WREATH PRODUCTS AND REPRESENTATIONS OF p-LOCAL FINITE GROUPS

NATÀLIA CASTELLANA AND ASSAF LIBMAN

Abstract. Given two finite p-local finite groups and a fusion preserving morphism between their Sylow subgroups, we study the question of extending it to a continuous map between their classifying spaces. The results depend on the construction of the wreath product of p-local finite groups which is also used to study p-local permutation representations.

1. Introduction

A fusion system $\mathcal{F}$ on a finite p-group P is a small category whose objects are the subgroups of P and whose morphisms are group monomorphisms which include all those homomorphisms obtained from conjugation by the elements of P. The idea of *saturated* fusion systems was formulated in the early 1980's by Puig [20] who studied representations of finite groups. Every block b of the group algebra kG, where k is an algebraically closed field of characteristic p, gives rise to a *saturated* fusion system on its defect group $P \leq G$. The principal block of kG gives rise to the fusion system $\mathcal{F}_S(G)$ whose objects are the subgroups of a Sylow p-subgroup S of G and its morphisms are induced by conjugation in G. Not all fusion systems have the form $\mathcal{F}_S(G)$, see e.g. [6, Examples 9.3-4] or [11].

The significance of $\mathcal{F}_S(G)$ in topology was recognized by Martino and Priddy in [15]. In [17, 18], Oliver shows that $\mathcal{F}_S(G)$ determines the homotopy type of the p-completion (in the sense of Bousfield and Kan [2]) of $BG = K(G, 1)$.

In order to understand self-homotopy equivalences of $BG^\wedge_p$, Broto, Levi and Oliver considered in [5] a category $\mathcal{L}_S(G)$ closely related to $\mathcal{F}_S(G)$. This category was studied earlier by Puig. Abstraction of this construction led them in [6] to the notion of a *centric linking system* $\mathcal{L}$ associated to a saturated fusion system $(S, \mathcal{F})$. The triple $(S, \mathcal{F}, \mathcal{L})$ is called a p-local finite group. Its *classifying space* is by definition the space $|\mathcal{L}|^\wedge_p$, a terminology justified by the fact that $|\mathcal{L}_S(G)|^\wedge_p \simeq BG^\wedge_p$ ([5, Lemma 1.2]). The spaces $|\mathcal{L}|^\wedge_p$ have many properties in common with p-completed classifying spaces of finite groups. Thus, p-local finite groups provide an important connection between group theory and topology via their linking systems.

This paper focuses on the following fundamental problem. In what way, if any, a fusion preserving map $(S, \mathcal{F}) \to (S', \mathcal{F}')$, see details below, gives rise to a map $|\mathcal{L}|^\wedge_p \to |\mathcal{L}'|^\wedge_p$ between the classifying spaces? A step forward is given in Theorem B below. It is related to the yet open problem of defining the concept of morphisms between p-local finite groups in a way which is compatible with maps between

Date: February 16, 2009.

Key words and phrases. Permutation representations, p-local finite groups.

The first author was supported by the FEDER/MEC grant MTM2007-61545. The second author was supported by an EPSRC grant EP/D506484/1.

their classifying spaces. It also gives a new insight to the study of maps between p-completed classifying spaces.

We will define a permutation representation of $(S,\mathcal{F},\mathcal{L})$ as a homotopy class of a map $|\mathcal{L}| \to (B\Sigma_n)^\wedge_p$ where Σ_n is a symmetric-group. In Theorem C below we will prove a p-local form of Cayley's theorem, namely the existence of p-local regular representations. We will then approach the notion of the homotopy-index of the Sylow subgroup S in $(S,\mathcal{F},\mathcal{L})$ through the regular representation. The index of a subgroup S in a finite group G is the number of the orbits of S in its action by translation on G. In other words, restriction of the regular representation of G to S results in $|G : S|$ copies of the regular representation of S. From the homotopy point of view, one could define the homotopy-index of S in $\mathcal{L}$ as the minimal n for which there is a map $|\mathcal{L}| \to (B\Sigma_{n\cdot|S|})^\wedge_p$ whose restriction to BS is homotopic to the map $BS \xrightarrow{n\cdot\mathrm{reg}_S} (B\Sigma_{n\cdot|S|})^\wedge_p$ induced by n copies of the regular representation of S. But this number is very difficult to compute, even for a p-local finite group associated to a finite group. Instead, we will define the lower homotopy-index of S in $\mathcal{L}$ as the smallest number p^k such that the map $BS \to (B\Sigma_{p^k\cdot|S|})^\wedge_p$ induced by $p^k\cdot\mathrm{reg}_S$ can be extended up to homotopy to a map $|\mathcal{L}| \to (B\Sigma_{p^k\cdot|S|})^\wedge_p$. This is a new invariant of p-local finite groups.

Let us now describe our results in greater detail. Suppose that $(S,\mathcal{F},\mathcal{L})$ and $(S',\mathcal{F}',\mathcal{L}')$ are p-local finite groups. Given a group homomorphism $\rho\colon S \to S'$ it is natural to ask if $B\rho\colon BS \to BS'$ can be extended, up to homotopy, to a map $\tilde{f}\colon |\mathcal{L}|^\wedge_p \to |\mathcal{L}'|^\wedge_p$ whose restriction to BS, namely $\tilde{f}\circ\Theta$, is homotopic to the composite $BS \xrightarrow{B\rho} BS' \xrightarrow{\Theta'} |\mathcal{L}'|^\wedge_p$ where Θ and Θ' are the maps described in .

Recall that ρ is called *fusion preserving* if for every $\varphi \in \mathcal{F}(P,Q)$ there exists some $\varphi' \in \mathcal{F}'(\rho(P),\rho(Q))$ such that $\rho\circ\varphi = \varphi'\circ\rho$. Ragnarsson shows in [22] that stably, namely in the homotopy category of spectra, the morphism $\tilde{f}$ above exists if and only if ρ is fusion preserving. Unstably this is unknown.

The content of Theorem B below is that $\tilde{f}$ exists provided the target $\mathcal{L}'$ is replaced with its wreath product with Σ_n for some n, a construction which we now describe.

The *wreath product* of a space X with a subgroup $G \leq \Sigma_n$, denoted $X \wr G$, is the homotopy orbit space $(X^n)_{hG}$ where G acts by permuting the factors (see Definition 3.4). This construction is equipped with a map $\Delta\colon X \to X \wr G$ which factors through the diagonal map $X \to X^n$. We prove in 3.6 below that if H is a discrete groups then there is a homotopy equivalence $(BH)\wr G \simeq B(H \wr G)$ such that $\Delta\colon BH \to (BH)\wr G$ is induced by the diagonal inclusion $H \leq H\wr G$. The next result should be compared with [3, Theorems D and E].

Theorem A. *Fix a p-local finite group $(S,\mathcal{F},\mathcal{L})$ where $S \neq 1$. Let K be a subgroup of Σ_n and let S' be a Sylow p-subgroup of $S \wr K$. Then there exists a p-local finite group $(S',\mathcal{F}',\mathcal{L}')$ which is equipped with a homotopy equivalence $|\mathcal{L}|\wr K \simeq |\mathcal{L}'|$ such that the composite*

$$BS' \xrightarrow{Bincl} B(S\wr K) \simeq (BS)\wr K \xrightarrow{\Theta\wr K} |\mathcal{L}|\wr K \simeq |\mathcal{L}'|$$

is homotopic to the natural map $\Theta'\colon BS' \to |\mathcal{L}'|$. Moreover, $(S',\mathcal{F}',\mathcal{L}')$ satisfying these properties is unique up to an isomorphism of p-local finite groups.

In Remark 5.3 we show that when Theorem A is applied to a p-local finite group $(S,\mathcal{F},\mathcal{L})$ of a finite group G then $(S',\mathcal{F}',\mathcal{L}')$ is the p-local finite group of $G \wr K$.

If $S = 1$ then $|\mathcal{L}| = *$ and we choose $(S', \mathcal{F}', \mathcal{L}')$ to be the p-local finite group associated to K and the map $\Delta\colon |\mathcal{L}| \to |\mathcal{L}'|$ is any map $* \to |\mathcal{L}'|$.

We prove Theorem A in §5 which is a technical section, however the remainder of the paper is completely independent of its proof.

1.1. **Definition.** We call the p-local finite group $(S', \mathcal{F}', \mathcal{L}')$ in the theorem above the *wreath product* of $(S, \mathcal{F}, \mathcal{L})$ with K and denote its fusion system and linking system by $\mathcal{F} \wr K$ and $\mathcal{L} \wr K$ respectively. Let $\Delta\colon |\mathcal{L}| \to |\mathcal{L}| \wr K \simeq |\mathcal{L}'|$ denote the diagonal inclusion followed by the homotopy equivalence in Theorem A.

Theorem B. *Let $(S, \mathcal{F}, \mathcal{L})$ and $(S', \mathcal{F}', \mathcal{L}')$ be p-local finite groups and suppose that $\rho\colon S \to S'$ is a fusion preserving homomorphism. Then there exists some $m \geq 0$ and a map $\tilde{f}\colon |\mathcal{L}|^\wedge_p \to |\mathcal{L}' \wr \Sigma_{p^m}|^\wedge_p$ such that the diagram below commutes up to homotopy*

$$\begin{tikzcd}
BS \arrow[r, "\eta\circ\Theta"] \arrow[d, "B\rho"'] & {|\mathcal{L}|^\wedge_p} \arrow[dr, "\tilde{f}"] & \\
BS' \arrow[r, "\eta\circ\Theta'"'] & {|\mathcal{L}'|^\wedge_p} \arrow[r, "\Delta^\wedge_p"'] & {|\mathcal{L}' \wr \Sigma_{p^m}|^\wedge_p.}
\end{tikzcd}$$

In Theorem 7.3 below we prove a more elaborate result.

A *permutation representation* of a finite group G is a homomorphism $\rho\colon G \to \Sigma_n$. The *rank* of ρ is n. Throughout, we will call ρ simply a "representation". Clearly G acts on itself by left (or right) translations giving rise to Cayley's embedding $\mathrm{reg}_G\colon G \to \Sigma_{|G|}$ which is called the *regular permutation representation* of G.

Two representations $\rho_1, \rho_2\colon G \to \Sigma_n$ are *equivalent* if they are conjugate in Σ_n, that is, if they differ by an inner automorphism of Σ_n. The set of equivalence classes of representations of G of rank n is denoted $\mathrm{Rep}_n(G)$. There are obvious inclusions $\Sigma_n \times \Sigma_m \leq \Sigma_{n+m}$ and $\Sigma_n \times \Sigma_m \leq \Sigma_{nm}$ obtained by taking the disjoint union and the product of sets of cardinality n and m. They give rise to commutative, associative and unital binary operations $+$ and $\times$ on the set $\coprod_{n\geq 0} \mathrm{Rep}_n(G)$. We shall write $k \cdot \rho$ for the k-fold sum $\rho + \cdots + \rho$.

Let $\mathcal{F}$ be a fusion system on S. A representation $\rho\colon S \to \Sigma_n$ is called *$\mathcal{F}$-invariant* if for every $P \leq S$ and every $\varphi \in \mathcal{F}(P, S)$ the representations $\rho|_P$ and $\rho \circ \varphi$ of P are equivalent. Let $\mathrm{Rep}_n(\mathcal{F})$ denote the subset of $\mathrm{Rep}_n(S)$ of all the equivalence classes of the $\mathcal{F}$-invariant representations of S of rank n. It is easy to see that $\coprod_{n\geq 0} \mathrm{Rep}_n(\mathcal{F})$ is closed under the operations $+$ and $\times$ on $\coprod_{n\geq 0} \mathrm{Rep}_n(S)$.

We define the set of *representations at p of rank n of a space X* as the set $\mathrm{Rep}_{n,p}(X) = [X, (B\Sigma_n)^\wedge_p]$ of unpointed homotopy classes of unpointed maps. Since $(B\Sigma_m)^\wedge_p \times (B\Sigma_n)^\wedge_p \simeq (B(\Sigma_m \times \Sigma_n))^\wedge_p$ (see [2, Theorem I.7.2]), the following maps $(B(\Sigma_m \times \Sigma_n))^\wedge_p \to (B\Sigma_{m+n})^\wedge_p$ and $(B(\Sigma_m \times \Sigma_n))^\wedge_p \to (B\Sigma_{mn})^\wedge_p$ induced by the inclusions equip $\coprod_{n\geq 0} \mathrm{Rep}_{n,p}(X)$ with commutative and associative binary operations $+$ and $\times$ such that $+$ is distributive over $\times$.

If P is a finite p-group then there are bijections

$$\mathrm{Rep}_n(P) \xrightarrow[\approx]{\rho \mapsto B\rho} [BP, B\Sigma_n] \xrightarrow[\approx]{f \mapsto \eta\circ f} [BP, (B\Sigma_n)^\wedge_p]$$

where $\eta\colon B\Sigma_n \to (B\Sigma_n)^\wedge_p$ is the completion map. The first bijection is a classical result going back to Hurewicz and the second was first shown by Mislin in [16, Proof of the main theorem]. In light of these bijections we make the following definition.

1.2. **Definition.** Fix a p-local finite group $(S, \mathcal{F}, \mathcal{L})$. We say that a permutation representation $f\colon |\mathcal{L}| \to (B\Sigma_n)^\wedge_p$ is *S-regular* if $n = m \cdot |S|$ for some $m \geq 0$ and the composite $BS \xrightarrow{\Theta} |\mathcal{L}| \xrightarrow{f} (B\Sigma_n)^\wedge_p$ is homotopic to $BS \xrightarrow{B(m \cdot \mathrm{reg}_S)^\wedge_p} (B\Sigma_n)^\wedge_p$.

We will deduce from Theorem B the following p-local form of Cayley's theorem.

Theorem C. *Every p-local finite group $(S, \mathcal{F}, \mathcal{L})$ admits an S-regular permutation representation $f\colon |\mathcal{L}| \to (B\Sigma_{p^m})^\wedge_p$.*

Recall from [5, Def. 2.2] that a continuous map $f\colon X \to Y$ is a *homotopy monomorphism at p* if $H^*(X; \mathbb{F}_p)$ is a finitely generated module over $H^*(Y; \mathbb{F}_p)$ via f^*. In Proposition 7.9 we show that every S-regular permutation representation is a homotopy monomorphism at p.

The reason we didn't define permutation representations as maps $|\mathcal{L}| \to B\Sigma_n$ (without p-completing the target) is that Theorem C would fail completely. For example, the nerve of the linking system of the Solomon p-local finite group, constructed by Levi and Oliver in [11], was shown to be simply connected in [8] and therefore [25, Theorem 8.1.11] implies that $[|\mathcal{L}_{\mathrm{Sol}}|, B\Sigma_n] = *$. In particular, the restriction of any $f\colon |\mathcal{L}_{\mathrm{Sol}}| \to B\Sigma_n$ to BS via Θ is induced by the trivial representation $\rho\colon S \to \Sigma_n$.

1.3. **Definition.** The *ring* $\mathrm{Rep}(|\mathcal{L}|)$ *of the virtual permutation representations* of a p-local finite group $(S, \mathcal{F}, \mathcal{L})$ is the Grothendieck group completion of the commutative monoid $(\coprod_{n \geq 0} \mathrm{Rep}_{n,p}(|\mathcal{L}|), +)$.

The *ring* $\mathrm{Rep}(\mathcal{F})$ *of the virtual $\mathcal{F}$-invariant representations of S* of a saturated fusion system $\mathcal{F}$ on S is the Grothendieck group completion of the commutative monoid $(\coprod_{n \geq 0} \mathrm{Rep}_n(\mathcal{F}), +)$.

Clearly $\mathrm{Rep}(\mathcal{F})$ is a subring of $\mathrm{Rep}(S)$. In §8 we will construct a ring homomorphism $\Phi\colon \mathrm{Rep}(\mathcal{L}) \to \mathrm{Rep}(\mathcal{F})$ which sends a map $f\colon |\mathcal{L}| \to (B\Sigma_n)^\wedge_p$ to the representation $\rho\colon S \to \Sigma_n$ such that $f \circ \Theta \simeq \eta \circ B\rho$ where f and Θ are as in Definition 1.2. We shall also see that $\mathrm{reg}_S\colon S \to \Sigma_{|S|}$ generates an ideal $\mathrm{Rep}^{\mathrm{reg}}(\mathcal{F})$ in $\mathrm{Rep}(\mathcal{F})$ whose underlying group is isomorphic to $\mathbb{Z}$.

The idea behind the next definition is that if H is a subgroup of index n in a finite group G then $\mathrm{reg}_G|_H \simeq n \cdot \mathrm{reg}_H$. Therefore the image of the restriction map $\mathrm{Rep}(G) \to \mathrm{Rep}(H)$ intersects $\mathrm{Rep}^{\mathrm{reg}}(H) := \{k \cdot \mathrm{reg}_H\}_{k \in \mathbb{Z}}$ in a subgroup of index divisible by n.

1.4. **Definition.** The lower p-local index of S in $\mathcal{L}$, denoted $\mathrm{Lind_p}(\mathcal{L}\colon S)$, is the index of $\mathrm{Im}(\Phi) \cap \mathrm{Rep}^{\mathrm{reg}}(\mathcal{F})$ in $\mathrm{Rep}^{\mathrm{reg}}(\mathcal{F})$.

We will prove in Lemma 8.5 that $\mathrm{Lind_p}(\mathcal{L}\colon S)$ is always a p-power. We conjecture that it is always equal to 1. A partial result is the theorem below.

Theorem D. *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group. Then* $\mathrm{Lind_p}(\mathcal{L}\colon S) = 1$ *if either*

(1) *$(S, \mathcal{F}, \mathcal{L})$ is associated with a finite group.*
(2) *$(S, \mathcal{F}, \mathcal{L})$ is one of the exotic examples in* [6, Examples 9.3 and 9.4] *or in* [23] *or in* [9] *or in* [7, Example 5.3].

Notation. The following notation will be used through the paper:

- $\eta\colon X \to X^\wedge_p$ is the Bousfield-Kan p-completion.

- If X is a G-space, $\kappa\colon X \to (X)_{hG} = EG \times_G X$ is the map $\kappa(x) = [e, x]$ for any $x \in X$ and e is the basepoint of EG.
- Given a map $f\colon X \to Y$ of spaces, let $\mathrm{map}^f(X, Y)$ denote the path component of f in $\mathrm{map}(X, Y)$. By convention f is the basepoint of this space.
- If $f : X \times Y \to Z$, the adjoint map is denoted by $f^\sharp\colon X \to \mathrm{map}(Y, Z)$.
- $\Theta\colon BS \to |\mathcal{L}|$ is the map from the Sylow subgroup introduced in 2.9.

We would like to thank Bob Oliver for pointing out an error in the proof of Theorem A which originated as an error in [3]. At the time this paper was revised he proved a result which generalises our theorem, see [19].

2. Preliminaries on p-local finite groups

We start with the notion of a saturated fusion system which is due to Puig [20] (see also [6]).

2.1. **Definition.** A fusion system $\mathcal{F}$ on a finite p-group S is a category whose objects are the subgroups of S and the set of morphisms $\mathcal{F}(P, Q)$ between two subgroups P, Q, satisfies the following conditions:

(a) $\mathcal{F}(P, Q)$ consists of group monomorphisms and contains the set $\mathrm{Hom}_S(P, Q)$ of all the homomorphisms $c_s\colon P \to Q$ which are induced by conjugation by elements $s \in S$.
(b) Every morphism in $\mathcal{F}$ factors as an isomorphism in $\mathcal{F}$ followed by an inclusion.

In a fusion system $\mathcal{F}$ over a p-group S, we say that two subgroups $P, Q \leq S$ are $\mathcal{F}$-conjugate if there is an isomorphism between them in $\mathcal{F}$. Let $\mathrm{Syl}_p(G)$ be the set of the Sylow p-subgroups of a group G. Given $P \leq G$ and $g \in G$, $c_g \in \mathrm{Hom}(P, G)$ is the monomorphism $c_g(x) = gxg^{-1}$. We write $\mathrm{Out}_{\mathcal{F}}(P) = \mathrm{Aut}_{\mathcal{F}}(P)/\mathrm{Inn}(P)$.

2.2. **Definition.** Let $\mathcal{F}$ be a fusion system on a p-group S. A subgroup $P \leq S$ is *fully centralized* in $\mathcal{F}$ if $|C_S(P)| \geq |C_S(P')|$ for all $P' \leq S$ which is $\mathcal{F}$-conjugate to P. A subgroup $P \leq S$ is *fully normalized* in $\mathcal{F}$ if $|N_S(P)| \geq |N_S(P')|$ for all $P' \leq S$ which is $\mathcal{F}$-conjugate to P.

