = \frac{1}{\pi} \int_0^1 r \oint_{\Gamma} \frac{\lambda_1 (z^2 + 1)^2}{iz((\lambda_1 - \lambda_2)z^4 + 2(\lambda_1 + \lambda_2)z^2 + (\lambda_1 - \lambda_2))} dz dr,$$

where Γ denotes the unit circle on the complex plane, and \oint_{Γ} the (closed curve) line integral along Γ . We apply the residue theorem to solve the inner line integral (e.g. Ahlfors, 1966, p. 149). The integrand is meromorphic and has no pole on the boundary of the unit circle. The residue theorem thus yields

$$\delta_1 = \frac{1}{\pi} \int_0^1 2\pi i r \sum_{a \in P} \text{Res}(\phi, a) dr,$$

where ϕ is the integrand of equation (5), P its set of poles within the unit circle, and $\text{Res}(\phi, a)$ the residue of ϕ in a . The integrand ϕ has three poles inside the unit circle

$$(6) \quad z_1 = 0, \quad z_{2/3} = \pm \frac{\sqrt{\lambda_1} - \sqrt{\lambda_2}}{\sqrt{\lambda_1 - \lambda_2}}$$

with residues

$$\text{Res}(\phi, 0) = \frac{-i\lambda_1}{(\lambda_1 - \lambda_2)}, \quad \text{Res}(\phi, z_{2/3}) = \frac{i\sqrt{\lambda_1\lambda_2}}{2(\lambda_1 - \lambda_2)}.$$

Hence we obtain

$$\delta_1 = \frac{1}{\pi} \int_0^1 2\pi r \frac{\sqrt{\lambda_1}}{\sqrt{\lambda_1} + \sqrt{\lambda_2}} dr = \frac{\sqrt{\lambda_1}}{\sqrt{\lambda_1} + \sqrt{\lambda_2}}.$$

The expression for δ_2 is obtained by exchanging λ_1 and λ_2 . The proof is complete. \square

Proof of Proposition 2. Parts (1) and (2) are proved in Dürre et al. (2014). In particular, Theorem 3 of Dürre et al. (2014) identifies W_S , under the conditions of part (2), as the asymptotic covariance matrix of the SSCM with known location, i.e.,

$$(7) \quad W_S = \text{Cov} \left(\text{vec} \left\{ \mathbf{s}(\mathbf{X} - \boldsymbol{\mu}) \mathbf{s}(\mathbf{X} - \boldsymbol{\mu})^T \right\} \right).$$

For proving part (3) it remains to evaluate (7) under the additional assumption $\mathbf{X} \sim F \in \mathcal{E}_2(\boldsymbol{\mu}, V)$. As in the proof of Proposition 1, we use the representation $\mathbf{X} = U\Lambda^{1/2}\mathbf{Y} + \boldsymbol{\mu}$, where $\mathbf{Y} = (Y_1, Y_2)^T$ is a spherical random vector. Again, we only consider the non-trivial case $\lambda_1 \neq \lambda_2$ and obtain

$$(8) \quad W_S = (U \otimes U) \text{Cov} \left\{ \text{vec} \left(\frac{\Lambda^{1/2} \mathbf{Y} \mathbf{Y}^T \Lambda^{1/2}}{\mathbf{Y}^T \Lambda \mathbf{Y}} \right) \right\} (U \otimes U)^T$$

The inner matrix on the right-hand side is

$$(9) \quad \mathbb{E} \left\{ \frac{1}{(\mathbf{Y}^T \Lambda \mathbf{Y})^2} \begin{pmatrix} \lambda_1^2 Y_1^4 & 0 & 0 & \lambda_1 Y_1^2 \lambda_2 Y_2^2 \\ 0 & \lambda_1 Y_1^2 \lambda_2 Y_2^2 & \lambda_1 Y_1^2 \lambda_2 Y_2^2 & 0 \\ 0 & \lambda_1 Y_1^2 \lambda_2 Y_2^2 & \lambda_1 Y_1^2 \lambda_2 Y_2^2 & 0 \\ \lambda_1 Y_1^2 \lambda_2 Y_2^2 & 0 & 0 & \lambda_2^2 Y_2^4 \end{pmatrix} \right\} \\ - \begin{pmatrix} \delta_1^2 & 0 & 0 & \delta_1 \delta_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \delta_1 \delta_2 & 0 & 0 & \delta_2^2 \end{pmatrix}$$