A fusion system $\mathcal{F}$ on S is *saturated* if:

(I) Each fully normalized subgroup $P \leq S$ is fully centralized and $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$.
(II) For $P \leq S$ and $\varphi \in \mathcal{F}(P, S)$ set
$$N_\varphi = \{g \in N_S(P) | \varphi c_g \varphi^{-1} \in \mathrm{Aut}_S(\varphi(P))\}.$$
If $\varphi(P)$ is fully centralized then there is $\bar{\varphi} \in \mathcal{F}(N_\varphi, S)$ such that $\bar{\varphi}|_P = \varphi$.

2.3. **Definition.** Let $\mathcal{F}$ be a fusion system on a p-group S. A subgroup $P \leq S$ is $\mathcal{F}$-*centric* if P and all its $\mathcal{F}$-conjugates contain their S-centralizers. A subgroup $P \leq S$ is $\mathcal{F}$-*radical* if $\mathrm{Out}_{\mathcal{F}}(P)$ has no non-trivial normal p-subgroup.

2.4. **Definition.** [6] Let $\mathcal{F}$ be a fusion system on a p-group S. A *centric linking system associated to* $\mathcal{F}$ is a category $\mathcal{L}$ whose objects are the $\mathcal{F}$-centric subgroups of S, together with a functor $\pi\colon \mathcal{L} \longrightarrow \mathcal{F}^c$ and monomorphisms $P \xrightarrow{\delta_P} \mathrm{Aut}_{\mathcal{L}}(P)$ for each $\mathcal{F}$-centric subgroup $P \leq S$, which satisfy the following conditions:

(A) π is the identity on objects. For each pair of objects $P, Q \in \mathcal{L}$, the action of $Z(P)$ on $\mathcal{L}(P,Q)$ via precomposition and $\delta_P\colon P \to \mathrm{Aut}_{\mathcal{L}}(P)$ is free and π induces a bijection $\mathcal{L}(P,Q)/Z(P) \xrightarrow{\cong} \mathcal{F}(P,Q)$.
(B) If $P \leq S$ is $\mathcal{F}$-centric then $\pi(\delta_P(g)) = c_g \in \mathrm{Aut}_{\mathcal{F}}(P)$ for all $g \in P$.
(C) For each $f \in \mathcal{L}(P,Q)$ and each $g \in P$, the following square commutes in $\mathcal{L}$:

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "\delta_P(g)"'] & Q \arrow[d, "\delta_Q(\pi(f)(g))"] \\ P \arrow[r, "f"'] & Q \end{tikzcd}.$$

A *p-local finite group* $(S, \mathcal{F}, \mathcal{L})$ consists of a saturated fusion systems $\mathcal{F}$ on S together with an associated linking system.

2.5. **Definition.** Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group. A *system of lifts* in $\mathcal{L}$ consists of functions $\delta_{P,Q}\colon N_S(P,Q) \to \mathcal{L}(P,Q)$, one for each pair $P, Q \leq S$ of $\mathcal{F}$-centric subgroups, such that:

(a) $\pi(\delta_{P,Q}(g)) = c_g \in \mathcal{F}(P,Q)$ for all $g \in N_S(P,Q)$.
(b) $\delta_{P,P}(g) = \delta_P(g)$ for all $g \in P$, namely $\delta_{P,P}$ extends the structure map δ_P.
(c) If $g \in N_S(P,Q)$ and $h \in N_S(Q,R)$ then $\delta_{P,R}(hg) = \delta_{Q,R}(h) \circ \delta_{P,Q}(g)$.

For any $P \leq Q$ set $\iota_P^Q = \delta_{P,Q}(e)$ where e is the identity of S.

2.6. **Remark.** Any p-local finite group admits a system of lifts by [6, Prop. 1.11].

We will write $\hat{g}$ for $\delta_{P,Q}(g)$. In this notation conditions (a) and (c) become $\pi(\hat{g}) = c_g$ and $\widehat{hg} = \hat{h} \circ \hat{g}$. Also $\iota_Q^R \circ \iota_P^Q = \iota_P^R$.

2.7. **Remark.** Every morphism in $\mathcal{L}$ is both a monomorphism and an epimorphism (but not necessarily an isomorphism). This is shown in [6, remarks after Lemma 1.10] and [3, Corollary 3.10]. We shall use this fact repeatedly throughout.

The orbit category of a p-local finite group $(S, \mathcal{F}, \mathcal{L})$ is denoted by $\mathcal{O}(\mathcal{F})$. This is the category whose objects are the subgroups of S and whose morphisms are

$$\mathcal{O}(\mathcal{F})(P,Q) = \mathrm{Rep}_{\mathcal{F}}(P,Q) \overset{def}{=} {}_{\mathrm{Inn}(Q)} \setminus \mathcal{F}(P,Q).$$

Also, $\mathcal{O}(\mathcal{F}^c)$ is the full subcategory of $\mathcal{O}(\mathcal{F})$ whose objects are the $\mathcal{F}$-centric subgroups of S.

2.8. **Proposition.** [6, Proposition 2.2] *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group. There exists a functor $\widetilde{B}\colon \mathcal{O}(\mathcal{F}^c) \to$ **Top** which is isomorphic in the homotopy category of spaces to the functor $P \mapsto BP$, and such that there is a homotopy equivalence*

$$\operatorname*{hocolim}_{\mathcal{O}(\mathcal{F}^c)} \widetilde{B} \xrightarrow{\simeq} |\mathcal{L}|.$$

2.9**. Notation.** For a finite group G, let $\mathcal{B}G$ denote the category with one object $\bullet_G$ and G as its set of automorphisms. For an $\mathcal{F}$-centric $P \leq S$ the monomorphism δ_P gives rise to a functor $\mathcal{B}P \to \mathcal{L}$ which, by abuse of notation, we denote by δ_P. For $P = S$, upon taking nerves of categories, we obtain a map

$$\Theta\colon BS \to |\mathcal{L}|$$

and we write $\Theta|_{BQ}$ for $\Theta \circ B\mathrm{incl}_Q^S$.

If Q is $\mathcal{F}$-centric, then the natural isomorphism of functors in Proposition 2.8 shows that $\Theta|_{BQ}$ is homotopic to $BQ \simeq \tilde{B}(Q) \to \operatorname{hocolim}_{\mathcal{O}(\mathcal{F}^c)} \tilde{B} = |\mathcal{L}|$. Therefore, for any $\mathcal{F}$-centric $Q \leq S$ and any morphism $\rho\colon Q \to S$ in $\mathcal{F}$ we have $\Theta \circ B\rho \simeq \Theta|_{BQ}$. In particular, $\Theta|_{BQ'} \circ B\psi \simeq \Theta|_{BQ}$ for any $\psi \in \operatorname{Iso}_{\mathcal{F}}(Q, Q')$. It follows from Alperin's fusion theorem for saturated fusion systems [6, Theorem A.10] that:

2.10. **Proposition.** *For any $Q, Q' \leq S$ and any $\rho \in \mathcal{F}(Q, Q')$ the maps $\Theta|_{BQ}$ and $\Theta|_{BQ'} \circ B\rho$ are homotopic.*

The following proposition on mapping spaces will be needed in §7. Here and elsewhere in this paper we use the letter η for the p-completion map $X \to X^\wedge_p$.

2.11. **Proposition.** *Fix a p-local finite group $(S, \mathcal{F}, \mathcal{L})$ and let P be a finite p-group. Given a homomorphism $\rho\colon P \to S$, set $Q = \rho(P) \leq S$. Then:*

(a) *There is a homotopy equivalence*

$$\operatorname{map}^{\eta\circ\Theta\circ B\rho}(BP, |\mathcal{L}|^\wedge_p) \simeq \operatorname{map}^{\eta\circ\Theta|_{BQ}}(BQ, |\mathcal{L}|^\wedge_p),$$

and this space is the p-completed classifying space of a p-local finite group.

(b) *After p-completion, the map*

$$\operatorname{map}^{\Theta|_{BQ}}(BQ, |\mathcal{L}|) \xrightarrow{\eta_*} \operatorname{map}^{\eta\circ\Theta|_{BQ}}(BQ, |\mathcal{L}|^\wedge_p).$$

induces a split surjection on homotopy groups.

Proof. (a) First of all, we can choose a fully centralized subgroup $Q' \leq S$ in $\mathcal{F}$ and an isomorphism $\psi\colon Q \to Q'$ in $\mathcal{F}$. Let $\rho'\colon P \to S$ denote the composite $P \xrightarrow{\rho} Q \xrightarrow{\psi} Q' \leq S$. By Proposition 2.10 observe that

$$\Theta|_{BQ} \simeq \Theta|_{BQ'} \circ B\psi. \tag{1}$$

Hence, $\Theta \circ B\rho \simeq \Theta \circ B\rho'$. It follows from [6, Theorem 6.3] that there are homotopy equivalences

$$\operatorname{map}^{\eta\circ\Theta\circ B\rho}(BP, |\mathcal{L}|^\wedge_p) \simeq \operatorname{map}^{\eta\circ\Theta\circ B\rho'}(BP, |\mathcal{L}|^\wedge_p) \simeq \\ \operatorname{map}^{\eta\circ\Theta|_{BQ'}}(BQ', |\mathcal{L}|^\wedge_p) \simeq \operatorname{map}^{\eta\circ\Theta|_{BQ}}(BQ, |\mathcal{L}|^\wedge_p)$$

where the first equivalence is implied by equation (1) and the third one follows since $B\psi\colon BQ \to BQ'$ is a homotopy equivalence. Also by [6, Theorem 6.3], this space is homotopy equivalent to the classifying space of a p-local finite group $|C_{\mathcal{L}}(Q')|^\wedge_p$.

(b) We can assume from (1), by replacing Q with Q' if necessary, that Q is fully centralised in $\mathcal{F}$. In [6, pp. 822] a functor

$$\Gamma\colon C_{\mathcal{L}}(Q) \times \mathcal{B}Q \to \mathcal{L}$$

is constructed where $C_{\mathcal{L}}(Q)$ is the centraliser linking system [6, Definition 2.4] of Q in $\mathcal{F}$. By p-completing the geometric realisation of Γ and taking adjoints we obtain a commutative square in which the bottom row is a homotopy equivalence by [6, Theorem 6.3]

$$\begin{tikzcd} |C_{\mathcal{L}}(Q)| \arrow[r, "|\Gamma|^\#"] \arrow[d, "\eta"'] & \operatorname{map}^{\Theta|_{BQ}}(BQ, |\mathcal{L}|) \arrow[d, "\eta_*"] \\ |C_{\mathcal{L}}(Q)|^\wedge_p \arrow[r, "(|\Gamma|^\wedge_p)^\#", "\simeq"'] & \operatorname{map}^{\eta\circ\Theta|_{BQ}}(BQ, |\mathcal{L}|^\wedge_p). \end{tikzcd} \tag{2}$$

Since $|C_{\mathcal{L}}(Q)|$ is p-good by [6, Proposition 1.12], upon p-completion of the diagram (2), we see that the vertical arrow on the left becomes an equivalence and therefore the composite $(\eta_*)^\wedge_p \circ (|\Gamma|^\#)^\wedge_p$ is a homotopy equivalence. In particular $(\eta_*)^\wedge_p$ is split surjective on homotopy groups. □

We end this section with a description of the product of p-local finite groups.

2.12**.** Let $\mathcal{F}_i$ be a saturated fusion system on a finite p-group S_i for $i = 1, \dots, n$. Define $S = \prod_{i=1}^n S_i$ and consider the product category $\prod_{i=1}^n \mathcal{F}_i$. Its objects are the subgroups of S of the form $\prod_i P_i$ where $P_i \leq S_i$, and morphisms have the form $\prod_i P_i \xrightarrow{\prod_i \varphi_i} \prod_i Q_i$ where $\varphi_i \in \mathcal{F}_i(P_i, Q_i)$.

2.13. **Notation.** For $P \leq S = \prod_{i=1}^n S_i$, we denote by $P^{(i)}$ the image of P under the projection $p^{(i)} \colon S \to S_i$. Clearly $P \leq \prod_{i=1}^n P^{(i)}$.

Let $\mathcal{F}$ be the fusion system on S generated by $\prod_i \mathcal{F}_i$. Thus, every morphism $\varphi \in \mathcal{F}(P, Q)$ is given by the restriction of a morphism $\prod_i P^{(i)} \xrightarrow{\prod_i \varphi_i} \prod_i Q^{(i)}$ in $\prod_i \mathcal{F}_i$. The φ_i's are unique in the sense that they are completely determined by φ because $p^{(i)}|_P \colon P \to P^{(i)}$ are by definition surjective and $p^{(i)}|_Q \circ \varphi = \varphi_i \circ p^{(i)}|_P$. We see that $\varphi \mapsto (\varphi_i)_{i=1}^n$ induces an inclusion $\mathcal{F}(P, Q) \subseteq \prod_i \mathcal{F}_i(P^{(i)}, Q^{(i)})$. In particular, $\prod_i \mathcal{F}_i$ is a full subcategory of $\mathcal{F}$.

We shall write $\times_{i=1}^n \mathcal{F}_i$ for the fusion system $\mathcal{F}$ just defined and we call it the product fusion system of the $\mathcal{F}_i$'s.

2.14. **Lemma.** *With the notation above, $(S, \mathcal{F})$ is a saturated fusion system. If $P \leq S$ is $\mathcal{F}$-centric then all the groups $P^{(i)}$ are $\mathcal{F}_i$-centric for $i = 1, \dots, n$.*

The assignment $P \mapsto \prod_i P^{(i)}$ and the inclusions $\mathcal{F}(P, Q) \subseteq \prod_i \mathcal{F}_i(P^{(i)}, Q^{(i)})$ give rise to a functor $r \colon \mathcal{F}^c \to \prod_i \mathcal{F}_i^c$ which is a retraction of the inclusion $\prod_i \mathcal{F}_i^c \subseteq \mathcal{F}^c$.

Proof. It is shown in [6, Lemma 1.5] that $\mathcal{F} = \times_i \mathcal{F}_i$ is a saturated fusion system on S.

The assignments $P \mapsto \prod_i P^{(i)}$ and $\varphi \mapsto \prod \varphi_i$ give rise to a functor $r \colon \mathcal{F} \to \prod_i \mathcal{F}_i$ which by inspection is a retraction to the inclusion $j \colon \prod_i \mathcal{F}_i \to \mathcal{F}$. It remains to show that j and r restrict to $\prod_i \mathcal{F}_i^c$ and $\mathcal{F}^c$.

Observe that $C_S(P) = \prod_i C_{S_i}(P^{(i)})$ for any $P \leq S$. If P is $\mathcal{F}$-centric then

$$(1) \qquad \prod_{i=1}^n C_{S_i}(P^{(i)}) = C_S(P) \leq P \leq \prod_{i=1}^n P^{(i)}.$$

Therefore $C_{S_i}(P^{(i)}) \leq P^{(i)}$ for all i. Now, if Q_i are $\mathcal{F}_i$-conjugate to $P^{(i)}$ via isomorphisms $\varphi_i \in \mathcal{F}_i(P^{(i)}, Q_i)$ then $(\varphi_1 \times \ldots \times \varphi_n)|_P$ is an $\mathcal{F}$-isomorphism onto some $Q \leq S$ such that $Q^{(i)} = Q_i$. By definition Q is also $\mathcal{F}$-centric and applying (1) to Q we obtain that $C_{S_i}(Q_i) \leq Q_i$ for all i. We deduce that $P^{(i)}$ are $\mathcal{F}_i$-centric.

Assume now that $P_i \leq S_i$ are $\mathcal{F}_i$-centric for all $i = 1, \dots, n$. Then $P = \prod_i P_i$ is $\mathcal{F}$-centric because if Q is $\mathcal{F}$-conjugate to P then it has the form $\prod_i Q_i$ where Q_i are $\mathcal{F}_i$-conjugate to P_i and therefore $C_S(Q) = \prod_i C_{S_i}(Q_i) \leq Q$. □

The construction of the product of saturated fusion systems appears in [6], but we were unable to find a reference for the product of p-local finite groups.

2.15. **Definition.** Let $(S_i, \mathcal{F}_i, \mathcal{L}_i)$ be p-local finite groups for $i = 1, \dots, n$. Their product $\times_{i=1}^n (S_i, \mathcal{F}_i, \mathcal{L}_i)$ is the p-local finite group $(S, \mathcal{F}, \mathcal{L})$ where $S = \prod_{i=1}^n S_i$ and

$\mathcal{F} = \times_{i=1}^n \mathcal{F}_i$. The centric linking system $\mathcal{L} = \times_{i=1}^n \mathcal{L}_i$ is defined as the following pullback of small categories where r is defined in Lemma 2.14

$$\begin{tikzcd} \times_{i=1}^n \mathcal{L}_i \arrow[r, "r_{\mathcal{L}}"] \arrow[d, "\pi"'] & \prod_{i=1}^n \mathcal{L}_i \arrow[d, "\prod_{i=1}^n \pi_i"] \\ (\times_{i=1}^n \mathcal{F}_i)^c \arrow[r, "r"'] & \prod_{i=1}^n \mathcal{F}_i^c. \end{tikzcd}$$

The functor $\pi \colon \mathcal{L} \to \mathcal{F}$ is defined by the pullback and the monomorphisms $\delta_P \colon P \to \mathrm{Aut}_{\mathcal{L}}(P)$ are defined by the composites

$$P \leq \prod_i P^{(i)} \xrightarrow{\prod_i \delta_{P^{(i)}}} \prod_i \mathrm{Aut}_{\mathcal{L}_i}(P^{(i)}).$$

We need to prove that axioms (A)-(C) of Definition 2.4 hold.

Proof. For any $\mathcal{F}$-centric subgroups $P, Q \leq S$ the set $\mathcal{L}(P,Q)$ is the pullback

(1)
$$\begin{tikzcd} \mathcal{L}(P,Q) \arrow[r, hook] \arrow[d, "\pi"'] & \prod_{i=1}^n \mathcal{L}_i(P^{(i)}, Q^{(i)}) \arrow[d, "\prod_i \pi"] \\ \times_{i=1}^n \mathcal{F}_i(P,Q) \arrow[r, hook, "r"] & \prod_{i=1}^n \mathcal{F}_i(P^{(i)}, Q^{(i)}). \end{tikzcd}$$

We start by proving that the monomorphisms δ_P are well-defined. That is, given $g = (g_i) \in P \leq S$ where P is $\mathcal{F}$-centric, $\prod_i \delta_{P^{(i)}}(g_i) \in \mathrm{Aut}_{\mathcal{L}}(P)$. The pullback diagram (1) shows that it is enough to check that $\prod \pi_i(\delta_{P^{(i)}}(g_i)) \in r((\times_{i=1}^n \mathcal{F}_i)^c)$. It follows from the fact that $\pi_i(\delta_{P^{(i)}}(g_i)) = c_{g_i} \in \mathrm{Aut}_{\mathcal{F}_i}(P^{(i)})$ and $r(c_g) = \prod c_{g_i}$. This also shows that axiom (B) holds since $\pi(\delta_P(g)) = \prod \pi_i(\delta_{P^{(i)}}(g_i))|_P = c_g|_P$.

We continue to prove that $(S, \mathcal{F}, \mathcal{L})$ satisfies axioms (A) and (C). It follows from the definition that π is the identity on objects. Observe that $\prod_i C_{S_i}(P^{(i)})$ acts transitively and freely on the fibre of the right-hand arrow in (1) because axiom (A) holds in $(S_i, \mathcal{F}_i, \mathcal{L}_i)$. Now, axiom (A) for $(S, \mathcal{F}, \mathcal{L})$ follows from the fact that $C_S(P) = \prod_i C_{S_i}(P^{(i)})$ and that diagram (1) is a pullback square so the fibres of the vertical arrows are isomorphic.

Finally, axiom (C) for $(S, \mathcal{F}, \mathcal{L})$ follows by applying axiom (C) to each component of a morphism $f \in \mathcal{L}(P,Q)$ and each $g \in P \leq \prod_i P^{(i)}$. $\square$

2.16. **Remark.** Using the notation of Definition 2.15, if $\{\delta^i_{P,Q}\}$ are systems of lifts in $\mathcal{L}_i$, there results a system of lifts in $\prod_i \mathcal{L}_i$ as follows. If $P, Q \leq S$ are $\mathcal{F}$-centric, then $\delta_{P,Q}$ is $\prod_i \delta_{P^{(i)},Q^{(i)}} \colon \prod_i N_{S_i}(P^{(i)}, Q^{(i)}) \to \prod_i \mathcal{L}_i(P^{(i)}, Q^{(i)})$.

2.17. **Proposition.** *Given p-local finite groups $(S_i, \mathcal{F}_i, \mathcal{L}_i)$ for $i = 1, \ldots, n$, the category $\prod_i \mathcal{L}_i$ is a full subcategory of $\times_i \mathcal{L}_i$ and the inclusion $j \colon \prod_i \mathcal{L}_i \to \times_i \mathcal{L}_i$ induces a homotopy equivalence on nerves. In particular, $\prod_{i=1}^n |\mathcal{L}_i| \simeq |\times_{i=1}^n \mathcal{L}_i|$.*

Proof. Set $\mathcal{L} = \times_{i=1}^n \mathcal{L}_i$. The category $\prod_i \mathcal{L}_i$ is a full subcategory of $\mathcal{L}$ by Definition 2.15 and the fact that $\prod_i \mathcal{F}_i$ is a full subcategory of $\times_i \mathcal{F}_i$. The assignment $P \mapsto \prod_i P^{(i)}$ and the inclusion $\mathcal{L}(P,Q) \subseteq \prod_{i=1}^n \mathcal{L}_i(P^{(i)}, Q^{(i)})$ give rise to a functor $r_{\mathcal{L}} \colon \mathcal{L} \to \prod_{i=1}^n \mathcal{L}_i$ (see the pullback diagram in Definition 2.15) which is a retract to the inclusion j by Lemma 2.14. Also there is a natural transformation $\mathrm{Id} \to j \circ r$ which is defined on an object $P \in \mathcal{L}$ by $\iota_P^{r(P)} \colon P \to r(P) = \prod_{i=1}^n P^{(i)}$ (see Rmk. 2.16 and Def. 2.5). This shows that $|r|$ is a homotopy inverse to $|j| \colon \prod_i |\mathcal{L}_i| \to |\mathcal{L}|$. $\square$

2.18. **Remark.** Given a p-local finite group $(S, \mathcal{F}, \mathcal{L})$, Definition 2.15 allows us to consider its n-fold product with itself denoted $(S^{\times n}, \mathcal{F}^{\times n}, \mathcal{L}^{\times n})$. By construction, the action of the symmetric group Σ_n on $S^{\times n}$ extends to an action on the fusion system $\mathcal{F}^{\times n}$ and the linking system $\mathcal{L}^{\times n}$ by permuting the factors. Moreover, the functor $\pi\colon \mathcal{L}^{\times n} \to \mathcal{F}^{\times n}$ and the distinguished monomorphisms $\delta_P\colon P \to \mathrm{Aut}_{\mathcal{L}^{\times n}}(P)$ for every $\mathcal{F}^{\times n}$-centric $P \leq S^{\times n}$ are Σ_n-equivariant from the construction in Definition 2.15. Therefore, also the inclusion $\mathcal{B}S^{\times n} \overset{\delta_{S^{\times n}}}{\to} \mathcal{B}\mathrm{Aut}_{\mathcal{L}^{\times n}}(S^{\times n}) \to \mathcal{L}^{\times n}$ is Σ_n-equivariant and so is the induced map $\Theta\colon BS^{\times n} \to |\mathcal{L}^{\times n}| \simeq |\mathcal{L}|^{\times n}$.

The choice of $\delta_{P,Q}$ in $\mathcal{L}^{\times n}$ made in Remark 2.16 is easily seen to be equivariant with respect to the action of Σ_n as well.

Finally, the functor j and the homotopy equivalence in Proposition 2.17 are also equivariant with respect to the action of Σ_n by permuting coordinates.

3. The wreath product of spaces

Let G be a finite group and X a G-space. The Borel construction X_{hG} is the orbit space of $EG \times X$ where EG is a contractible space on which G acts freely on the right. Recall from 2.9 that $\mathcal{B}G$ is the small category with one object and G as a morphism set. Then X can be viewed as a functor $X\colon \mathcal{B}G \to \mathit{Top}$ and the Borel construction is a model for $\mathrm{hocolim}_{\mathcal{B}G} X$. There is a natural map $X_{hG} \to X/G$ to the orbit space of X induced by the map $EG \to *$.

A standard model for EG is the geometric realisation of the simplicial set $\mathcal{E}G$ whose set of n-simplices is the $n+1$-fold product $G \times \cdots \times G$ with face and degeneracy maps defined using deletion and duplication and where G acts diagonally via right translations. The identity element of G equips EG with a natural choice of a basepoint (which is not invariant under G.) This basepoint provides an augmentation map $\kappa(X)\colon X \to X_{hG}$ which is an inclusion map and it fits into the fibration sequence

$$X \xrightarrow{\kappa(X)} X_{hG} \to BG. \tag{3.1}$$

We will tend to simply write κ instead of $\kappa(X)$ whenever X is understood from the context. A fixed point $x \in X$ corresponds to a G-map $* \to X$ and gives rise to a section $s\colon BG \to X_{hG}$ for this fibration.

Now assume that G is a semidirect product $H \ltimes N$. Consider spaces, namely, simplicial sets X, Y and Z such that X has a left action of G and Z has a right action of H. Assume further that Y has a left action of H and a right action of N such that $h \cdot (y \cdot n) = (h \cdot y) \cdot (hnh^{-1})$ for all $y \in Y$, $h \in H$ and $n \in N$. Note that the actions of N and H on Y do not commute. Then $Z \times Y$ admits a right G-action defined by $(z, y) \cdot (h, n) = (z \cdot h, (h \cdot y) \cdot n)$ where $g = hn \in H \ltimes N$ and $(z, y) \in Z \times Y$. Moreover, by inspection, there is a homeomorphism

$$(Z \times Y) \times_G X \cong Z \times_H (Y \times_N X).$$

Taking $Z = EH$ and $Y = EN$ where the left H-action on EN is via conjugation, we obtain a homeomorphism

$$(EH \times EN) \times_G X \xrightarrow{\approx} EH \times_H (EN \times_N X). \tag{3.2}$$

Moreover there is an obvious isomorphism of simplicial sets

$$\mathcal{E}H \times \mathcal{E}N \xrightarrow{\approx} \mathcal{E}G$$

which in turn induces a homeomorphism $EH \times EN \approx EG$ of G-spaces. It now becomes clear that

(3.3) $$X \xrightarrow{\kappa} X_{hN} \xrightarrow{\kappa} (X_{hN})_{hH} \underset{\approx}{\longrightarrow} X_{hG} \qquad \text{is equal to} \qquad X \xrightarrow{\kappa} X_{hG}.$$

3.4. **Definition.** The wreath product of a space X with a subgroup G of Σ_k is the space
$$X \wr G := (X^{\times k})_{hG}$$
where G acts by permuting the factors of $X^{\times k}$. The diagonal map $\Delta_X\colon X \to X^{\times k}$ and $\kappa\colon X^{\times k} \to X \wr G$ give rise to a natural map
$$\Delta(X)\colon X \to X \wr G.$$

We shall use a left normed notation for iteration of the wreath product construction. That is, by convention, $X \wr G_1 \wr G_2 \wr \cdots \wr G_n$ denotes $(\cdots((X \wr G_1) \wr G_2) \wr \cdots) \wr G_n$. Applying (3.2) and (3.3) iteratively it is left as an easy exercise to prove

3.5. **Proposition.** *Given permutation groups $G_i \leq \Sigma_{k_i}$ where $i = 1, \dots, n$, there is a homeomorphism*
$$\alpha_n\colon X \wr G_1 \wr G_2 \wr \cdots \wr G_n \xrightarrow{\approx} X \wr (G_1 \wr G_2 \wr \cdots \wr G_n)$$
which is natural in X. Moreover, the composite
$$X \xrightarrow{\Delta} X \wr G_1 \xrightarrow{\Delta} (X \wr G_1) \wr G_2 \xrightarrow{\Delta} \cdots \xrightarrow{\Delta} X \wr G_1 \wr G_2 \wr \cdots \wr G_n \underset{\approx}{\xrightarrow{\alpha_n}} X \wr (G_1 \wr G_2 \wr \cdots \wr G_n)$$
is equal to $\Delta\colon X \to X \wr (G_1 \wr G_2 \wr \cdots \wr G_n)$ via the above homeomorphism.

3.6. **Remark.** Clearly Σ_k fixes all the points in the image of the diagonal map $X \to X^k$. If $X \neq \emptyset$, then the fibre sequence (3.1) $X^k \to X \wr G \to BG$ splits for any $G \leq \Sigma_k$ and the long exact sequence in homotopy groups gives rise to isomorphisms
$$\begin{aligned} \pi_1(X \wr G) &\cong (\pi_1 X) \wr G && \text{and} \\ \pi_i(X \wr G) &\cong (\pi_i X)^k && \text{for all } i \geq 2. \end{aligned}$$
Moreover, $\kappa\colon X^k \to X \wr G$ induces inclusions $\prod_k \pi_* X \leq \pi_*(X \wr G)$ on which $G \leq \pi_1(X \wr G)$ acts on higher homotopy groups by permuting the factors.

In particular, if $X = BH$ for a discrete group H, there is a homotopy equivalence $(BH) \wr G \simeq B(H \wr G)$ and $\Delta\colon BH \to (BH) \wr G \simeq B(H \wr G)$ is homotopic to the map induced by the diagonal inclusion $H \leq H \wr G$.

Let Y be a G-space. For any space X, $\mathrm{map}(X, Y)$ becomes a G-space, and the evaluation map $X \times \mathrm{map}(X, Y) \xrightarrow{\mathrm{ev}} Y$ is clearly G-equivariant. Therefore it gives rise to a map $\mathrm{ev}_{hG}\colon X \times \mathrm{map}(X, Y)_{hG} \to Y_{hG}$ whose adjoint is denoted
$$(\mathrm{ev}_{hG})^{\#}\colon \mathrm{map}(X, Y)_{hG} \to \mathrm{map}(X, Y_{hG}).$$
If the component $\mathrm{map}^f(X, Y)$ of some $f\colon X \to Y$ is invariant under the G-action then inspection of the adjunction shows that $(\mathrm{ev}_{hG})^{\#}$ restricts to
$$(\mathrm{ev}_{hG})^{\#}\colon \mathrm{map}^f(X, Y)_{hG} \to \mathrm{map}^{\kappa(Y) \circ f}(X, Y_{hG}).$$
Moreover, the composite

(3.7) $$\mathrm{map}^f(X, Y) \xrightarrow{\kappa} \mathrm{map}^f(X, Y)_{hG} \xrightarrow{(\mathrm{ev}_{hG})^{\#}} \mathrm{map}^{\kappa \circ f}(X, Y_{hG})$$

coincides with the natural map induced by $Y \xrightarrow{\kappa(Y)} Y_{hG}$.

3.8. **Proposition.** *Fix a map* $f\colon A \to X$ *and* $G \le \Sigma_k$. *Denote the adjoint of*

$$A \times (\mathrm{map}^f(A,X) \wr G) = A \times \mathrm{map}^{\Delta_X \circ f}(A, X^k)_{hG} \xrightarrow{\mathrm{ev}_{hG}} (X^k)_{hG} = X \wr G$$

by $\gamma\colon \mathrm{map}^f(A,X) \wr G \to \mathrm{map}^{\Delta(X)\circ f}(A, X \wr G)$. *Then:*

(a) *The triangle below is commutative.*

$$\begin{tikzcd}
\mathrm{map}^f(A,X) \arrow[d, "\Delta"'] \arrow[dr, "\mathrm{map}(A,\Delta(X))"] & \\
\mathrm{map}^f(A,X) \wr G \arrow[r, "\gamma"'] & \mathrm{map}^{\Delta(X)\circ f}(A, X \wr G)
\end{tikzcd}$$

(b) *If* A *is a non-empty path connected CW-complex then* γ *is a homotopy equivalence.*

Proof. (a) Note that $\prod_k \mathrm{map}^f(A,X) = \mathrm{map}^{\Delta_X \circ f}(A, X^k)$ and that this component is invariant under the action of $G \le \Sigma_k$. The commutativity of the triangle follows from (3.7) and Definition 3.4.

(b) First, we check that the evaluation $ev\colon \mathrm{map}^c(A, BG) \to BG$ at some $a \in A$ is a homotopy equivalence where the domain is the path component of the null-homotopic maps. Since this map between connected spaces has a section $\mathrm{const}\colon BG \to \mathrm{map}^c(A, BG)$, its homotopy fibre $\mathrm{map}^c_*(A, BG)$ is connected. But it is in fact contractible because $\Omega \mathrm{map}_*(A, BG) \simeq \mathrm{map}_*(A, G) \simeq *$. Then the section is also a homotopy equivalence.

Now consider the following ladder in which the rows are fibre sequences and π_* is induced by $X \to *$.

$$\begin{tikzcd}
\mathrm{map}^f(A,X)^k \arrow[r] \arrow[d, "\mathrm{incl}"'] & \mathrm{map}^f(A,X) \wr G \arrow[r] \arrow[d, "\gamma"'] & BG \arrow[d, "\simeq"', "\mathrm{const}"] \\
F \arrow[r] & \mathrm{map}^{\Delta(X)\circ f}(A, X \wr G) \arrow[r, "\pi_*"'] & \mathrm{map}^c(A, BG).
\end{tikzcd} \tag{1}$$

It commutes because the right hand square commutes as a consequence of the commutativity of the following square and adjunction

$$\begin{tikzcd}
A \times \mathrm{map}^{\Delta_X \circ f}(A, X^k)_{hG} \arrow[r] \arrow[d, "\mathrm{ev}_{hG}"'] & A \times \mathrm{map}(A, *)_{hG} \arrow[d, "\mathrm{proj}=\mathrm{ev}_{hG}"] \\
(X^{\times k})_{hG} \arrow[r, "\pi"'] & *_{hG} = BG.
\end{tikzcd}$$

Now, F is a union of path components of $\mathrm{map}(A, X^k)$ because it is the fibre of the fibration $\mathrm{map}(A, X \wr G) \to \mathrm{map}(A, BG)$ over the component of the constant map. Moreover, F clearly contains the component $\mathrm{map}^{\Delta_X \circ f}(A, X^k)$ and inspection of γ shows that the map between the fibres is simply the inclusion. Comparison of the long exact sequences in homotopy of the fibre sequences in (1) shows that F is connected, whence $F = \mathrm{map}^f(A,X)^{\times k}$. Application of the five lemma to the exact sequences in homotopy now yields the result. $\square$

4. Killing homotopy groups

The aim of this section is to study the effect on homotopy groups of the map $X \xrightarrow{\Delta(X)} X \wr \Sigma_k \xrightarrow{\eta} (X \wr \Sigma_k)^\wedge_p$ where $\Delta(X)$ was defined in the last section and η is the p-completion map.

4.1. **Proposition.** *Let X be a pointed space. Then the kernel of $\pi_* X \to \pi_*(X^\wedge_p)$ contains all the elements whose order is prime to p.*

Proof. Let $[\Theta] \in \pi_*(X)$ be an element of order k prime to p. Then the map $\Theta\colon S^n \to X$ factors through the Moore space $M(\mathbb{Z}/k, n)$, which is a nilpotent space with the mod p homology of a point. It follows that $\eta \circ \Theta\colon S^n \to X^\wedge_p$ factors through $M(\mathbb{Z}/k, n)^\wedge_p \simeq *$ (see [2, Ch. VI.5]), and therefore is null-homotopic. □

An element of exponent n in a group G is an element whose order divides n. For the proof of the next result, recall that for any space, $\pi_1(X)$ acts on the groups $\pi_* X$, see e.g. [25, Corollary 7.3.4] or [27, Ch. III]. We write α^ω for the image of the action of $\omega \in \pi_1 X$ on $\alpha \in \pi_n X$.

4.2. **Lemma.** *Fix an integer $n \geq 3$ and a pointed space X. Then the kernel of*

$$\pi_* X \xrightarrow{\Delta(X)_*} \pi_*(X \wr \Sigma_n) \xrightarrow{\eta_*} \pi_*((X \wr \Sigma_n)^\wedge_p)$$

contains all the elements of exponent n in $\pi_* X$.

Proof. We recall from Remark 3.6 that

$$\begin{aligned} \pi_1(X \wr \Sigma_n) &= (\pi_1 X) \wr \Sigma_n \\ \pi_i(X \wr \Sigma_n) &= \oplus_n \pi_i X \qquad \text{for } i \geq 2. \end{aligned}$$

Furthermore, $\kappa\colon \prod_n X \to X \wr \Sigma_n$ induces the inclusion $\prod_n \pi_* X \leq \pi_*(X \wr \Sigma_n)$. The section $s\colon B\Sigma_n \to X \wr \Sigma_n$ defined by the fixed point $(*, \dots, *) \in X^n$ induces the inclusion $\Sigma_n \leq \pi_1(X \wr \Sigma_n)$ which acts by permuting the factors of $\pi_*(X^n) \leq \pi_*(X \wr \Sigma_n)$.

We can choose elements $\omega_k \in \Sigma_n$ whose order is prime to p and $\omega_k(1) = k$ for all $k = 1, \dots, n$. Indeed, if $p > 2$ we can choose the involutions $\omega_k = (1, k)$. If $p = 2$ we can choose ω_k to be 3-cycles (note that $n \geq 3$.) In both cases we choose $\omega_1 = \mathrm{id}$.

For every $k = 1, \dots, n$ let $j_k\colon X \to \prod_n X$ denote the inclusion into the kth factor with respect to the basepoint of X. Note that $\Delta_X\colon X \to X^n$ induces $(\Delta_X)_*(\theta) = (\theta, \dots, \theta) \in \prod_n \pi_* X$. By inspection of the action of $\omega_k \in \pi_1(X \wr \Sigma_n)$, it follows that for any $\theta \in \pi_i X$, $(\kappa \circ j_k)_*(\theta) = ((\kappa \circ j_1)_*(\theta))^{\omega_k} \in \pi_i(X \wr \Sigma_n)$. Now fix some $\theta \in \pi_i X$ of exponent n. Since $\Delta(X)$ is defined as the composite $X \xrightarrow{\Delta_X} \prod_n X \xrightarrow{\kappa} X \wr \Sigma_n$, we have

$$\Delta(X)_*(\theta) = \prod_{k=1}^n (\kappa \circ j_k)_*(\theta) = \prod_{k=1}^n ((\kappa \circ j_1)_*(\theta))^{\omega_k}.$$

Now consider the p-completion map $X \wr \Sigma_n \xrightarrow{\eta} (X \wr \Sigma_n)^\wedge_p$ and note that it maps ω_k to the trivial element by Proposition 4.1. By applying η_* and using the naturality of the action of the fundamental group we see that

$$\begin{aligned} (\eta \circ \Delta(X))_*(\theta) = \prod_{k=1}^n \eta_*\big(((\kappa \circ j_1)_*(\theta))^{\omega_k}\big) &= \prod_{k=1}^n \eta_*\big((\kappa \circ j_1)_*(\theta)\big)^{\eta_*(\omega_k)} \\ &= (\eta_*((\kappa \circ j_1)_*(\theta)))^n = \eta_*((\kappa \circ j_1)_*(\theta^n)) = 0. \end{aligned}$$

□

4.3. **Lemma.** *Fix some $k \geq 3$ and consider a map $f\colon X \to Y$. Assume that every element of $\pi_i \mathrm{map}^f(X,Y)$ has exponent k and that $\mathrm{map}^{\eta\circ\Delta(Y)\circ f}(X, (Y \wr \Sigma_k)^\wedge_p)$ is p-complete. Then the homomorphism*

$$\pi_i \mathrm{map}^f(X,Y) \xrightarrow{\mathrm{map}(X,\eta\circ\Delta(Y))_*} \pi_i \mathrm{map}^{\eta\circ\Delta(Y)\circ f}(X, (Y \wr \Sigma_k)^\wedge_p)$$

is trivial.

Proof. According to Proposition 3.8(a) the triangle in the diagram below commutes up to homotopy.

$$\begin{tikzcd}
\mathrm{map}^f(X,Y) \arrow[r, "\Delta(Y)_*"] \arrow[dr, "\Delta"'] & \mathrm{map}^{\Delta\circ f}(X, Y \wr \Sigma_k) \arrow[r, "\eta_*"] & \mathrm{map}^{\eta\circ\Delta\circ f}(X, (Y \wr \Sigma_k)^\wedge_p) \\
& \mathrm{map}^f(X,Y) \wr \Sigma_k \arrow[u, "\gamma"] \arrow[r, "\eta"'] & (\mathrm{map}^f(X,Y) \wr \Sigma_k)^\wedge_p \arrow[u, dotted]
\end{tikzcd}$$

Since $\mathrm{map}^{\eta\circ\Delta(Y)\circ f}(X, (Y \wr \Sigma_k)^\wedge_p)$ is p-complete, the map $(\eta_* \circ \gamma)^\wedge_p$ gives rise to a choice of a map for the dotted arrow so that the square is homotopy commutative. We can now apply Lemma 4.2 to the diagonal arrow Δ and the bottom arrow η. □

5. The wreath product of p-local finite groups

Given a finite group G, the space $(BG) \wr \Sigma_k$ is the classifying space of the group $G \wr \Sigma_k$ (see 3.6). In this section we prove an analogous result for p-local finite groups.