It remains to solve the three integrals

$$I_1 = \mathbb{E} \left\{ \frac{\lambda_1^2 Y_1^4}{(\lambda_1 Y_1^2 + \lambda_2 Y_2^2)^2} \right\}, \quad I_2 = \mathbb{E} \left\{ \frac{\lambda_2^2 Y_2^4}{(\lambda_1 Y_1^2 + \lambda_2 Y_2^2)^2} \right\}, \quad I_3 = \mathbb{E} \left\{ \frac{\lambda_1 Y_1^2 \lambda_2 Y_2^2}{(\lambda_1 Y_1^2 + \lambda_2 Y_2^2)^2} \right\},$$

where I_1 and I_2 are of the same type: I_2 is obtained from I_1 by simply exchanging λ_1 and λ_2 . We start with I_1 . By a fully analogous chain of arguments and manipulations as in the proof of Proposition 1, we arrive at

$$I_1 = \frac{1}{\pi} \int_0^1 r \oint_{\Gamma} \frac{\lambda_1^2 (z^2 + 1)^4}{iz((\lambda_1 - \lambda_2)z^4 + 2(\lambda_1 + \lambda_2)z^2 + (\lambda_1 - \lambda_2))^2} dz dr.$$

and apply again the residue theorem to solve the inner line integral. We call the integrand ϕ_1 and observe that it has the same singularities as the integrand ϕ in the proof of Proposition 1, cf. (6). However, the poles z_2 and z_3 are of order two here, resulting in the residues

$$\text{Res}(\phi_1, 0) = \frac{-i\lambda_1^2}{(\lambda_1 - \lambda_2)^2}, \quad \text{Res}(\phi_1, z_{2/3}) = \frac{i\sqrt{\lambda_1 \lambda_2}(3\lambda_1 - \lambda_2)}{4(\lambda_1 - \lambda_2)^2}.$$

Hence we obtain

$$I_1 = \frac{1}{\pi} \int_0^1 2\pi r \frac{\lambda_1^2 - \frac{1}{2}\sqrt{\lambda_1 \lambda_2}(3\lambda_1 - \lambda_2)}{(\lambda_1 - \lambda_2)^2} dr = \frac{\lambda_1^2 - \frac{1}{2}\sqrt{\lambda_1 \lambda_2}(3\lambda_1 - \lambda_2)}{(\lambda_1 - \lambda_2)^2}.$$

It remains to solve I_3 , which we transform, again, by the same chain of arguments as in the proof of Proposition 1, to

$$I_3 = -\frac{1}{\pi} \int_0^1 r \oint_{\Gamma} \frac{\lambda_1 \lambda_2 (z^4 - 1)^2}{iz((\lambda_1 - \lambda_2)z^4 + 2(\lambda_1 + \lambda_2)z^2 + (\lambda_1 - \lambda_2))^2} dz dr.$$

We call the integrand ϕ_3 . Its poles are also given by (6) with z_2 and z_3 being of order two, resulting in the residues

$$\text{Res}(\phi_3, 0) = \frac{-i\lambda_1 \lambda_2}{(\lambda_1 - \lambda_2)^2}, \quad \text{Res}(\phi_3, z_{2/3}) = \frac{i\sqrt{\lambda_1 \lambda_2}(\lambda_1 + \lambda_2)}{4(\lambda_1 - \lambda_2)^2}.$$

We finally arrive at

$$I_3 = -\frac{1}{\pi} \int_0^1 2\pi r \frac{\lambda_1 \lambda_2 - \frac{1}{2} \sqrt{\lambda_1 \lambda_2} (\lambda_1 + \lambda_2)}{(\lambda_1 - \lambda_2)^2} = \frac{-\lambda_1 \lambda_2 + \frac{1}{2} \sqrt{\lambda_1 \lambda_2} (\lambda_1 + \lambda_2)}{(\lambda_1 - \lambda_2)^2}.$$

Plugging the obtained expressions for I_1 , I_2 and I_3 into the matrix (9) and observing (8) yields the expression for W_S given in Proposition 2 (3). The proof is complete. \square

Towards the proof of Propostion 3, we consider as an intermediate step the SSCM-based shape estimator \hat{V}_n defined at the beginning of Section 3. Precisely, we give the asymptotic distribution of the estimator

$$\hat{V}_{0,n} = \begin{pmatrix} \hat{v}_{0,11} & \hat{v}_{0,12} \\ \hat{v}_{0,12} & \hat{v}_{0,22} \end{pmatrix} = \frac{1}{\sqrt{\hat{v}_{11} \hat{v}_{22}}} \hat{V}_n.$$