Recall that a p-local finite group $(S, \mathcal{F}, \mathcal{L})$ admits an S-system of lifts $\{\delta_{P,Q}\}$, see Definition 2.5 and the remarks below it. Thus, an element $s \in S$ permutes the set of all morphisms $\mathcal{L}$, by either pre-composition with $\widehat{s^{-1}}$ (i.e. $\varphi \mapsto \varphi \circ \widehat{s^{-1}}$) or by post-composition with $\hat{s}$ (i.e $\varphi \mapsto \hat{s} \circ \varphi$) where $s \in N_S(Q, sQs^{-1})$. These assignments form a left and right action of S on $\mathcal{L}$ and we obtain an action of S on $\mathcal{L}$ by conjugation of the subgroups $P \leq S$ and by conjugation of morphisms $\varphi \mapsto \hat{s} \circ \varphi \circ \widehat{s^{-1}}$.

5.1. **Definition.** The action of a group G on S is called fusion preserving if the image of $G \xrightarrow{\tau} \mathrm{Aut}(S)$ consists of fusion preserving automorphisms, that is, for every $\varphi \in \mathcal{F}(P,Q)$ and every $g \in G$ the composite $\tau_g \circ \varphi \circ \tau_g^{-1}$ belongs to $\mathcal{F}(\tau_g(P), \tau_g(Q))$.

In this section we prove Theorem 5.2 which is a variant of [3, Theorem 4.6]. While condition (2) of Theorem 5.2 offers some simplifications, we relax the assumption imposed in [3] that G is a finite p-group. The main idea of the proof remains the same but some new arguments were needed. We also felt that some details are missing in [3] and we therefore decided to present a complete proof of Theorem 5.2.

5.2. **Theorem.** *Let G be a finite group which acts on the centric linking system $\mathcal{L}_0$ of a p-local finite group $(S_0, \mathcal{F}_0, \mathcal{L}_0)$. The action of $g \in G$ on $\varphi \in \mathrm{Mor}(\mathcal{L}_0)$ is denoted by $\varphi \mapsto g \cdot \varphi \cdot g^{-1}$. Assume that $S_0 \lhd G$ and let S be a Sylow p-subgroup of G. Assume further that:*

1. *Each $g \in G$ acts on $Ob(\mathcal{L})$ by sending P to gPg^{-1}. For each $g \in G$ and each $\varphi \in \mathcal{L}_0(P,Q)$, $\pi_0(g\varphi g^{-1}) = c_g \circ \pi_0(\varphi) \circ c_{g^{-1}} \in \mathcal{F}_0(gPg^{-1}, gQg^{-1})$.*
2. *If $P_0 \leq S_0$ is $\mathcal{F}_0$-centric and if a homomorphism $c_g\colon P_0 \to S_0$ for some $g \in G$ belongs to $\mathcal{F}_0$, then $g \in S_0$.*

(3) *The action of G on $\mathcal{L}_0$ extends the conjugation action of S_0 on $\mathcal{L}_0$.*
(4) *There is a G-equivariant system of lifts in $\mathcal{L}_0$, that is, $g \cdot \hat{s} \cdot g^{-1} = \widehat{gsg^{-1}}$ for any $g \in G$ and any $s \in N_{S_0}(P, Q)$.*
(5) *If $Q \leq S_0$ is not $\mathcal{F}_0$-centric but $\bar{Q} := N_{S_0}(Q)$ is $\mathcal{F}_0$-centric, then there exists $\tilde{\varphi} \in \mathcal{L}_0(\bar{Q}, S_0)$ such that $\pi_0(\tilde{\varphi})(Q)$ does not contain its S_0-centraliser and moreover, for any $x \in N_S(Q)$ there exists some $s \in S_0$ such that $x\tilde{\varphi}x^{-1} = \hat{s} \circ \tilde{\varphi}$.*

Then, there exists a p-local finite group $(S, \mathcal{F}, \mathcal{L})$ with the following properties:

(a) *There are inclusions $\mathcal{F}_0 \subseteq \mathcal{F}$, $\mathcal{F}_0^c \subseteq \mathcal{F}^c$ and $\mathcal{L}_0 \subseteq \mathcal{L}$ in such a way that the distinguished monomorphisms δ_P in $\mathcal{L}$ extend the ones in $\mathcal{L}_0$. The map $i\colon |\mathcal{L}_0| \to |\mathcal{L}|$ induced by the inclusion fits in a homotopy fibre sequence*

$$|\mathcal{L}_0| \xrightarrow{i} |\mathcal{L}| \to B(G/S_0).$$

Moreover, if S_0 has a complement K in G, that is $G = S_0 \rtimes K$, then:

(b) *There is a homotopy equivalence $|\mathcal{L}_0|_{hK} \xrightarrow{\simeq} |\mathcal{L}|$ such that the composite $|\mathcal{L}_0| \to |\mathcal{L}_0|_{hK} \simeq |\mathcal{L}|$ is homotopic to $|\mathcal{L}_0| \xrightarrow{i} |\mathcal{L}|$ and such that $\Theta\colon BS \to |\mathcal{L}|$ is homotopic to the composite*

$$BS \xrightarrow{Bincl} BG \simeq (BS_0)_{hK} \xrightarrow{(\Theta_0)_{hK}} |\mathcal{L}_0|_{hK} \simeq |\mathcal{L}|.$$

(c) *Up to isomorphism $(S, \mathcal{F}, \mathcal{L})$ is the unique p-local finite group with the properties in (b).*

As a corollary we obtain the proof of Theorem A in the Introduction.

Proof of Theorem A. By Remark 2.18 there is an action of Σ_n on the n-fold product $(S_0, \mathcal{F}_0, \mathcal{L}_0) = (S^{\times n}, \mathcal{F}^{\times n}, \mathcal{L}^{\times n})$ by permuting the factors.

The action of S_0 on $\mathcal{L}_0$ by conjugation clearly extends to an action of $S_0 \rtimes \Sigma_n$ because $S_0 = S^{\times n}$ acts on every coordinate of $\mathcal{L}_0 = \mathcal{L}^{\times n}$ and Σ_n acts by permuting the factors of $\mathcal{L}_0$ and the factors of $S_0 = S^{\times n}$. Set $G = S \wr K = S_0 \rtimes K$. We shall now show that the action of G on $\mathcal{L}_0$ satisfies hypotheses (1)–(1) of Theorem 5.2.

Hypothesis (1) is clearly satisfied because K acts on S_0 by permuting the factors which is an automorphism of $\mathcal{F}_0 = \mathcal{F}^{\times n}$. Note that $\pi\colon \mathcal{L}_0 \to \mathcal{F}_0$ is Σ_n-equivariant and it is also S_0-equivariant since $\pi(\hat{s}) = c_s$ for any $s \in S$. Hypothesis (3) holds by the definition of the action of $G = S_0 \rtimes K$ on $\mathcal{L}_0$. For Hypothesis (4) choose a system of lifts $\{\delta_{P,Q}\}$ in $\mathcal{L}$ (see Remark 2.6) and use Remarks 2.16 and 2.18 together with the obvious fact that the system $\{\delta_{P,Q}\}$ is S_0-equivariant.

We now check hypothesis (2). Fix an $\mathcal{F}_0$-centric subgroup $P_0 \leq S_0$ and let $P_0^{(i)}$ be defined as in 2.13. Since $P_0^{(i)}$ are $\mathcal{F}$-centric for $i = 1, \dots, n$ by Lemma 2.14 and $S \neq 1$, it follows that $P_0^{(i)} \neq 1$ whence $Z(P_0^{(i)}) \neq 1$ for all $i = 1, \dots, n$. Also note that $\prod_i Z(P_0^{(i)}) = \prod_i C_S(P_0^{(i)}) = C_{S_0}(P_0) \leq P_0$ because P_0 is $\mathcal{F}_0$-centric. Fix some $g = (s_1, \dots, s_n; \sigma) \in G = S \wr K$ and assume that $g \notin S_0$, namely $\sigma \neq 1$. Without loss of generality we can assume that $\sigma(1) = 2$. Choose $1 \neq z_1 \in C_S(P_0^{(1)})$ and consider $(z_1, 1, \dots, 1; \mathrm{id}) \in \prod_{i=1}^n Z(P_0^i) \leq P_0$. Then

$$\begin{aligned} c_g((z_1, 1, \dots, 1; \mathrm{id})) = (s_1, \dots, s_n; \sigma)(z_1, 1, \dots, 1; \mathrm{id})(s^{-1}_{\sigma^{-1}(1)}, \dots, s^{-1}_{\sigma^{-1}(n)}; \sigma^{-1}) \\ = (1, s_2 z_1 s_2^{-1}, 1, \dots, 1; \mathrm{id}). \end{aligned}$$

Therefore $c_g \notin \mathcal{F}_0(P_0, S_0)$ because it cannot be a restriction of a morphism in $\prod_n \mathcal{F}$.

Finally we prove that hypothesis (5) is satisfied. Assume that $Q \leq S_0$ is not $\mathcal{F}_0$-centric but $\bar{Q} := N_{S_0}(Q)$ is $\mathcal{F}_0$-centric. Observe that $N_S(Q^{(i)})$ are all $\mathcal{F}$-centric because $N_{S_0}(Q)^{(i)}$ are all $\mathcal{F}$-centric by Lemma 2.14 and $N_{S_0}(Q)^{(i)} \leq N_S(Q^{(i)})$.

For every i we choose a morphism $\varphi_i \in \mathcal{L}(N_S(Q^{(i)}), S)$ such that $\pi(\varphi)(Q^{(i)})$ is fully $\mathcal{F}$-centralised (see [6, A.2(b)]), and define a morphism $(\varphi_1, \ldots, \varphi_n) \in \mathcal{L}_0(\prod_i N_S(Q^{(i)}), S_0)$. Let $\tilde{\varphi} \in \mathcal{L}_0(\bar{Q}, S_0)$ be its restriction to $\bar{Q}$. Then $\pi(\tilde{\varphi})(Q)$ is fully centralised since $\pi_0(\tilde{\varphi})(Q)^{(i)} = \pi(\varphi_i)(Q^{(i)})$ are fully centralised for all i. By assumption Q is not $\mathcal{F}_0$-centric, hence $\pi_0(\tilde{\varphi})(Q)$ does not contain its S_0-centralizer.

It remains to show that for any $g \in N_G(Q)$ there exists some $s \in S_0$ such that $g\tilde{\varphi}g^{-1} = \hat{s} \circ \tilde{\varphi}$. Set $W = N_G(Q)/N_{S_0}(Q) \leq \Sigma_n$. Choose $u\sigma \in N_G(Q)$ where $u \in \prod_i N_S(Q)^{(i)}$ and $\sigma \in W$ and assume that $\sigma(i) = j$. Given $x \in Q^{(i)}$, choose $\underline{x} \in Q \leq \prod_i Q^{(i)}$ with $x_i = x$. Note that

$$u\sigma \cdot \underline{x} \cdot \sigma^{-1}u^{-1} = u \cdot (x_{\sigma(i)}) \cdot u^{-1}.$$

Thus, $u_j x u_j^{-1} \in Q^{(j)}$. It follows then that $u_j Q^{(i)} u_j^{-1} \subseteq Q^{(j)}$, that is, $Q^{(i)}$ is S-conjugate to a subgroup of $Q^{(j)}$. By symmetry $Q^{(i)}$ and $Q^{(j)}$ are S-conjugate. Thus, after conjugating by an apropriate element in S_0 we may assume that $Q^{(i)} = Q^{(j)}$ whenever $\sigma(i) = j$ for some $\sigma \in W$. Note that this does not change W. Moreover, in the definition of $\tilde{\varphi}$, we can take $\varphi_i = \varphi_j$ if $\sigma(i) = j$ for some $\sigma \in W$. Finally, for any $g \in N_G(Q)$ we can write $g = y\sigma$ for some $\sigma \in W$ and $y \in N_{S_0}(Q) \subseteq \prod_i N_S(Q^{(i)})$. By the choice of the morphisms φ_i, it is clear that $\sigma\tilde{\varphi}\sigma^{-1} = \tilde{\varphi}$, hence

$$g\tilde{\varphi}g^{-1} = y\sigma\tilde{\varphi}\sigma^{-1}y^{-1} = y\tilde{\varphi}y^{-1} = \hat{y} \circ \tilde{\varphi} \circ \widehat{y^{-1}} = \hat{y} \circ \widehat{\tilde{\varphi}(y^{-1})} \circ \tilde{\varphi} = \hat{s} \circ \tilde{\varphi}$$

where $s \in S_0$.

Now we apply Theorem 5.2(b) to conclude that there exists a p-local finite group $(S', \mathcal{F}', \mathcal{L}')$ with $(|\mathcal{L}_0|)_{hK} \simeq |\mathcal{L}'|$ such that

$$BS' \xrightarrow{B\mathrm{incl}} BG \simeq (BS_0)_{hK} \xrightarrow{(\Theta_0)_{hK}} |\mathcal{L}_0|_{hK} \simeq |\mathcal{L}'| \tag{1}$$

is homotopic to $\Theta' \colon BS' \to |\mathcal{L}'|$. Also observe that the horizontal arrows in

$$\begin{array}{ccc} (BS)^{\times n} & = & BS_0 \\ {\scriptstyle \Theta^{\times n}}\downarrow & & \downarrow{\scriptstyle \Theta_0} \\ |\mathcal{L}|^{\times n} & \xrightarrow[\simeq]{} & |\mathcal{L}_0| \end{array}$$

form a Σ_n-equivariant map of the vertical arrows. It follows that the composite in (1) is homotopic to the map

$$BS' \xrightarrow{B\mathrm{incl}} BG \simeq (BS) \wr K \xrightarrow{\Theta \wr K} |\mathcal{L}| \wr K \simeq |\mathcal{L}'|.$$

which is therefore homotopic to $\Theta' \colon BS' \to |\mathcal{L}'|$. The uniqueness of $(S', \mathcal{F}', \mathcal{L}')$ with this property is guaranteed by part (c) of Theorem 5.2. □

5.3. **Remark.** If the p-local finite group in Theorem A is associated with a finite group G then $(S', \mathcal{F}', \mathcal{L}')$ satisfies $|\mathcal{L}'|^\wedge_p \simeq (|\mathcal{L}|^\wedge_p \wr K)^\wedge_p \simeq (BG^\wedge_p \wr K)^\wedge_p \simeq B(G \wr K)^\wedge_p$. Those equivalences follow from the Serre spectral sequence associated to $|\mathcal{L}|^n \times_K EK$ and [2, Lemma I.5.5] since the spaces involved are p-good ([6, Proposition 1.12]). Thus, $\mathcal{L}'$ is the linking system associated to $G \wr K$.

In the remainder of this section we will prove Theorem 5.2. **From now on, the hypotheses and notation set up in Theorem 5.2 are in force.** Its proof, namely the construction of $(S, \mathcal{F}, \mathcal{L})$, is obtained in a sequence of definitions and claims 5.4–5.16. Their proofs are given after 5.16.

5.4. **Definition.** Let $\mathcal{H}_0$ denote the set of all the $\mathcal{F}_0$-centric subgroups of S_0. Fix once and for all a Sylow p-subgroup S of G and for every $P \leq S$ let P_0 denote $P \cap S_0$.

The action of G on the set of all subgroups of S_0 by conjugation restricts to an action on the set $\mathcal{H}_0$ of all the $\mathcal{F}_0$-centric subgroups of S_0 because G acts via fusion preserving automorphisms of S_0 by hypothesis (1).

5.5. **Definition.** Let $\mathcal{F}_1$ be the fusion system on S_0 generated by $\mathcal{F}_0$ and $\mathrm{Aut}_G(S_0)$. Define a category $\mathcal{L}_1$ whose object set is $\mathcal{H}_0$ and

$$\mathrm{Mor}(\mathcal{L}_1) = (G \times \mathrm{Mor}(\mathcal{L}_0)) \Big/ (gs, \varphi) \sim (g, \hat{s} \circ \varphi) \qquad (s \in S_0).$$

The morphism set $\mathcal{L}_1(P_0, Q_0)$ where $P_0, Q_0 \in \mathcal{H}_0$ consists of the equivalence classes $[g : \varphi]$ such that $g \in G$ and $\varphi \in \mathcal{L}_0(P_0, Q_0^g)$. Composition is given by the formula

$$[g : \varphi] \circ [h : \psi] = [gh : (h^{-1}\varphi h) \circ \psi],$$

and identities are the elements of the form $[1 : \mathrm{id}_{P_0}]$. We check later that composition is well-defined.

Define a functor $\pi_1 \colon \mathcal{L}_1 \to \mathcal{F}_1$ which is the identity on the set of objects and

$$\pi_1([g : \varphi]) = c_g \circ \pi_0(\varphi).$$

We also define functions $\hat{\delta}_{P_0,Q_0} \colon N_G(P_0, Q_0) \to \mathcal{L}_1(P_0, Q_0)$ by $g \mapsto [g : \iota_{P_0}^{Q_0^g}]$ and denote the image of g by $\hat{g}$.

We will prove the following properties relating $\mathcal{L}_1$ and $\mathcal{L}_0$.

5.6. **Lemma.** *The category $\mathcal{L}_1$ satisfies the following properties:*

(a) *There is an inclusion functor $j \colon \mathcal{L}_0 \to \mathcal{L}_1$ which is the identity on objects and $\varphi \mapsto [1 : \varphi]$ on morphisms.*

(b) *Every morphism in $\mathcal{L}_1$ has the form $\hat{g} \circ \varphi$ where φ is a morphism in $\mathcal{L}_0 \subseteq \mathcal{L}_1$. If $\varphi \in \mathcal{L}_0(P_0, Q_0)$ and $x \in N_G(P_0)$, then $\varphi \circ \hat{x} = \hat{x} \circ (x^{-1}\varphi x)$.*

(c) *There is a homotopy fibre sequence*

$$|\mathcal{L}_0| \xrightarrow{|j|} |\mathcal{L}_1| \to B(G/S_0).$$

If S_0 admits a complement K in G then there is a homotopy equivalence $|\mathcal{L}_0|_{hK} \simeq |\mathcal{L}_1|$ such that the composite $|\mathcal{L}_0| \to |\mathcal{L}_0|_{hK} \simeq |\mathcal{L}_1|$ is homotopic to the map induced by the inclusion j. Moreover, the composite

$$BG \simeq (BS_0)_{hK} \xrightarrow{(\Theta_0)_{hK}} |\mathcal{L}_0|_{hK} \simeq |\mathcal{L}_1|$$

is homotopic to the map $BG \to |\mathcal{L}_1|$ induced by the functor $k \colon \mathcal{B}G \to \mathcal{L}_1$ with $k(\bullet_G) = S_0$ and $k(g) = [g : 1_{S_0}]$.

The next step in our construction is to define the following category.

5.7. **Definition.** Define a category $\mathcal{L}_2$ whose object set is

$$\mathcal{H} = \{P \le S \ : \ P_0 = P \cap S_0 \in \mathcal{H}_0\}$$

and whose morphism sets are defined by

$$\mathcal{L}_2(P,Q) = \{\psi \in \mathcal{L}_1(P_0,Q_0) \ : \ \forall x \in P \ \exists y \in Q \text{ s.t. } (\psi \circ \hat{x} = \hat{y} \circ \psi)\}.$$

By construction $\mathcal{L}_2(P,Q) \subseteq \mathcal{L}_1(P_0,Q_0)$ and composition of morphisms is obtained by composing them in $\mathcal{L}_1$. Identities id_P have the form $[1 : \mathrm{id}_{P_0}]$. Also define maps $\hat{\delta}_{P,Q} \colon N_G(P,Q) \to \mathcal{L}_2(P,Q)$ by $g \mapsto [g : \iota_{P_0}^{Q_0^g}]$ and denote the image of g by $\hat{g}$.

The main properties of the category $\mathcal{L}_2$ and its relation to the previously defined $\mathcal{L}_1$ are contained in next two lemmas.

5.8. **Lemma.** *The category $\mathcal{L}_1$ is the full subcategory of $\mathcal{L}_2$ on the objects $\mathcal{H}_0$ and the inclusion $j \colon \mathcal{L}_1 \to \mathcal{L}_2$ induces a homotopy equivalence on nerves.*

5.9. **Lemma.** *Let $P,Q \le S$. The category $\mathcal{L}_2$ satisfies the following properties:*

(a) *For every morphism $\psi \in \mathcal{L}_2(P,Q)$ there exists a unique group monomorphism $\pi_2(\psi) \colon P \to Q$ which satisfies $\psi \circ \hat{x} = \widehat{\pi_2(\psi)(x)} \circ \psi$ in $\mathcal{L}_2$ for all $x \in P$. Moreover, $\pi_2(\psi)|_{P_0} = \pi_1(\psi)$.*

(b) *$\pi_2(id_{P_0}) = id_P$ and $\pi_2(\lambda) \circ \pi_2(\psi) = \pi_2(\lambda \circ \psi)$ for every $P \xrightarrow{\psi} Q \xrightarrow{\lambda} R$ in $\mathcal{L}_2$.*

(c) *For every $\hat{g} \in \mathcal{L}_2(P,Q)$ with $g \in N_G(P,Q)$, we have $\pi_2(\hat{g}) = c_g$.*

(d) *Given $\psi \in \mathcal{L}_2(P,Q)$, if $\pi_2(\psi)$ is an isomorphism of groups then ψ is an isomorphism in $\mathcal{L}_2$.*

Lemma 5.9 justifies the following definition.