We have remarked at the end of Section 1 that, for analyzing the scale-invariant properties of the shape of an elliptical distribution, fixing the overall scale of the shape matrix V is not necessary, and we view the shape as an equivalence class of positive definite matrices being proportional to each other. For explicit computations, however, it is at some point necessary to fix the scale, that is, picking one specific representative from the equivalence class. Various ways of standardizing the shape can be found in the literature. Paindaveine (2008) argues to choose $\det(V) = 1$, which corresponds to our choice of \hat{V}_n in Section 3. However, for our purposes, it is most convenient to standardize V such that the product of its diagonal elements is 1, which corresponds to $\hat{V}_{0,n}$ described above. Accordingly, we denote by V_0 the representative of the equivalence class with this property (reciprocal diagonal elements) and parametrize it as

$$(10) \quad V_0 = \begin{pmatrix} a & \rho \\ \rho & a^{-1} \end{pmatrix},$$

where the parameters a and ρ have the same meaning as in Section 3, that is, the ratio of the diagonal elements of V and the correlation, respectively. Lengthy but straightforward calculus yields

$$(11) \quad \hat{v}_{0,12} = \frac{c \hat{s}_{12} b}{\sqrt{(\hat{s}_{12}^2 + b^2)^2 + (\hat{s}_{12} c b)^2}},$$

$$(12) \quad \hat{v}_{0,11} = \frac{2\mathbf{s}(\hat{s}_{12}) \sqrt{-\rho(4\rho\hat{s}_{12}^2 + 4\sqrt{1 - \tilde{s}_{12}^2} \hat{s}_{12} - \tilde{s}_{12}) + 2\tilde{s}_{1,2} - 4\sqrt{1 - \tilde{s}_{12}^2} \hat{s}_{12}}}{4\hat{s}_{12} b}$$

where, as before, \hat{s}_{ij} , $i, j = 1, 2$, denote the elements of the SSCM $\hat{S}(\mathbb{X}_n; \boldsymbol{\mu}_n)$, and b , c and d are defined in (3). The following proposition summarizes the asymptotic behavior of the estimator $\hat{V}_{0,n}$.

Proposition 5. *Under the assumptions of Proposition 3, we have for $n \rightarrow \infty$ that*

- (1) $\hat{V}_{0,n} \xrightarrow{a.s.} V_\rho$ and
 - (2) $\sqrt{n} \{(\hat{v}_{0,11}, \hat{v}_{0,12})^T - (a, \rho)^T\} \xrightarrow{\mathcal{L}} N_2(\mathbf{0}, W_{V_0})$, where $W_{V_0} = G W_S G^T$
- with

$$G = \frac{\left((a^2 + 1) \sqrt{1 - \rho^2} + 2a(1 - \rho^2) \right)}{\sqrt{1 - \rho^2} (4a^2 \rho^2 + (a^2 - 1)^2)} \begin{pmatrix} g_{1,1} & g_{1,2} & 0 & 0 \\ g_{2,1} & g_{2,2} & 0 & 0 \end{pmatrix}$$

and

$$\begin{aligned} g_{1,1} &= (a^2 - 1)^2 \sqrt{1 - \rho^2} + 2a(a^2 + 1)\rho^2, \\ g_{1,2} &= (a - 1)(a + 1)\rho \left\{ 2a\sqrt{1 - \rho^2} - a^2 - 1 \right\}, \\ g_{2,1} &= \frac{1}{a} \left\{ (a^2 + 1)\sqrt{1 - \rho^2} - 2a(1 - \rho^2) \right\}, \\ g_{2,2} &= 2(a^2 + 1)\rho^2\sqrt{1 - \rho^2} + a^{-1}(a^2 - 1)^2(1 - \rho^2). \end{aligned}$$