5.10. **Definition.** Let $\mathcal{F}_2$ be the category whose object set is $\mathcal{H}$, see Definition 5.7, and whose morphism sets $\mathcal{F}_2(P,Q)$ are the set of group monomorphisms $\pi_2(\mathcal{L}_2(P,Q))$ defined by Lemma 5.9. By the properties shown in this lemma, there results a projection functor $\pi_2 \colon \mathcal{L}_2 \to \mathcal{F}_2$ which is the identity on objects.

5.11. **Lemma.** *The category $\mathcal{F}_2$ satisfies the following properties:*

(a) *For every $P,Q \in \mathcal{H}$, $\mathrm{Hom}_G(P,Q) \subseteq \mathcal{F}_2(P,Q)$. In particular, $\mathcal{F}_2$ contains all the inclusions $P \le Q$ of groups in $\mathcal{H}$.*

(b) *Every morphism in $\mathcal{F}_2$ factors as an isomorphism in $\mathcal{F}_2$ followed by an inclusion. In particular, every isomorphism of groups $f \colon P \to Q$ in $\mathcal{F}_2$ is an isomorphism in $\mathcal{F}_2$.*

Thus, $\mathcal{F}_2$ falls short of being a fusion system on S only because its set of objects $\mathcal{H}$ need not contain all the subgroups of S.

5.12. **Definition.** Let $\mathcal{F}$ denote the fusion system on S generated by $\mathcal{F}_2$.

5.13. **Lemma.** *The fusion system $\mathcal{F}$ satisfies the following properties:*

(a) *$\mathcal{F}_2$ is the full subcategory of $\mathcal{F}$ generated by the objects in $\mathcal{H}$.*

(b) *Every $P \in \mathcal{H}$ is $\mathcal{F}$-centric. In particular, $\mathcal{H}_0 \subseteq \mathcal{F}^c$.*

(c) *Every morphism $f \in \mathcal{F}(P,Q)$ restricts to a morphism $f|_{P_0} \in \mathcal{F}(P_0,Q_0)$.*

5.14. **Lemma.** *The functor $\pi_2 \colon \mathcal{L}_2 \to \mathcal{F}$ satisfies all the axioms of a centric linking system on the object set $\mathcal{H}$.*

Finally, the last step in the proof is to show that the fusion system $(S, \mathcal{F})$ defined in 5.12 is saturated and that $\mathcal{L}_2$ can be extended to a unique centric linking system $\mathcal{L}$ associated to $\mathcal{F}$.

5.15. **Lemma.** *$\mathcal{F}$ is a saturated fusion system on S.*

5.16. **Lemma.** *There exists a p-local finite group $(S, \mathcal{F}, \mathcal{L})$ such that $\mathcal{L}_2$ is a full subcategory of $\mathcal{L}$ and $\pi_2\colon \mathcal{L}_2 \to \mathcal{F}$ is the restriction of $\pi\colon \mathcal{L} \to \mathcal{F}$. Moreover, $\hat{\delta}_P\colon P \to \mathrm{Aut}_{\mathcal{L}_2}(P)$ are the distinguished monomorphisms of $(S, \mathcal{F}, \mathcal{L})$ for all $P \in \mathcal{H}$, and the inclusion $\mathcal{L}_2 \subseteq \mathcal{L}$ induces a homotopy equivalence on nerves.*

Assuming definitions and lemmas 5.4–5.16, we can now prove Theorem 5.2.

Proof of Theorem 5.2. The p-local finite group $(S, \mathcal{F}, \mathcal{L})$ is constructed in Lemma 5.16. Together with Lemma 5.8 we obtain inclusions of full subcategories $\mathcal{L}_1 \subseteq \mathcal{L}_2 \subseteq \mathcal{L}$ which induce homotopy equivalences on nerves. By Lemma 5.6(c), there results the homotopy fibre sequence of part (a).

Now assume that S_0 has a complement K in G and we prove points (b) and (c). Lemma 5.6(c) shows that there are homotopy equivalences $|\mathcal{L}_0|_{hK} \simeq |\mathcal{L}_1| \simeq |\mathcal{L}|$ such that $|\mathcal{L}_0| \to |\mathcal{L}_0|_{hK} \simeq |\mathcal{L}|$ is homotopic to the map induced by the inclusion $\mathcal{L}_0 \subseteq_j \mathcal{L}_1 \subseteq \mathcal{L}$. Moreover the map

$$BS \xrightarrow{B\mathrm{incl}} BG \simeq (BS_0)_{hK} \xrightarrow{(\Theta_0)_{hK}} |\mathcal{L}_0|_{hK} \simeq |\mathcal{L}|$$

is induced by the functor $\Lambda_0\colon \mathcal{B}S \to \mathcal{L}$ which sends $\bullet_S$ to S_0 and defined on morphisms by $s \mapsto [s : 1_{S_0}] = \hat{s} \in \mathrm{Aut}_{\mathcal{L}}(S_0)$ (see Lemmas 5.16, 5.6 and Definition 5.7). The map $\Theta\colon BS \to |\mathcal{L}|$ is the realisation of the functor $\Lambda_1\colon \mathcal{B}S \to \mathcal{B}\mathrm{Aut}_{\mathcal{L}}(S) \to \mathcal{L}$ where $s \mapsto \hat{s} \in \mathrm{Aut}_{\mathcal{L}}(S)$, then the lift of the inclusion $\iota_{S_0}^S \in \mathcal{L}(S_0, S)$ provides a natural transformation $\Lambda_0 \to \Lambda_1$ because $\hat{s} \circ \iota_{S_0}^S = \iota_{S_0}^S \circ \hat{s}$, see Def. 2.5. Therefore $|\Lambda_0|$ and $|\Lambda_1|$ are homotopic and the proof of point (b) is complete.

Now assume that $(S, \mathcal{F}', \mathcal{L}')$ is another p-local finite group which satisfies the properties in point (b). Let λ denote the composite $BS \to BG = (BS_0)_{hK} \to |\mathcal{L}_0|_{hK}$. By assumption there is a homotopy commutative diagram

$$\begin{tikzcd} & BS \arrow[dl, "\Theta"'] \arrow[d, "\lambda"] \arrow[dr, "\Theta'"] & \\ |\mathcal{L}| & |\mathcal{L}_0|_{hK} \arrow[l, "\simeq"] \arrow[r, "\simeq"'] & |\mathcal{L}'|. \end{tikzcd}$$

The isomorphism of $(S, \mathcal{F}, \mathcal{L})$ and $(S, \mathcal{F}', \mathcal{L}')$ follows from [6, Theorem 7.7] $\square$

In the rest of the section we fill in the details needed for the construction in 5.5–5.16.

Proof that Def. 5.5 makes $\mathcal{L}_1$ a small category and makes $\pi_1\colon \mathcal{L}_1 \to \mathcal{F}_1$ a functor. The verification that composition of morphisms is well defined is similar to the one in [3, Theorem 4.6]. Specifically, for any $g_0, h_0 \in S_0$

$$[gg_0 : \varphi] \circ [hh_0 : \psi] = [gg_0hh_0 : (h_0^{-1}h^{-1}\varphi hh_0) \circ \psi] = \qquad \text{by hypothesis (3)}$$

$$[gg_0h : (h^{-1}\varphi h) \circ \widehat{h_0} \circ \psi] = [gh : \widehat{h^{-1}g_0h} \circ (h^{-1}\varphi h) \circ \widehat{h_0} \circ \psi] = \qquad \text{by hypothesis (4)}$$

$$[gh : h^{-1}(\widehat{g_0} \circ \varphi)h \circ \widehat{h_0} \circ \psi] = [g : \widehat{g_0} \circ \varphi] \circ [h : \widehat{h_0} \circ \psi].$$

Associativity is straightforward as well as checking that $[1 : 1_{P_0}]$ are identity morphisms $P_0 \to P_0$.

It is evident from the definition that π_1 maps identity morphisms in $\mathcal{L}_1$ to identities in $\mathcal{F}_1$. It also respects compositions by the following calculation which uses hypothesis (1) in the third equality

$$\begin{aligned}\pi_1([g:\varphi]) \circ \pi_1([h:\psi]) &= c_g \circ \pi_0(\varphi) \circ c_h \circ \pi_0(\psi) \\ &= c_{gh} \circ (c_{h^{-1}} \circ \pi_0(\varphi) \circ c_h) \circ \pi_0(\psi) = c_{gh} \circ \pi_0(h^{-1}\varphi h) \circ \pi_0(\psi) \\ &= c_{gh} \circ \pi_0(h^{-1}\varphi h \circ \psi) = \pi_1([gh : h^{-1}\varphi h \circ \psi]) = \pi_1([g:\varphi] \circ [h:\psi]).\end{aligned}$$

□

Proof of Lemma 5.6. (a) By Definition 5.5 we have $[1:\varphi] \circ [1:\varphi'] = [1:\varphi \circ \varphi']$ so j is clearly associative and unital. It is an inclusion functor because $[1:\varphi] = [1:\varphi']$ if and only if $\varphi = \varphi'$ by the definition of morphisms in $\mathcal{L}_1$.

(b) Clearly, every morphism ψ in $\mathcal{L}_1$ has the form $[g:\varphi] = [g:1] \circ [1:\varphi] = \hat{g} \circ \varphi$. Given φ and x as in the statement, by Definition 5.5

$$\varphi \circ \hat{x} = [1:\varphi] \circ [x:1] = [x : x^{-1}\varphi x] = [x : 1_{Q_0^x}] \circ [1 : x^{-1}\varphi x] = \hat{x} \circ x^{-1}\varphi x.$$

(c) Set $\bar{G} = G/S_0$ and denote its elements by $\bar{g} = gS_0$. There is a functor $\Pi\colon \mathcal{L}_1 \to \mathcal{B}(\bar{G})$ which sends every object of $\mathcal{L}_1$ to $\bullet_{\bar{G}}$ and maps $[g:\varphi] \mapsto \bar{g}$.

Now, consider the comma category $(\bullet_{\bar{G}} \downarrow \Pi)$. Its objects are pairs $(\bar{g}, P_0)$ and morphisms $(\bar{g}, P_0) \to (\bar{h}, Q_0)$ are morphisms $[x:\lambda] \in \mathcal{L}_1(P_0, Q_0)$ such that $\bar{x} = \bar{h}\bar{g}^{-1}$. We can easily check that $\hat{g}\colon P_0^g \to P_0$ provides an isomorphism $(\bar{e}, P_0^g) \to (\bar{g}, P_0)$ in $(\bullet_{\bar{G}} \downarrow \Pi)$. Therefore, the set of objects of the form $(\bar{e}, P_0)$ form a skeletal full subcategory of $(\bullet_{\bar{G}} \downarrow \Pi)$, that is, it contains an element from every isomorphism class of objects. This subcategory is clearly isomorphic to $\mathcal{L}_0$ and moreover the composite $\mathcal{L}_0 \subseteq (\bullet_{\bar{G}} \downarrow \Pi) \to \mathcal{L}_1$ is the inclusion j in part (a).

Moreover, any morphism $\bar{g} \in \mathcal{B}\bar{G}$ clearly induces an automorphism of the category $(\bullet_{\bar{G}} \downarrow \Pi)$. Therefore, Quillen's theorem B [21] applies in this situation to show that $|(\bullet_{\bar{G}} \downarrow \Pi)| \to |\mathcal{L}_1| \to |\mathcal{B}(G/S_0)|$ is a homotopy fibre sequence. Finally, using the homotopy equivalence $|j|$ we obtain the homotopy fibre sequence $|\mathcal{L}_0| \xrightarrow{|j|} |\mathcal{L}_1| \xrightarrow{|\Pi|} BG/S_0$.

Now suppose that S_0 has a complement K in G. Recall that G acts on the category $\mathcal{L}_0$ and we view the restriction of this action to K as a functor $\mathcal{B}K \to \mathbf{Cat}$. Let $\mathrm{Tr}_K(\mathcal{L}_0)$ denote the transporter category (or Grothendieck construction) of this functor; See e.g. [26]. The object set of $\mathrm{Tr}_K(\mathcal{L}_0)$ is $\mathcal{H}_0$, and the morphisms $P_0 \to Q_0$ are pairs (k, φ) where $\varphi \in \mathcal{L}_0({}^kP_0, Q_0)$. Composition is given by the following formula: $(k_2, \varphi_2) \circ (k_1, \varphi_1) = (k_2k_1, \varphi_2 \circ k_2\varphi_1k_2^{-1})$. Define a functor $\Phi\colon \mathrm{Tr}_K(\mathcal{L}_0) \to \mathcal{L}_1$ which is the identity on objects and

$$\Phi\colon \mathrm{Tr}_K(\mathcal{L}_0)(P_0, Q_0) \to \mathcal{L}_1(P_0, Q_0) \qquad \text{is defined by } (k, \varphi) \mapsto [k : k^{-1}\varphi k].$$

It is clear that $\Phi(1, \mathrm{id}) = [1 : \mathrm{id}]$ and for any pair of composable morphisms (k_2, φ_2) and (k_2, φ_2) in $\mathrm{Tr}_K(\mathcal{L}_0)$,

$$\begin{aligned}\Phi(k_2, \varphi_2) \circ \Phi(k_1, \varphi_1) &= [k_2 : k_2^{-1}\varphi_2k_2] \circ [k_1 : k_1^{-1}\varphi_1k_1] \\ &= [k_2k_1 : k_1^{-1}k_2^{-1}\varphi_2k_2k_1 \circ k_1^{-1}\varphi_1k_1] = \Phi(k_2k_1, \varphi_2 \circ k_2\varphi_1k_2^{-1}).\end{aligned}$$

By definition Φ is bijective on the object set. It is also bijective on morphism sets because $K \cap S_0 = 1$ so every morphism in $\mathcal{L}_1(P_0, Q_0)$ has a unique representative of the form $[k\colon \varphi]$ where $k \in K$ and $\varphi \in \mathcal{L}_0$.

Thomason [26] constructed a homotopy equivalence $|\mathcal{L}_0|_{hK} \xrightarrow{\beta} |\operatorname{Tr}_K(\mathcal{L}_0)|$ such that $|\mathcal{L}_0| \to |\mathcal{L}_0|_{hK} \simeq |\operatorname{Tr}_K(\mathcal{L}_0)|$ is homotopic to the map induced by the inclusion $\mathcal{L}_0 \subseteq \operatorname{Tr}_K(\mathcal{L}_0)$ via $\varphi \mapsto [\bar{e} : \varphi]$. Furthermore, by inspection Φ carries the subcategory of $\mathcal{L}_0$ in $\operatorname{Tr}_K(\mathcal{L}_0)$ onto $\mathcal{L}_0 \subseteq \mathcal{L}_1$ via the identity map. We deduce that $|\Phi| \circ \beta$ is a homotopy equivalence $|\mathcal{L}_0|_{hK} \to |\mathcal{L}_1|$ whose composition with $|\mathcal{L}_0| \to |\mathcal{L}_0|_{hK}$ is homotopic to the map induced by the inclusion $j\colon \mathcal{L}_0 \to \mathcal{L}_1$.

To complete the proof we now consider the subcategory $\mathcal{B}S_0$ of $\mathcal{B}\operatorname{Aut}_{\mathcal{L}_0}(S_0) \subseteq \mathcal{L}_0$ via the monomorphism $\delta_{S_0}\colon S_0 \to \operatorname{Aut}_{\mathcal{L}_0}(S_0)$ and observe that it is invariant under the action of K by hypothesis (4). Thus, there is an inclusion of subcategories $\operatorname{Tr}_K \mathcal{B}S_0 \subseteq \operatorname{Tr}_K \mathcal{L}_0$ induced by $\operatorname{Tr}_K(\delta_{S_0})$. By inspection there is an isomorphism of categories $\operatorname{Tr}_K \mathcal{B}S_0 \cong \mathcal{B}G$ via the functor $(k, s) \mapsto sk$ such that the composite

$$\mathcal{B}G \cong \operatorname{Tr}_K(\mathcal{B}S_0) \subseteq \operatorname{Tr}_K(\mathcal{L}_0) \xrightarrow{\Phi} \mathcal{L}_1$$

is the functor which sends $\bullet_G$ to S_0 and $g \mapsto [g : 1] \in \operatorname{Aut}_{\mathcal{L}_1}(S_0)$. $\square$

Here are more properties of $\mathcal{L}_1$ that we will need later in order to study the properties of the category $\mathcal{L}_2$.

5.17. **Lemma.** *Let $P_0, Q_0, R_0 \in \mathcal{H}_0$. Then*

(a) *For every $g \in N_G(P_0, Q_0)$ and $h \in N_G(Q_0, R_0)$ the equality $\hat{h} \circ \hat{g} = \widehat{hg}$ holds in $\mathcal{L}_1$.*

(b) *Fix $\psi \in \mathcal{L}_1(P_0, Q_0)$ of the form $[g\colon \varphi]$. Then, for every $x \in N_G(P_0)$ there exists at most one $y \in N_G(Q_0)$ such that $\psi \circ \hat{x} = \hat{y} \circ \psi$. In this case $y = gxg^{-1}s_0$ for a unique $s_0 \in S_0$. Moreover, if $x \in P_0$ then $y = \pi_1(\psi)(x)$ satisfies $\psi \circ \hat{x} = \hat{y} \circ \psi$.*

(c) *Every morphism in $\mathcal{L}_1(P_0, Q_0)$ is both a monomorphism and an epimorphism.*

(d) *Fix $\psi \in \mathcal{L}_1(P_0, Q_0)$ such that $\pi_1(\psi)(P_0) \leq R_0$ for some $R_0 \leq Q_0$. Then there exists $\lambda \in \mathcal{L}_1(P_0, R_0)$ such that $\psi = \iota \circ \lambda$ where $\iota = \hat{e} \in \mathcal{L}_1(R_0, Q_0)$.*

(e) *If $\pi_1(\psi) = \pi_1(\psi')$ where $\psi, \psi' \in \mathcal{L}_1(P_0, Q_0)$ then $\psi' = \psi \circ \hat{z}$ for a unique $z \in Z(P_0)$.*

(f) *Fix $P_0 \in \mathcal{H}_0$ and set $H := \{g \in G \mid gP_0g^{-1}$ is $\mathcal{F}_0$-conjugate to $P_0\}$. Then H is a subgroup of G which contains S_0 and $|\operatorname{Aut}_{\mathcal{L}_1}(P_0) : \operatorname{Aut}_{\mathcal{L}_0}(P_0)| = |H : S_0|$.*

Proof. (a) $\hat{h} \circ \hat{g} = [h : \hat{e}] \circ [g : \hat{e}] = [hg : \hat{e}] = \widehat{hg}$ by Defn. 5.5 and hypothesis (4).

(c) Every morphsm $[g\colon \varphi]$ factors as $[g\colon 1] \circ [1\colon \iota_{\varphi(P_0)}^{Q_0}] \circ [1\colon \varphi']$ for some isomorphism $\varphi'\colon P_0 \to \varphi(P_0)$ in $\mathcal{L}_0$. Since $[g\colon 1]$ and $[1\colon \varphi']$ are isomorphisms it is enough to show that morphisms of the form $[1\colon \iota_{P_0}^{Q_0}] \in \mathcal{L}_1(P_0, Q_0)$, which we denote by ι, are monomorphisms and epimorphisms.

Consider morphisms $[h : \varphi], [h' : \varphi'] \in \mathcal{L}_1(R_0, P_0)$ and assume that $\iota \circ [h : \varphi] = \iota \circ [h' : \varphi']$. Then $[h, \hat{e} \circ \varphi] = [h'\colon \hat{e} \circ \varphi']$ and therefore $h' = hs$ for some $s \in S_0$ and $\hat{e} \circ \varphi = \hat{e} \circ \hat{s} \circ \varphi'$ where $\hat{e} \in \mathcal{L}_0(\varphi(P_0), hQ_0h{-}1)$. Since $\hat{e}$ is a monomorphism in $\mathcal{L}_0$ it follows that $\varphi = \hat{s} \circ \varphi'$ and therefore $[h\colon \varphi] = [h'\colon \varphi']$. This shows that ι is a monomorphism.

If $[h:\varphi]\circ\iota = [h':\varphi']\circ\iota$ then a direct claculation shows that $[h\colon \varphi\circ\hat{e}] = [h'\colon \varphi'\circ\hat{e}]$. A similar argument to the one above using the fact that $\hat{e}$ is an epimorphism in $\mathcal{L}_0$ shows that $h' = hs$ and $\varphi = \hat{s}\circ\varphi'$, whence $[h\colon\varphi] = [h'\colon\varphi']$.

(b) If y exists then it is unique becuase by part (c), ψ is an epimorphism. Since $\psi\circ\hat{x} = [gx\colon x^{-1}\varphi x]$ and $\hat{y}\circ\psi = [yg\colon\varphi]$ we see that there is a unique $s\in S_0$ such that $gx = ygs^{-1}$, whence $y = gxg^{-1}\cdot gsg^{-1}$.

If $x\in P_0$ then axiom (C) satisfied by $\mathcal{L}_0$, see Definition 2.4, implies that

$$\psi\circ\hat{x} = [g:\varphi]\circ[x:1] = [gx:\widehat{x^{-1}}\circ\varphi\circ\hat{x}] = [g:\varphi\circ\hat{x}] = [g:\widehat{\pi_0(\varphi)(x)}\circ\varphi] =$$

$$= [c_g(\pi_0(\varphi)(x))\cdot g:\varphi] = c_g(\widehat{\pi_0(\varphi)(x)})\cdot g\circ\psi = \pi_1(\psi)(x)\circ\psi.$$

(d) Write $\psi = [g:\varphi]$ for some $\varphi\in\mathcal{L}_0(P_0,Q_0^g)$. Then $\varphi = \hat{e}\circ\bar{\varphi}$ for some $\bar{\varphi}\in\mathcal{L}_0(P_0,gR_0g^{-1})$ and $e\in N_{S_0}(P_0,gR_0g^{-1})$. By inspection $\psi = [1\colon\hat{e}]\circ[g\colon\bar{\varphi}]$.

(e) Write $\psi = [g:\varphi]$ and $\psi' = [g':\varphi']$ in $\mathcal{L}_1(P_0,Q_0)$. By assumption and Definition 5.5, $c_g\circ\pi_0(\varphi) = c_{g'}\circ\pi_0(\varphi')$, whence $\pi_0(\varphi) = c_{g^{-1}g'}\circ\pi_0(\varphi')$. Since $\pi_0(\varphi),\pi_0(\varphi')\in\mathcal{F}_0$, we obtain that $c_{g^{-1}g'}\in\mathcal{F}_0(P_0,P_0)$ so hypothesis (2) implies that $g^{-1}g'\in S_0$, namely $g' = gs$ for some $s\in S_0$. Then $\pi_0(\varphi) = c_s\circ\pi_0(\varphi')$ implies that $\hat{s}\circ\varphi' = \varphi\circ\hat{z}$ for some $z\in Z(P_0)$. Therefore, $\psi\circ\hat{z} = [g\colon\varphi\circ\hat{z}] = [g\colon\hat{s}\circ\varphi'] = [g'\colon\varphi'] = \psi'$.

(f) By hypothesis (1) in Theorem 5.2, if Q_0 is $\mathcal{F}_0$-conjugate to Q_0' then gQ_0g^{-1} is $\mathcal{F}_0$-conjugate to $gQ_0'g^{-1}$ for any $g\in G$. This implies that H is a subgroup of G and it contains S_0 because $\mathcal{F}_{S_0}(S_0)\subseteq\mathcal{F}_0$.

Let $g_1,\cdots,g_n$ be representatives for the cosets of S_0 in H. By Definition 5.5 every element $\psi\in\mathrm{Aut}_{\mathcal{L}_1}(P_0)$ can be written uniquely as $\psi = [g_i:\varphi]$ for some $i = 1,\dots,n$ where $\varphi\in\mathcal{L}_0(P_0,{}^{g_i}P_0)$. Also note that $|\mathcal{L}_0(P_0,{}^{g_i}P_0)| = |\mathrm{Aut}_{\mathcal{L}_0}(P_0)|$ because ${}^{g_i}P_0$ is $\mathcal{F}_0$-conjugate to P_0. This shows that $|\mathrm{Aut}_{\mathcal{L}_1}(P_0)| = n\cdot|\mathrm{Aut}_{\mathcal{L}_0}(P_0)| = |H:S_0|\cdot|\mathrm{Aut}_{\mathcal{L}_0}(P_0)|$. □

We now turn to the study of the properties of the category $\mathcal{L}_2$.

Proof that Defn. 5.7 makes $\mathcal{L}_2$ a small category. Given morphisms $\psi\in\mathcal{L}_2(P,Q)$ and $\rho\in\mathcal{L}_2(Q,R)$, we leave it as an easy exercise to check that $\rho\circ\psi\in\mathcal{L}_1(P_0,R_0)$ belongs to $\mathcal{L}_2(P,R)$. Thus, composition of morphisms in $\mathcal{L}_2$ is well defined. It is easily seen to be unital and associative because this is the case in $\mathcal{L}_1$.

Since $S_0\lhd G$ it follows that $N_G(P,Q)\subseteq N_G(P_0,Q_0)$. Now fix some $g\in N_G(P,Q)$ and $x\in P$ and set $y = gxg^{-1}\in Q$. It follows from Lemma 5.17(a) that $\hat{g}\circ\hat{x} = \widehat{gx} = \widehat{yg} = \hat{y}\circ\hat{g}$. Therefore $\hat{g}\in\mathcal{L}_2(P,Q)$. □

We are now ready to prove the lemmas 5.8 and 5.9.

Proof of Lemma 5.8. By construction $\mathcal{L}_2(P_0,Q_0)\subseteq\mathcal{L}_1(P_0,Q_0)$ for any $P_0,Q_0\in\mathcal{H}_0$. For every $x\in P_0$ and every $\psi = [g:\varphi]\in\mathcal{L}_1(P_0,Q_0)$, it follows from Lemma 5.17(b) that $\psi\circ\hat{x} = \hat{y}\circ\psi$ in $\mathcal{L}_1$ where $y = \pi_1(\psi)(x)\in Q_0$. Therefore $\psi\in\mathcal{L}_2(P_0,Q_0)$ and we conclude that $\mathcal{L}_1(P_0,Q_0) = \mathcal{L}_2(P_0,Q_0)$.

The inclusion functor $j\colon\mathcal{L}_1\to\mathcal{L}_2$ has a left inverse $r\colon\mathcal{L}_2\to\mathcal{L}_1$ which maps an object P to P_0 and maps morphisms via the inclusions $\mathcal{L}_2(P,Q)\subseteq\mathcal{L}_1(P_0,Q_0)$. Observe that $r\circ j = \mathrm{Id}_{\mathcal{L}_1}$ because $\mathcal{L}_2(P_0,Q_0) = \mathcal{L}_1(P_0,Q_0)$.

By Lemma 5.17(b) we see that $\mathcal{L}_2(P_0,P)$ contains $[e:1_{P_0}] = \hat{e}$. These morphisms define a natural transformation $j\circ r\to\mathrm{Id}$. This is because we recall that $[e:1_{P_0}]$ and

$[e : 1_{Q_0}]$ are the identities of P_0 and Q_0 in $\mathcal{L}_1$ and for any $\psi \in \mathcal{L}_2(P,Q) \subseteq \mathcal{L}_1(P_0, Q_0)$

$$\psi \circ [e : 1_{P_0}] = [e : 1_{Q_0}] \circ \psi.$$

Then it follows that j and r yield homotopy equivalences on nerves. □

Proof of Lemma 5.9. (a) By Definition 5.7, for every $x \in P$ there exists some $y \in Q$ such that $\psi \circ \hat{x} = \hat{y} \circ \psi$. Since $P \leq N_G(P_0)$ and $Q \leq N_G(Q_0)$, Lemma 5.17(b) implies that y is unique. There results a well defined function $\pi_2(\psi) \colon P \to Q$ defined by $\pi_2(\psi)(x) = y$. In addition, since $\hat{x}$ and $\hat{y} = \widehat{\pi_2(\psi)(x)}$ are morphisms in $\mathcal{L}_2$ (see Definition 5.7) and $\mathcal{L}_2(P,Q) \subseteq \mathcal{L}_1(P_0, Q_0)$, we deduce that the equation $\psi \circ \hat{x} = \widehat{\pi_2(\psi)(x)} \circ \psi$ holds in $\mathcal{L}_2$. Moreover, $\pi_2(\psi) \colon P \to Q$ is the unique function that satisfies this equality for all $x \in P$. The fact that $\pi_2(\psi)|_{P_0} = \pi_1(\psi)$ follows from the last assertion in Lemma 5.17(b).

Given $x, x' \in P$, set $y = \pi_2(\psi)(x)$ and $y' = \pi_2(\psi)(x')$. Then, by Lemma 5.17(a)

$$\psi \circ \widehat{xx'} = \psi \circ \hat{x} \circ \hat{x}' = \hat{y} \circ \psi \circ \hat{x}' = \hat{y} \circ \hat{y}' \circ \psi = \widehat{yy'} \circ \psi.$$

This shows that $\pi_2(\psi)$ is a homomorphism. If $x \in \ker \pi_2(\psi)$ then $\psi \circ \hat{x} = \hat{1} \circ \psi = \psi$. Since ψ is a monomorphism by Lemma 5.17(c), we deduce that $\hat{x} = \mathrm{id}$, hence $x = 1$. Therefor $\pi_2(\psi)$ is a monomorphism.

(b) Clearly $\pi_2([e : 1_{P_0}]) = \mathrm{Id}_{P_0}$. Now given $P \xrightarrow{\psi} Q \xrightarrow{\lambda} R$ in $\mathcal{L}_2$, set $y = \pi_2(\psi)(x)$ and $z = \pi_2(\lambda)(y)$. Then $\psi \circ \hat{x} = \hat{y} \circ \psi$ and $\lambda \circ \hat{y} = \hat{z} \circ \lambda$ so $\lambda \circ \psi \circ \hat{x} = \hat{z} \circ \lambda \circ \psi$ whence, by the uniqueness statement in Lemma 5.17(b), we conclude that $z = \pi_2(\lambda \circ \psi)(x)$.

(c) This follows from Lemma 5.17(a) because for any $x \in P$ we have $\hat{g} \circ \hat{x} = \widehat{gx} = \widehat{c_g(x)g} = \widehat{c_g(x)} \circ \hat{g}$ in $\mathcal{L}_1$ so $\pi_2(\hat{g}) = c_g$.

(d) Write $\psi = [g : \varphi]$. Observe that $\pi_2(\psi)(P_0) = \pi_1(\psi)(P_0) \leq Q_0$ by statement (a). Since $\pi_2(\psi) \colon P \to Q$ is an isomorphism, for every $y_0 \in Q_0 \leq Q$ there exists some $x \in P$ such that $\pi_2(\psi)(x) = y_0$, namely $\psi \circ \hat{x} = \widehat{y_0} \circ \psi$. By Lemma 5.17(b) we know that $y_0 = gxg^{-1}s_0$ for some $s_0 \in S_0$. We deduce then that $x \in S_0 \cap P = P_0$ because $S_0 \lhd G$. This shows that $\pi_2(\psi)(P_0) = Q_0$ and therefore $\pi_1(\psi)$ is an isomorphism of groups.

Since $\pi_1(\psi)$ is an isomorphism, φ is an isomorphism in $\mathcal{L}_0$ and therefore ψ is an isomorphism in $\mathcal{L}_1$. Given any $y \in Q$ there is a unique $x \in P$ with $\psi \circ \widehat{x^{-1}} = \widehat{y^{-1}} \circ \psi$ because $\pi_2(\psi)$ is an isomorphism. By taking inverses one sees that ψ^{-1} belongs to $\mathcal{L}_2$ so ψ is an isomorphism in $\mathcal{L}_2$. □

For later use we also need the following technical lemma.

5.18. **Lemma.** *Fix $P \in \mathcal{H}$ and consider $N_S(P_0)$ as a subgroup of $\mathrm{Aut}_{\mathcal{L}_1}(P_0)$ via the monomorphism $\hat{\delta}_{P_0,P_0} \colon N_S(P_0) \to \mathrm{Aut}_{\mathcal{L}_1}(P_0)$. Let $Q \leq N_S(P_0)$ and assume that $Q = \psi P \psi^{-1}$ for some $\psi \in \mathrm{Aut}_{\mathcal{L}_1}(P_0)$. Then $P_0 = Q_0$ and ψ is an isomorphism in $\mathcal{L}_2$ from P to Q.*

Proof. Recall from Lemma 5.8 that $\mathrm{Aut}_{\mathcal{L}_1}(P_0) = \mathrm{Aut}_{\mathcal{L}_2}(P_0)$. For $x \in P_0$ set $\hat{y} = \psi \hat{x} \psi^{-1} \in Q$. Thus $\psi \circ \hat{x} = \hat{y} \circ \psi$ and by Definition 5.10, $y = \pi_2(\psi)(x) \in P_0$. This shows that $P_0 = \psi P_0 \psi^{-1}$ and, in particular, $P_0 \leq Q_0$. Moreover .

Since $P_0 \leq Q_0$, we may consider $\iota := \hat{e} \in \mathcal{L}_1(P_0, Q_0)$ where $e \in G$ is the identity element, and define $\lambda = \iota \circ \psi \in \mathcal{L}_1(P_0, Q_0)$. For every $x \in P$, set $\hat{y} = \psi \hat{x} \psi^{-1}$. By definition $y \in Q$. Note that $P_0 \lhd Q$ because $P_0 \lhd P$. So Lemma 5.17(a) implies

$$\lambda \circ \hat{x} = \iota \circ \psi \circ \hat{x} = \iota \circ \hat{y} \circ \psi = \hat{y} \circ \hat{e} \circ \psi = \hat{y} \circ \psi.$$

We conclude from Definition 5.7 that $\lambda \in \mathcal{L}_2(P,Q)$. Furthermore, $\pi_2(\lambda)$ is an isomorphism because it is a monomorphism by Lemma 5.9(a) and $|P| = |Q|$. Lemma 5.9(d) now shows that λ is an isomorphism in $\mathcal{L}_2$ and, in particular, it is an isomorphism of the objects P_0 and Q_0 in $\mathcal{L}_1$. In particular $|P_0| = |Q_0|$ and therefore $\lambda = \psi$. $\square$

We now check the main properties of the category $\mathcal{F}_2$.

Proof of Lemma 5.11. (a) This is immediate from Lemma 5.9(c). By taking $e \in N_G(P,Q)$ for any inclusion $P \leq Q$ in $\mathcal{H}$ we obtain $\mathrm{incl}_P^Q \in \mathcal{F}_2(P,Q)$.

(b) Fix a homomorphism $f\colon P \to Q$ in $\mathcal{F}_2$ and set $R = f(P)$. By definition, $f = \pi_2(\psi)$ for some $\psi \in \mathcal{L}_2(P,Q)$. Also note that every $y \in R$ must normalise $f(P_0)$ because f is an isomorphism and that by Lemma 5.9(a), $f(P_0) = \pi_1(\psi)(P_0)$.

Write $\psi = [g\colon \varphi]$. Then there is an isomorphism $\bar{\varphi}$ in $\mathcal{L}_0$ such that $\psi = [1\colon \iota_{f(P_0)}^{Q_0}] \circ [g\colon \bar{\varphi}]$. Since $\psi \in \mathcal{L}_2$, for every $x \in P$ there exists $y \in R$ such that

$$[1\colon \iota_{f(P_0)}^{Q_0}] \circ [g\colon \bar{\varphi}] \circ \hat{x} = \hat{y} \circ [1\colon \iota_{f(P_0)}^{Q_0}] \circ [g\colon \bar{\varphi}] = [1\colon \iota_{f(P_0)}^{Q_0}] \circ \hat{y} \circ [g\colon \bar{\varphi}].$$

By Lemma 5.17(c), $[1\colon \iota_{f(P_0)}^{Q_0}]$ is a monomorphism and we deduce that $[g\colon \bar{\varphi}]$ is an isomorphism $P \to R$ in $\mathcal{L}_2$. Also $f = \mathrm{incl}_R^Q \pi_2([g\colon \bar{\varphi}])$ This completes the proof. $\square$

5.19. **Lemma.** *Consider $P \leq S$ such that $P_0 \in \mathcal{H}_0$. Then $C_G(P) = C_{S_0}(P) = Z(P_0)^P$ where P acts on $Z(P_0)$ by conjugation.*

Proof. If $g \in C_G(P)$ then $c_g|_{P_0} = \mathrm{id}_{P_0} \in \mathrm{Aut}_{\mathcal{F}_0}(P_0)$. By hypothesis (2), $g \in S_0$, and it follows that $C_G(P) = C_{S_0}(P)$. Now, $C_{S_0}(P) \leq C_{S_0}(P_0) = Z(P_0)$ because P_0 is $\mathcal{F}_0$-centric. Therefore, $C_G(P) = C_{Z(P_0)}(P) = Z(P_0)^P$. $\square$

Lemmas 5.13 and 5.14 state the main properties of the fusion system $\mathcal{F}$.

Proof of Lemma 5.13. (a) Clearly $\mathcal{H}$ is closed to taking supergroups because $\mathcal{H}_0$ is closed to taking supergroups in S_0. Since $\mathcal{F}$ is generated by inclusions and restriction of homomorphisms in $\mathcal{F}_2$, Lemma 5.11 shows that for any $P, Q \in \mathcal{H}$ the inclusion $\mathcal{F}_2(P,Q) \subseteq \mathcal{F}(P,Q)$ is an equality.

(b) By definition $P_0 \in \mathcal{H}_0$. By Lemma 5.19, $C_S(P) = Z(P_0)^P \leq P$. Assume that Q is $\mathcal{F}$-conjugated to P. By part (a) there exists some $\psi \in \mathcal{L}_2(P,Q)$ such that $\pi_2(\psi)(P) = Q$. Parts (a) and (d) of Lemma 5.9 imply that ψ is an isomorphism in $\mathcal{L}_2$. From Definition 5.7 it is clear that ψ is an isomorphism in $\mathcal{L}_1(P_0, Q_0)$ and in particular $Q_0 \in \mathcal{H}_0$, namely Q_0 is $\mathcal{F}_0$-centric. It follows from Lemma 5.19 that $C_S(Q) = Z(Q_0)^Q \cong Z(P_0)^P$, whence P is $\mathcal{F}$-centric.

(c) For any $f \in \mathcal{F}(P,Q)$ where $P, Q \in \mathcal{H}$, part (a) implies that $f = \pi_2(\psi)$ for some $\psi \in \mathcal{L}_2(P,Q) \subseteq \mathcal{L}_2(P_0, Q_0)$. The result follows from Lemma 5.9(a) which shows that $f|_{P_0} = \pi_1(\psi)$ whose image is contained in Q_0 by Definition 5.5. $\square$

Proof of Lemma 5.14. The monomorphisms $\delta_P\colon P \to \mathrm{Aut}_{\mathcal{L}_2}(P)$ are the restrictions of the maps $\hat{\delta}_{P,Q}\colon N_G(P,Q) \to \mathcal{L}_2(P,Q)$, i.e. $\delta_P(g) = [g : 1_{P_0}]$.

To verify axiom (A) in [6, Definition 1.7], see also 2.4, we need to show that for any $P, Q \in \mathcal{H}$ the set $\pi_2^{-1}(f)$ where $f \in \mathcal{F}(P,Q)$ admit a transitive free action of $C_S(P)$ via $\delta_P\colon N_S(P) \to \mathrm{Aut}_{\mathcal{L}_2}(P)$. Note that $\mathcal{F}(P,Q) = \mathcal{F}_2(P,Q)$ by Lemma 5.13. Consider $\psi, \psi' \in \mathcal{L}_2(P,Q) \subseteq \mathcal{L}_1(P_0, Q_0)$ such that $\pi_2(\psi) = \pi_2(\psi')$. By restriction to P_0, Lemma 5.9(a) shows that $\pi_1(\psi) = \pi_1(\psi')$. Lemma 5.17(f) shows that there exists $z \in Z(P_0)$ such that $\psi' = \psi \circ \hat{z}$ in $\mathcal{L}_1$. Note that $\hat{z} \in \mathrm{Aut}_{\mathcal{L}_2}(P_0)$ by Definition 5.5 so the equality $\psi' = \psi \circ \hat{z}$ also holds in $\mathcal{L}_2$. Furthermore, Lemma 5.17(c) implies that

$$\pi_2(\psi) = \pi_2(\psi') = \pi_2(\psi \circ \hat{z}) = \pi_2(\psi) \circ c_z.$$

As a consequence $z \in C_S(P)$ and we conclude that $C_S(P)$ acts transitively on the fibres of $\pi_2\colon \mathcal{L}_2(P,Q) \to \mathcal{F}(P,Q)$. The action is free by Lemma 5.19 and the uniqueness assertion in Lemma 5.17(e).

Axiom (B) holds by Lemma 5.9(c). To verify axiom (C) we fix a morphism $\psi \in \mathcal{L}_2(P,Q)$ and an element $g \in P$. Set $f = \pi_2(\psi) \in \mathcal{F}(P,Q)$. By the definition of the morphisms in $\mathcal{L}_2$, see Lemma 5.9(a) we have $\psi \circ \hat{g} = \widehat{f(g)} \circ \psi$, which is what we need. $\square$

Notation. We shall write $P \simeq_{\mathcal{F}} Q$ for the statement that $P, Q \leq S$ are $\mathcal{F}$-conjugate.