Proof of Proposition 5. Part (1) is a consequence of the continuous mapping theorem, part (2) follows with the delta method. Note that V_0 is specified by the two elements $\hat{v}_{0,11}$ and $\hat{v}_{0,12}$, and, likewise, $\hat{S}_n = \hat{S}_n(\mathbb{X}_n; \boldsymbol{\mu}_n)$ by the two elements \hat{s}_{11} and \hat{s}_{12} . Let H be the function that maps $(\hat{s}_{11}, \hat{s}_{12})$ to $(\hat{v}_{0,11}, \hat{v}_{0,12})$ and (s_{11}, s_{12}) to (a, ρ) . It is given explicitly by the formulas (11) and (12), from which we can compute its derivative. However, due to the complex structure of H , it is a cumbersome task to compute its derivative. It is much easier to compute the derivative of its inverse and apply the inverse function theorem. With $\{(x, y) \mid 0 < x, |y| < x\}$ and $\{(x, y) \mid 0 < x < 1, |y| < x\}$ being its domain and image, respectively, the function H is invertible and continuously differentiable. Let J denote its inverse. The function J maps (a, ρ) to (s_{11}, s_{12}) and is described in Proposition 1. In the following, we will compute its derivative, for which we require an explicit form of J . The eigenvalue decomposition of V_0 is given by

$$\lambda_{1/2} = (2a)^{-1} (a^2 + 1 \pm \sqrt{q})$$

and

$$U = \begin{pmatrix} \frac{2a|\rho|}{\{(\sqrt{q}-a^2+1)^2+4a^2\rho^2\}^{1/2}} & \frac{2a|\rho|}{\{(\sqrt{q}+a^2-1)^2+4a^2\rho^2\}^{1/2}} \\ \frac{\mathbf{s}(\rho)(\sqrt{q}-a^2+1)}{\{(\sqrt{q}-a^2+1)^2+4a^2\rho^2\}^{1/2}} & -\frac{\mathbf{s}(\rho)(\sqrt{q}+a^2-1)}{\{(\sqrt{q}+a^2-1)^2+4a^2\rho^2\}^{1/2}} \end{pmatrix},$$

where $q = 4a^2\rho^2 + (a^2 - 1)^2$. By Proposition 1 (1) and (2) we find

$$s_{11} = \frac{\sqrt{k}\{4a^2\rho^2 + \sqrt{q}(a^2 - 1) + (a^2 - 1)^2\} + \sqrt{m}\{4a^2\rho^2 + \sqrt{q}(1 - a^2) + (a^2 - 1)^2\}}{2q(\sqrt{m} + \sqrt{k})}$$

and $s_{12} = (2q)^{-1}a\rho(\sqrt{k} - \sqrt{m})^2$, where $k = a^2 + 1 + \sqrt{q}$ and $m = a^2 + 1 - \sqrt{q}$. The derivative of J is

$$\mathbb{D}J(a, \rho) = \begin{pmatrix} \frac{2a(a^2+1)\rho^2\sqrt{1-\rho^2}+(a^2-1)(1-\rho^2)}{q((a^2+1)\sqrt{1-\rho^2}+2a(1-\rho^2))} & -\frac{(a-1)a(a+1)\rho(2a\sqrt{1-\rho^2}-a^2-1)}{q((a^2+1)\sqrt{1-\rho^2}+2a(1-\rho^2))} \\ -\frac{(a-1)(a+1)\rho((a^2+1)\sqrt{1-\rho^2}-2a(1-\rho^2))}{q((a^2+1)\sqrt{1-\rho^2}+2a(1-\rho^2))} & \frac{a((a-1)^2\sqrt{1-\rho^2}+2a(a^2+1)\rho^2)}{q((a^2+1)\sqrt{1-\rho^2}+2a(1-\rho^2))} \end{pmatrix}.$$

The determinant of this matrix is

$$\det \mathbb{D}J(a, \rho) = a\sqrt{1 - \rho^2} \left\{ (a^2 + 1) \sqrt{1 - \rho^2} + 2a(1 - \rho^2) \right\}^{-2}.$$

By virtue of the inverse function theorem, we have $\mathbb{D}H(s_{11}, s_{12}) = (\mathbb{D}J(a, \rho))^{-1}$. Hence we obtain $\mathbb{D}H(s_{11}, s_{12})$ by inverting the 2×2 matrix $\mathbb{D}J(a, \rho)$. It can be seen to be (except for the zero columns) the matrix G in Proposition 5. The proof is complete. \square

Proof of Proposition 3. Proposition 3 is an immediate corollary of Proposition 5, noting that $\hat{\rho}_n = \hat{v}_{0,12}$. The asymptotic variance of $\hat{\rho}_n$ is the lower diagonal element of W_{V_0} given in Proposition 5. \square

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