The next step is to prove Lemma 5.15 which shows that $\mathcal{F}$ is a saturated fusion system.

Clearly S_0 acts on $\mathcal{H}_0$ by conjugation and $[P_0]_{S_0}$ denotes the orbit of P_0, i.e. the conjugacy class of P_0 in S_0. Since G acts via fusion preserving automorphisms, it acts on the set $\mathcal{H}_0/\mathcal{F}_0$ of the $\mathcal{F}_0$-conjugacy classes of the subgroups $P_0 \in \mathcal{H}_0$ which we denote $[P_0]_{\mathcal{F}_0}$. The stabiliser of $[P_0]_{\mathcal{F}_0}$ under this action of G is denoted, as usual, by $G_{[P_0]_{\mathcal{F}_0}}$. Now, $G_{[P_0]_{\mathcal{F}_0}}$ acts on the set $[P_0]_{\mathcal{F}_0} \subseteq \mathcal{H}_0$. Clearly, $S_0 \leq G_{[P_0]_{\mathcal{F}_0}}$ because $\mathcal{F}_{S_0}(S_0) \subseteq \mathcal{F}_0$. Moreover, since $S_0 \lhd G$, this action induces an action of $G_{[P_0]_{\mathcal{F}_0}}$ on the set $\mathcal{P}$ of all the S_0-conjugacy classes of the subgroups of S_0 that are $\mathcal{F}_0$-conjugate to P_0.

5.20. **Lemma.** *For every $P \in \mathcal{H}$ there exist $\bar{P}, P' \in \mathcal{H}$ such that:*

(a) *$\bar{P} = {}^aP$ for some $a \in G$ and $\bar{P} \simeq_{\mathcal{F}} P'$, whence $P \simeq_{\mathcal{F}} P'$, and*

(b) *P'_0 is fully $\mathcal{F}_0$-normalised and $P'_0 \simeq_{\mathcal{F}_0} \bar{P}_0$.*

In addition, $\bar{S} := N_S(P'_0)S_0$ is a Sylow p-subgroup of $G_{[\bar{P}_0]_{\mathcal{F}_0}}$ and $\bar{S}/S_0$ fixes the S_0-conjugacy class $[P'_0]_{S_0}$.

Proof. The argument follows the one in the proof of step 3 in [3, Theorem 4.6].

Clearly $S_0 \cdot P \leq G_{[P_0]_{\mathcal{F}_0}}$ because $P \leq N_G(P_0)$ and $\mathcal{F}_{S_0}(S_0) \subseteq \mathcal{F}_0$. Choose $S' \in \mathrm{Syl}_p(G_{[P_0]_{\mathcal{F}_0}})$ which contains $S_0 \cdot P$. By Sylow's theorems, there exists some $a \in G$ such that $S' = G_{[P_0]_{\mathcal{F}_0}} \cap S^a$. Set $\bar{P} = {}^aP$ and observe that

$$\bar{P} = {}^aP \leq {}^a(G_{[P_0]_{\mathcal{F}_0}} \cap S^a) \leq S.$$

Also $\bar{P}_0 = {}^aP_0 \in \mathcal{H}_0$, so $\bar{P} \in \mathcal{H}$. In addition, $G_{[\bar{P}_0]_{\mathcal{F}_0}} = {}^a(G_{[P_0]_{\mathcal{F}_0}})$. It follows that

$$\bar{S} := S \cap G_{[\bar{P}_0]_{\mathcal{F}_0}} = {}^a(S') \in \mathrm{Syl}_p(G_{[\bar{P}_0]_{\mathcal{F}_0}}).$$

Consider now the set $\mathcal{P}_{fn}$ of all the S_0-conjugacy classes of the fully $\mathcal{F}_0$-normalised subgroups $R \leq S_0$ which are $\mathcal{F}_0$-conjugate to $\bar{P}_0$. Since G normalises S_0 and it is fusion preserving, it carries fully $\mathcal{F}_0$-normalised subgroups of S_0 to ones, and therefore $G_{[\bar{P}_0]_{\mathcal{F}_0}}$ acts on $\mathcal{P}_{fn}$.

We now restrict the action of $G_{[\bar{P}_0]_{\mathcal{F}_0}}$ on $\mathcal{P}_{fn}$ to $\bar{S}$. By [3, Proposition 1.16] we know that $|\mathcal{P}_{fn}| \neq 0 \mod p$. Therefore $\bar{S}/S_0$ must have some fixed point $[R_0]_{S_0}$. Thus, R_0 is fully $\mathcal{F}_0$-normalised and is $\mathcal{F}_0$-conjugate to $\bar{P}_0$. For every $g \in \bar{S} \leq S$ we have $gR_0g^{-1} \simeq_{S_0} R_0$ so $\bar{S} \leq N_S(R_0)S_0$. On the other hand

$S_0 N_S(R_0) \leq G_{[R_0]_{\mathcal{F}_0}} = G_{[\bar{P}_0]_{\mathcal{F}_0}}$ and $\bar{S}$ is a Sylow p-subgroup of the latter group, hence

$$\bar{S} = S_0 \cdot N_S(R_0).$$

It remains to find some $P' \in \mathcal{H}$ such that $P' \simeq_{\mathcal{F}} \bar{P}$ and such that $P'_0 = R_0$. Now, since $\bar{P} \leq \bar{S}$, it must stabilise $[R_0]_{S_0}$. We conclude that $\bar{P}/\bar{P}_0$ acts on

$$X := \{[f] \in \mathrm{Rep}_{\mathcal{F}_0}(\bar{P}_0, S_0) \ : \ \mathrm{Im}\, f \text{ is } S_0\text{-conjugate to } R_0\}$$

via $[f_0] \mapsto [c_g \circ f_0 \circ c_{g^{-1}}]$. Clearly X is not empty because by construction $\bar{P}_0 \simeq_{\mathcal{F}_0} R_0$. Choose some $f \in \mathcal{F}_0(\bar{P}_0, R_0)$. Then every element of X has the form $[\alpha \circ f]$ for some $\alpha \in \mathrm{Aut}_{\mathcal{F}_0}(R_0)$. Moreover $[\alpha \circ f] = [\beta \circ f]$ if and only if $\alpha^{-1}\beta \in \mathrm{Aut}_{S_0}(R_0)$. Therefore

$$|X| = \frac{|\mathrm{Aut}_{\mathcal{F}_0}(R_0)|}{|\mathrm{Aut}_{S_0}(R_0)|} \neq 0 \mod p$$

because R_0 is fully $\mathcal{F}_0$-normalised. Since $\bar{P}$ is a finite p-group, there is some $[f_0] \in X^{\bar{P}}$ where $f_0 \in \mathcal{F}_0(\bar{P}_0, S_0)$ and $\mathrm{Im}\, f_0 = R_0$. Let $\psi_0 \in \mathcal{L}_0(\bar{P}_0, S_0)$ be a lift of f_0.

Recall from Lemma 5.6(a) that we may consider ψ_0 as a morphism in $\mathcal{L}_1(\bar{P}_0, S_0)$ via an inclusion $\mathcal{L}_0 \subseteq \mathcal{L}_1$. Fix some $x \in \bar{P}$. Since $\bar{P}$ fixes $[f_0]$, there exists some $s \in S_0$ such that

$$c_x^{-1} \circ f_0 \circ c_x = c_s \circ f_0.$$

Lifting to $\mathcal{L}_0$ and using hypothesis (1), we see that there exists a unique $z \in C_{S_0}(\bar{P}_0) = Z(\bar{P}_0)$ such that

$$x^{-1}\psi_0 x = \hat{s} \circ \psi_0 \circ \hat{z} = \widehat{s f_0(z)} \circ \psi_0 \qquad \text{in } \mathcal{L}_0. \tag{1}$$

Set $y := x s f_0(z)$ and note that $y \in \bar{P} \cdot S_0 \cdot Z(R_0) \leq S$. Lemma 5.6(c), equation (1) and the properties of S-systems of lifts (see Def. 2.5) imply that

$$\psi_0 \circ \hat{x} = \hat{x} \circ (x^{-1}\psi_0 x) = \hat{x} \circ \widehat{s f_0(z)} \circ \psi_0 = \hat{y} \circ \psi_0.$$

Therefore, by definition, $\psi_0 \in \mathcal{L}_2(\bar{P}, S)$. Consider $f = \pi_2(\psi_0) \in \mathcal{F}(\bar{P}, S)$ and set $P' = f(\bar{P})$. By Lemmas 5.13(a) and 5.11(b), f restricts to an isomorphism $f \colon \bar{P} \to P'$ in $\mathcal{F}$. By Lemma 5.9(a) and Lemma 5.6(a) we see that $f|_{\bar{P}_0} = \pi_0(\psi_0) = f_0 \in \mathcal{F}_0(\bar{P}_0, R_0)$. Since $f \in \mathcal{F}(\bar{P}, P')$ is an isomorphism we deduce from Lemma 5.13(c) that $P'_0 = f(\bar{P}_0) = R_0$. This completes the proof since f is an $\mathcal{F}$-isomorphism between $\bar{P}$ and P' which restricts to an $\mathcal{F}_0$-isomorphism f_0 between $\bar{P}_0$ and $R_0 = P'_0$. $\square$

5.21. **Lemma.** *If $P \leq S$ is $\mathcal{F}$-centric but $P \notin \mathcal{H}$, then there exists $P' \leq S$ which is $\mathcal{F}$-conjugate to P such that*

$$\mathrm{Out}_S(P') \cap O_p(\mathrm{Out}_{\mathcal{F}}(P')) \neq 1.$$

Proof. The argument is almost repeated from step 4 in the proof of [3, Theorem 4.6] if we find a subgroup $\hat{P} \leq S$ which is $\mathcal{F}$-conjugate to P and such that $\hat{P}_0$ does not contain its S_0-centraliser.

Assume to the contrary that there is some P which is $\mathcal{F}$-centric, P_0 is not $\mathcal{F}_0$-centric, and for which there does not exist $\hat{P}$ as above. Choose $P \leq S$ so that P_0 has the maximal possible order. If $N_{S_0}(P_0)$ is $\mathcal{F}_0$-centric we choose $\tilde{\varphi} \in \mathcal{L}_0(N_{S_0}(P_0), S_0)$ as in hypothesis (5) of Theorem 5.2. Then $\tilde{\varphi}$ represents a morphism $N_S(P_0) \to S$ in $\mathcal{L}_2$ because for any $x \in N_S(P_0)$

$$\tilde{\varphi} \circ \hat{x} = [1 \colon \tilde{\varphi}] \circ [x \colon 1] = [x \colon x^{-1}\tilde{\varphi}x] = [x \colon \hat{s} \circ \hat{\varphi}] = [xs \colon \tilde{\varphi}] = [xs \colon 1] \circ [1 \colon \tilde{\varphi}] = \widehat{xs} \circ \tilde{\varphi}$$

Thus, $\pi_2(\tilde{\varphi})$ is a morphism $f\colon N_S(P_0) \to S$ in $\mathcal{F}$ whose restriction to P gives rise to an $\mathcal{F}$-conjugate $\hat{P}$ and by Lemmas 5.13(c) and 5.9(a), $\hat{P}_0 = f(P_0)$ does not contain its S_0-centraliser. This is a contradiction, hence $N_{S_0}(P_0)$ cannot be $\mathcal{F}_0$-centric.

Now, P must normalise $N_{S_0}(P_0)$ and we consider $Q := PN_{S_0}(P_0)$. Clearly $Q_0 = N_{S_0}(P_0)$. By the maximality of $|P_0|$ we deduce that Q is $\mathcal{F}$-conjugate to some $\hat{Q}$ such that $\hat{Q}_0$ does not contain its S_0-centraliser. By restriction to $P \leq Q$ we see that P is $\mathcal{F}$-conjugate to some $\hat{P} \leq \hat{Q}$ and $\hat{P}_0$ cannot contain its S_0-centraliser because $\hat{Q}_0 \geq \hat{P}_0$ does not contain its S_0-centraliser. This is again a contradiction.

Finally, our notation was chosen in such a way that the argument in Step 4 in [3, proof of Theorem 4.6] can be now read verbatim to complete the proof. □

Proof of 5.15. By [4, Theorem 2.2] and Lemma 5.21, $\mathcal{F}$ is saturated if the saturation axioms of Definition 2.2 hold for all subgroups in $\mathcal{H}$. To show this, we slightly modify the argument in [3, Thm 4.6].

Condition I. Fix $P \in \mathcal{H}$ which is fully $\mathcal{F}$-normalised. We have to show that it is fully $\mathcal{F}$-centralised and that $\mathrm{Aut}_S(P)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(P)$. By Lemma 5.13(b) we know that P is $\mathcal{F}$-centric and in particular fully $\mathcal{F}$-centralised.

Consider $\bar{P}$ and P' as in Lemma 5.20. Recall that $\bar{S} = N_S(P_0')S_0$ is a Sylow p-subgroup of $G_{[\bar{P}_0]_{\mathcal{F}_0}}$. Lemma 5.6(a) shows that $\mathrm{Aut}_{\mathcal{L}_0}(\bar{P}_0) \leq \mathrm{Aut}_{\mathcal{L}_1}(\bar{P}_0)$ and by Lemma 5.17(f) and Lemmas 5.11 and 5.13

$$|\mathrm{Aut}_{\mathcal{L}_1}(P_0') : \mathrm{Aut}_{\mathcal{L}_0}(P_0')| = |G_{[\bar{P}_0]_{\mathcal{F}_0}} : S_0|. \tag{5.22}$$

By definition $N_{S_0}(P_0') = S_0 \cap N_S(P_0')$ so

$$|N_S(P_0')/N_{S_0}(P_0')| = |N_S(P_0')S_0/S_0| = |\bar{S}/S_0|. \tag{5.23}$$

Now, P_0' is fully $\mathcal{F}_0$-normalised and is $\mathcal{F}_0$-centric so

$$|\mathrm{Aut}_{\mathcal{L}_0}(P_0') : N_{S_0}(P_0')| \neq 0 \mod p. \tag{5.24}$$

Since $|G_{[P_0']_{\mathcal{F}_0}} : \bar{S}| \neq 0 \mod p$, we deduce from (5.22), (5.23) and (5.24) that

$$|\mathrm{Aut}_{\mathcal{L}_1}(P_0') : N_S(P_0')| = \frac{|\mathrm{Aut}_{\mathcal{L}_1}(P_0')|}{|\mathrm{Aut}_{\mathcal{L}_0}(P_0')|} \cdot \frac{|\mathrm{Aut}_{\mathcal{L}_0}(P_0')|}{|N_{S_0}(P_0')|} \cdot \frac{|N_{S_0}(P_0')|}{|N_S(P_0')|} \neq 0 \mod p,$$

namely $N_S(P_0') \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{L}_1}(P_0'))$.

Fix $\psi \in \mathrm{Aut}_{\mathcal{L}_1}(P_0')$ such that

$$\psi^{-1}N_S(P_0')\psi \supseteq R \in \mathrm{Syl}_p(N_{\mathrm{Aut}_{\mathcal{L}_1}(P_0')}(P')) \tag{5.25}$$

and set

$$P'' = \psi P' \psi^{-1} \leq N_S(P_0').$$

Lemma 5.18 shows that $P_0' = P_0''$ and that $\psi \in \mathcal{L}_2(P', P'')$ is an isomorphism. In particular, P'' is $\mathcal{F}$-conjugate to P', hence also to P because $P' = {}^aP$ for some $a \in G$ and $\hat{a} \in \mathcal{L}_2(P, P')$ is an isomorphism. We now claim that

(i) $\mathrm{Aut}_{\mathcal{L}_2}(P'') = N_{\mathrm{Aut}_{\mathcal{L}_1}(P_0')}(P'')$ and (ii) $N_S(P'') = N_{N_S(P_0)}(P'')$.

Clearly (i) follows from the definition of the morphisms in $\mathcal{L}_2$ because

$$\begin{aligned} \lambda \in \mathrm{Aut}_{\mathcal{L}_2}(P'') &\iff \forall x \in P''\, \exists y \in P''(\lambda \circ \hat{x} \circ \lambda^{-1} = \hat{y}) \\ &\iff \lambda \in N_{\mathrm{Aut}_{\mathcal{L}_1}(P_0')}(P''). \end{aligned}$$

For (ii), note that $P'' \subseteq N_S(P_0') \subseteq \mathrm{Aut}_{\mathcal{L}_1}(P_0')$ so by the choice of ψ in equation (5.25),

$$N_{N_S(P_0')}(P'') = N_S(P_0') \cap N_{\mathrm{Aut}_{\mathcal{L}_1}(P_0')}(P'') \in \mathrm{Syl}_p(N_{\mathrm{Aut}_{\mathcal{L}_1}(P_0')}(P'')).$$

On the other hand

$$N_{N_S(P_0')}(P'') \leq N_S(P'') \leq N_{\mathrm{Aut}_{\mathcal{L}_1}(P_0')}(P''),$$

hence $N_S(P'') = N_{N_S(P_0')}(P'')$. We deduce that $N_S(P'') \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{L}_2}(P'')$. Finally, $\mathrm{Aut}_{\mathcal{L}_2}(P) \cong \mathrm{Aut}_{\mathcal{L}_2}(P'')$ because P'' and P are isomorphic in $\mathcal{L}_2$ (via $\psi \circ \hat{a}$). Also, $|N_S(P)| \geq |N_S(P'')|$ because P is fully $\mathcal{F}$-normalised. Therefore $N_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{L}_2}(P))$ and Lemma 5.14 implies that $\mathrm{Aut}_S(P)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(P)$.

Condition II. Fix $P \in \mathcal{H}$ and $f \in \mathcal{F}(P, S)$. Parts (a) and (b) of Lemma 5.13 show that $f(P) \in \mathcal{H}$ and that $f(P)$ is $\mathcal{F}$-centric and in particular it is fully $\mathcal{F}$-centralised. We have to prove that f extends to some morphism $N_f \to S$ in $\mathcal{F}$ where

$$N_f = \{g \in N_S(P) \ : \ f \circ c_g = c_s \circ f \text{ for some } s \in S\}.$$

Note that s in the defintion of N_f belongs to $N_S(\mathrm{Im}\, f)$. Set $Q = N_f$. We observe that

$$Q \leq N_S(Q_0) \qquad \text{and} \qquad Q \leq N_S(P) \leq N_S(P_0). \tag{5.26}$$

By construction of $\mathcal{F}_2$, there exists $\varphi \in \mathcal{L}_2(P, S)$ such that $f = \pi_2(\varphi)$. Now φ in a morphism in $\mathcal{L}_1(P_0, S_0)$ and we write $\varphi = [g \colon \varphi_0]$. By definition of $Q = N_f$, for any $q \in Q$ there exists $t \in S$ such that $f \circ c_q = c_t \circ f$. Lemma 5.13(c) and (5.26) imply that $f_0 \circ c_q = c_t \circ f_0$ where $f_0 \colon P_0 \to S_0$ is the restriction of f. By Lemma 5.17(b), $f_0 = \pi_1(\varphi)$ so part (e) of that lemma implies that $\varphi \circ \hat{q} = \hat{t} \circ \varphi \circ \hat{z}$ for some $z \in Z(P_0)$. Part (b) of that Lemma applies again to show that $\varphi \circ \hat{q} = \widehat{s_q} \circ \varphi$ for some $s_q \in S$.

Now, if $q \in Q_0$ then $\varphi \circ \hat{q} = [gq \colon q^{-1}\varphi_0 q]$ and $\widehat{s_q} \circ \varphi = [s_q g \colon \varphi_0]$. Therefore there is $s \in S_0$ such that

$$gqs = s_q g \qquad \text{and} \qquad \hat{s} \circ \varphi_0 = \widehat{q^{-1}} \circ \varphi_0 \circ \widehat{q}.$$

In particular $s_q \in S_0$ and $\pi_0(\varphi_0) \circ c_q = c_{qs} \circ \pi_0(\varphi_0)$. This shows that $Q_0 \subseteq N_{\pi_0(\varphi_0)}$ and we may extend φ_0 to some $\psi_0 \in \mathcal{L}_0(Q_0, S_0)$ because $Q_0 \supseteq P_0$ which is $\mathcal{F}_0$-centric. Define $\psi = [g \colon \psi_0]$ and note that $\varphi = \psi \circ [1 \colon \iota_{P_0}^{Q_0}]$. From (5.26), for any $q \in Q$

$$\psi \circ \hat{q} \circ [1 \colon \iota_{P_0}^{Q_0}] = \psi \circ [1 \colon \iota_{P_0}^{Q_0}] \circ \hat{q} = \varphi \circ \hat{q} = \widehat{s_q} \circ \varphi = \widehat{s_q} \circ \psi \circ [1 \colon \iota_{P_0}^{Q_0}].$$

Since $[1 \colon \iota_{P_0}^{Q_0}]$ is an epimorphism in $\mathcal{L}_1$ by Lemma 5.17(c), we deduce that $\psi \in \mathcal{L}_2(Q, S)$. Finally, $f = \pi_2(\varphi) = \pi_2(\psi) \circ \mathrm{incl}_P^Q$. This completes the proof. $\square$

Proof of Lemma 5.16. Our notation was chosen in such a way that the argument in [3, Theorem 4.6, Step 7] can be read verbatim and we shall therefore avoid reproducing it. $\square$

6. Maps from a homotopy colimit

Let $\mathcal{C}$ be a small category, and $X\colon \mathcal{C} \to \mathbf{Top}$ be a diagram of spaces over $\mathcal{C}$. The values taken by the functor will be denoted by $X(c)$ and $X(\varphi)$ where $c \in \mathcal{C}$, $\varphi \in \mathrm{Mor}_{\mathcal{C}}(c, c')$. The homotopy colimit of the diagram X is the space

$$\mathrm{hocolim}_{\mathcal{C}} X = \Big(\coprod_{n \geq 0} \coprod_{c_0 \to \cdots \to c_n} X(c_0) \times \Delta^n \Big) / \sim$$

where we divide by the usual face and degeneracy identifications [2, Ch. XII].

There is a filtration of $\mathrm{hocolim}_{\mathcal{C}} X$ by spaces $F_n X$ where $F_n X$ is the image of the union of $X(c) \times \Delta^m$ in $\mathrm{hocolim}_{\mathcal{C}} X$ for all $m \leq n$. Notice that $F_0 X$ is just $\coprod_{c \in \mathcal{C}} X(c)$ and $F_1 X$ is the union of the mapping cylinders of all $\varphi \in \mathrm{Mor}(\mathcal{C})$. Observe that a map $f_1\colon F_1 X \to Y$ is the same as a set of maps $f_1(c)\colon X(c) \to Y$ together with homotopies $f_1(c') \circ X(\varphi) \simeq f_1(c)$ for every $\varphi \in \mathcal{C}(c, c')$. Equivalently, these are paths $f(c) \rightsquigarrow f_1(c') \circ X(\varphi)$ in $\mathrm{map}^{f(c)}(X(c), Y)$. A set of maps $X(-) \xrightarrow{f(-)} Y$ which admits such homotopies is called a *system of homotopy compatible maps* and it gives rise to an element in the set $\varprojlim_{\mathcal{C}} [X(c), Y]$.

Fix a system of homotopy compatible maps $X(-) \xrightarrow{f(-)} Y$. By the remark above it gives rise to a map $f_1\colon F_1 X \to Y$ where $f_1|_{X(c)} = f(c)$. Wojtkowiak [28] addressed the question whether f_1 can be extended, up to homotopy, to a map $\tilde{f}\colon \mathrm{hocolim}_{\mathcal{C}} X \to Y$. The method is to extend f_1 by induction on the spaces $F_n X$.

Given a map $\tilde{f}_n\colon F_n X \to Y$ whose restriction to $X(c)$ is homotopic to $f(c)$, Wojtkowiak developed an obstruction theory for extending it to $F_{n+1} X$ without changing it on $F_{n-1} X$. The existence of such an extension depends on the vanishing of a certain obstruction class in $\varprojlim^{n+1}_{\mathcal{C}} \pi_n(\mathrm{map}^{f(c)}(X(c), Y))$. The extension from $F_1 X$ to $F_2 X$ involves in general a functor into the category of groups and representations, whose $\varprojlim^2$ term is described in Wojtkowiak's work. Fortunately, if these groups are abelian then Wojtkowiak's definition of $\varprojlim^2$ coincides with the usual one from homological algebra. Once the map has been extended to $F_2 X$, there are homotopies between the paths $f_1(c) \rightsquigarrow f_1(c'') \circ X(\psi \circ \varphi)$ and $f(c) \rightsquigarrow f_1(c') \circ X(\varphi) \rightsquigarrow f_1(c'') \circ X(\psi) \circ X(\varphi)$ for all $c \xrightarrow{\varphi} c' \xrightarrow{\psi} c''$. Thanks to these homotopies there are functors $c \mapsto \pi_n(\mathrm{map}^{f(c)}(X(c), Y))$ into $\mathbf{Ab}$ for all $n > 1$.

Given two maps $\tilde{f}_1, \tilde{f}_2\colon \mathrm{hocolim}_{\mathcal{C}} X \to Y$ whose restrictions to $X(c)$ are homotopic to $f(c)$ for all $c \in \mathcal{C}$, Wojtkowiak also studies an obstruction theory for the construction of a homotopy $\tilde{f}_1 \simeq \tilde{f}_2$. Clearly, $\tilde{f}_1$ and $\tilde{f}_2$ give rise to a homotopy $\tilde{f}_1|_{F_0 X} \overset{H_0}{\simeq} \tilde{f}_2|_{F_0 X}$. The idea is to extend the homotopy H_0 inductively to $I \times F_n X$. Given a homotopy $\tilde{f}_1|_{F_{n-1} X} \overset{H_{n-1}}{\simeq} \tilde{f}_2|_{F_{n-1} X}$, the possibility of extending it to a homotopy $\tilde{f}_1|_{F_n X} \overset{H_n}{\simeq} \tilde{f}_2|_{F_n X}$ without changing it on $F_{n-2} X$, depends on the vanishing of an obstruction class in $\varprojlim^n \pi_n(\mathrm{map}^{f(c)}(X(c), Y))$.

6.1. **Definition** ([6, Definition 3.3])**.** Fix a prime p. A small category $\mathcal{C}$ has *p-height d* if for every functor $F\colon \mathcal{C} \to \mathbb{Z}_{(p)}$-mod the groups $\varprojlim^i_{\mathcal{C}} F$ vanish for all $i > d$. The p-height of $\mathcal{C}$ is infinite if no such d exists and it is finite otherwise.

6.2. **Theorem.** *Let $\mathcal{C}$ be a finite category of p-height $d < \infty$. Consider a sequence of maps $Y_0 \xrightarrow{g_0} Y_1 \xrightarrow{g_1} \cdots \xrightarrow{g_d} Y_{d+1}$ and let $y_i = g_i \circ \cdots \circ g_0\colon Y_0 \to Y_{i+1}$. Given a functor $X\colon \mathcal{C} \to$ **Top** and a system of homotopy compatible maps $f(-)\colon X(-) \to Y_0$, define* *new systems of homotopy compatible maps $f_i(-) = y_i \circ f(-)\colon X(-) \to Y_{i+1}$ for all $i = 0, \dots, d$. Assume that*

(i) *For every $c \in \mathcal{C}$ and every $i = 1, \dots, d$ the induced map*

$$\pi_i \mathrm{map}^{f_{i-1}(c)}(X(c), Y_i) \xrightarrow{(g_i)_*} \pi_i \mathrm{map}^{f_i(c)}(X(c), Y_{i+1})$$

is the trivial homomorphism between abelian groups.

(ii) *The groups $\pi_{*>0}\mathrm{map}^{f_i(c)}(X(c), Y_i)$ are $\mathbb{Z}_{(p)}$-modules for all $c \in \mathcal{C}$ and all i.*

Then

(a) *There exists a map $\tilde{f}\colon \operatorname{hocolim}_{\mathcal{C}} X \to Y_d$ which renders the following square homotopy commutative for all $c \in \mathcal{C}$,*

$$\begin{tikzcd} X(c) \arrow[r, "f(c)"] \arrow[d, "\iota(c)"'] & Y_0 \arrow[d, "y_{d-1}"] \\ \operatorname{hocolim}_{\mathcal{C}} X \arrow[r, "\tilde{f}"'] & Y_d. \end{tikzcd}$$

(b) *If $\tilde{f}_1, \tilde{f}_2\colon \operatorname{hocolim}_{\mathcal{C}} X \to Y_0$ satisfy $\tilde{f}_1|_{X(c)} \simeq \tilde{f}_2|_{X(c)} \simeq f(c)$ for all $c \in \mathcal{C}$ then the composites $\operatorname{hocolim}_{\mathcal{C}} X \xrightarrow{\tilde{f}_1, \tilde{f}_2} Y_0 \xrightarrow{y_d} Y_{d+1}$ are homotopic.*

Proof. (a) We will define by induction maps $\tilde{f}_i\colon F_i X \to Y_i$ for all $i = 1, \dots, d$ such that $\tilde{f}_i|_{X(c)} \simeq f_{i-1}(c)$ for all $c \in \mathcal{C}$.

Note that, by definition of a system of homotopy compatible maps, we can construct a map $\tilde{f}_0\colon F_1 X \to Y_0$. Let $\tilde{f}_1 = g_0 \circ \tilde{f}_0$ Assume by induction that $\tilde{f}_i\colon F_i X \to Y_i$ with $\tilde{f}_i|_{X(c)} \simeq f_{i-1}$ has been constructed for some $1 \leq i < d$. The obstruction class Θ'_{i+1} for the extension of $\tilde{f}_i$ to $F_{i+1}X$ is mapped by the homomorphism

$$\varprojlim_{\mathcal{C}^{\mathrm{op}}}{}^{i+1} \pi_i \mathrm{map}^{f_{i-1}(c)}(X(c), Y_i) \xrightarrow{(g_i)_*} \varprojlim_{\mathcal{C}^{\mathrm{op}}}{}^{i+1} \pi_i \mathrm{map}^{f_i(c)}(X(c), Y_{i+1})$$

to the obstruction class Θ_{i+1} for the extension of $g_i \circ \tilde{f}_i$ to $F_{i+1}X$. When $i \geq 1$, by hypothesis (i) the groups are abelian and this homomorphism is trivial, whence $\Theta_{i+1} = 0$. Wojtkowiak's obstruction theory guarantees the existence of a map $\tilde{f}_{i+1}\colon F_{i+1}X \to Y_{i+1}$ which agrees with $g_i \circ \tilde{f}_i$ on $F_{i-1}X$ and such that $\tilde{f}_{i+1}|_{X(c)} \simeq g_i \circ f_{i-1}(c) = f_i(c)$. This completes the induction step.

Hypothesis (ii) and the assumption on $\mathcal{C}$ imply that the groups

$$\varprojlim_{\mathcal{C}^{\mathrm{op}}}{}^{i} \pi_{i-1} \mathrm{map}^{f_{d-1}}(X(c), Y_d)$$

are trivial for all $i \geq d+1$. Thus, the obstructions to the extension of $\tilde{f}_d$ to $F_i X$ where $i > d$ must all vanish. We can therefore construct by induction on $i \geq d+1$ maps $\tilde{f}_i\colon F_i X \to Y_d$ such that $\tilde{f}_i|_{X(c)} \simeq f_{d-1}(c)$ for all $c \in \mathcal{C}$ and such that $\tilde{f}_{i+1}$ agrees with $\tilde{f}_i$ on $F_{i-1}X$. We can finally define $\tilde{f}\colon \operatorname{hocolim}_{\mathcal{C}} X = \bigcup_i F_i X \to Y_d$ with the required properties. In fact, $\tilde{f}|_{F_n X} = \tilde{f}_{n+1}|_{F_n X}$ for all $n > d$.

(b) First, we construct by induction homotopies $y_i \circ \tilde{f}_1|_{F_i X} \overset{H_i}{\simeq} y_i \circ \tilde{f}_2|_{F_i X}$ for all $i = 0, \dots, d$. Recall that $F_0 X = \coprod_{c \in \mathcal{C}} X(c)$ and we define H_0 as the sum of the homotopies $y_0 \circ \tilde{f}_1|_{X(c)} \simeq y_0 \circ \tilde{f}_2|_{X(c)}$.

Assume by induction that $H_i\colon y_i\circ\tilde{f}_1|_{F_iX}\simeq y_i\circ\tilde{f}_2|_{F_iX}$ has been constructed where $0\leq i<d$. The obstruction Υ'_i for the extension of H_i to a homotopy $y_i\circ\tilde{f}_1|_{F_{i+1}X}\simeq y_i\circ\tilde{f}_2|_{F_{i+1}X}$ is mapped by the homomorphism

$$\varprojlim_{\mathcal{C}^{\mathrm{op}}}{}^{i+1}\pi_{i+1}\mathrm{map}^{f_i(c)}(X(c),Y_{i+1})\xrightarrow{(g_{i+1})_*}\varprojlim_{\mathcal{C}^{\mathrm{op}}}{}^{i+1}\pi_{i+1}\mathrm{map}^{f_{i+1}(c)}(X(c),Y_{i+2})$$

to the obstruction class Υ_i for the extension of $g_{i+1}\circ H_i\colon I\times F_iX\to Y_{i+2}$ to $I\times F_{i+1}X$. This homomorphism is trivial by hypothesis (i). Therefore $\Upsilon_i=0$, and by Wojtkowiak's theory there is a homotopy $y_{i+1}\circ\tilde{f}_1|_{F_{i+1}X}\overset{H_{i+1}}{\simeq}y_{i+1}\circ\tilde{f}_2|_{F_{i+1}X}$. This completes the induction step.

Now, the hypothesis on $\mathcal{C}$ together with (ii) imply that the groups

$$\varprojlim_{\mathcal{C}^{\mathrm{op}}}{}^{i}\pi_i\mathrm{map}^{f_d(c)}(X(c),Y_{d+1})$$

are trivial for all $i\geq d+1$. We can therefore construct by induction on $i\geq d+1$ homotopies $y_d\circ\tilde{f}_1|_{F_iX}\overset{H_i}{\simeq}y_d\circ\tilde{f}_2|_{F_iX}$ such that H_{i+1} and H_i agree on $I\times F_{i-1}X$. There results a homotopy $y_d\circ\tilde{f}_1\simeq y_d\circ\tilde{f}_2$. □

7. Maps between p-local finite groups

7.1. **Definition.** Let $(S,\mathcal{F})$ be a fusion system. A map $f\colon BS\to X$ is called $\mathcal{F}$-invariant, if for every $\varphi\in\mathcal{F}(P,S)$ the composite $BP\xrightarrow{B\varphi}BS\xrightarrow{f}X$ is homotopic to $f|_{BP}=f\circ B\mathrm{incl}_P^S$.

7.2. **Example.** Let $(S,\mathcal{F},\mathcal{L})$ be a p-local finite group. The map $\Theta\colon BS\to|\mathcal{L}|$ of 2.9 is $\mathcal{F}$-invariant by Proposition 2.10.

Given a p-local finite group $(S,\mathcal{F},\mathcal{L})$, the question we address in this section is when an $\mathcal{F}$-invariant map $f\colon BS\to X$ can be extended to a map $|\mathcal{L}|\to X$. Here is the main result of this section which uses the constructions in §3.

7.3. **Theorem.** *Let $(S,\mathcal{F},\mathcal{L})$ and $(S',\mathcal{F}',\mathcal{L}')$ be p-local finite groups and consider an $\mathcal{F}$-invariant map $f\colon BS\to|\mathcal{L}'|^\wedge_p$. Then:*

(a) *There exists $m\geq 0$ and a map $\tilde{f}\colon|\mathcal{L}|\to(|\mathcal{L}'|\wr\Sigma_{p^m})^\wedge_p$ which renders the following square homotopy commutative*

$$\begin{tikzcd} BS \arrow[r, "f"] \arrow[d, "\Theta"'] & {|\mathcal{L}'|^\wedge_p} \arrow[d, "\Delta^\wedge_p"] \\ {|\mathcal{L}|} \arrow[r, "\tilde{f}"'] & (|\mathcal{L}'|\wr\Sigma_{p^m})^\wedge_p \end{tikzcd}$$

(b) *There exists $e>0$ such that for any two maps $\tilde{f}_1,\tilde{f}_2\colon|\mathcal{L}|\to|\mathcal{L}'|^\wedge_p$ with $\tilde{f}_1\circ\Theta\simeq\tilde{f}_2\circ\Theta\simeq f$, the composites $|\mathcal{L}|\xrightarrow{\tilde{f}_1,\tilde{f}_2}|\mathcal{L}'|^\wedge_p\xrightarrow{\Delta^\wedge_p}(|\mathcal{L}'|\wr\Sigma_{p^e})^\wedge_p$ are homotopic.*

7.4. **Example.** If $f=\Theta\colon BS\to|\mathcal{L}|$ then $\tilde{f}$ can be chosen as the identity on $|\mathcal{L}|^\wedge_p$.

The main tool for proving Theorem 7.3 is Theorem 6.2 but we will need some preliminary facts about the homotopy groups of mapping spaces and p-completion of wreath products of spaces.

Consider a group G. Its abelianisation is denoted by G_{ab}. Its maximal p-perfect subgroup [2, Ch. VII.3] is denoted by $O^p(G)$. This is the maximal subgroup of K such that $H_1(K;\mathbb{F}_p)=0$. It is clearly characteristic in G. It is also clear that $O^p(G)$ contains every element of G with finite order prime to p. In particular, if G is finite then $G/O^p(G)$ is a finite p-group. For a finite abelian group A, set $A_{(p)}=A\otimes\mathbb{Z}_{(p)}$; this is the set of p-power order elements in A.

7.5. **Proposition.** *Set $H=G\wr\Sigma_k$ for some group G. If either $p>2$ and $k\geq 2$ or if $p=2$ and $k\geq 3$ then $H/O^p(H)$ is an abelian p-group. In particular, if G is finite then $H/O^p(H)$ is a finite abelian p-group.*

Proof. Note that H contains $G^{\times k}$ as a normal subgroup and we will write $g_{(i)}$ for the element $(1,\dots,1,g,1,\dots,1)\in H$ with $g\in G$ in the ith position. To prove the result it suffices to show that $O^p(H)$ contains all the elements of the form $g_{(1)}g_{(i)}^{-1}$ for all $i>1$, all the elements of the form $g_{(1)}$ where $g\in[G,G]$ is a commutator in G and that it contains $A_k\leq\Sigma_k$. Indeed, the quotient of H by the normal subgroup generated by the elements of the first and second type is $G_{\mathrm{ab}}\times\Sigma_k$, so throwing in A_k would guarantee that the quotient $H/O^p(H)$ is abelian.

If $p>2$ then clearly the involutions $\tau=(1,i)\in\Sigma_k$ belong to $O^p(H)$. Therefore, for any $g\in G$ also $g_{(1)}\tau g_{(1)}^{-1}$ belongs to $O^p(H)$. As a consequence we also have $g_{(1)}\tau g_{(1)}^{-1}\tau^{-1}=g_{(1)}g_{(i)}^{-1}\in O^p(H)$. These are the elements of the first type.

Now, given $a,b\in G$ we observe that

$$\{a_{(1)}^{-1}a_{(2)}\}\cdot\{b_{(1)}^{-1}b_{(2)}\}\cdot\{(ab)_{(1)}(ab)_{(2)}^{-1}\}=(a^{-1}b^{-1}ab)_{(1)},$$

so $O^p(H)$ contains all the elements of the form $g_{(1)}$ with $g\in[G,G]$. Finally, since Σ_k is generated by involutions, Σ_k is also contained in $O^p(H)$. This completes the proof in this case.

If $p=2$ and $k\geq 3$, consider some $g\in G$ and a 3-cycle $(1,i,j)$. By inspection $[g_{(1)},\tau]=g_{(1)}g_{(i)}^{-1}$ so $O^p(H)$ contains all the elements of the first type. Also $O^p(H)$ contains all the elements of the second type, namely $g_{(1)}\in H$ where $g\in[G,G]$ by the same argument we used for odd p. Finally, $O^p(H)$ contains all the 3-cycles in Σ_k, whence it contains A_k. □

7.6. **Corollary.** *Let X be a p-good space, then $\pi_1((X\wr\Sigma_k)^\wedge_p)$ is abelian if $k\geq 3$. It is a finite abelian p-group if $\pi_1(X^\wedge_p)$ is finite.*

Proof. We may replace X with $X^\wedge_p$ by next Lemma 7.7. Set $Y=X\wr\Sigma_k$ and $\pi=\pi_1Y$. By Remark 3.6 and Proposition 7.5 we see that $\pi/O^p(\pi)$ is an abelian group and that it is a finite abelian p-group if π_1X is finite. Let $E\to Y$ be the principal fibration obtained by pulling back the covering map $B(O^p(\pi))\to B\pi$ along the first Postnikov section $Y\to B\pi$. Clearly, $\pi_1E=O^p(\pi)$ and we obtain a fibre sequence $E\to Y\to B(\pi/O^p(\pi))$. By [2, Ch. VII.3.2], E is p-good and $E^\wedge_p$ is simply connected. By fibrewise p-completion [2, Ch. I.8.3], there is a fibre sequence $E^\wedge_p\to\dot{Y}^\wedge_p\to B(\pi/O^p(\pi))$ where $Y\to\dot{Y}^\wedge_p$ is a mod-p equivalence because E is p-good. We deduce that $\pi_1(\dot{Y}^\wedge_p)=\pi/O^p(\pi)$ which is abelian. The description in [1, Prop 5.5, 5.6] of the fundamental group of the p-completion of a space implies that $\pi_1(Y^\wedge_p)=\pi_1((\dot{Y}^\wedge_p)^\wedge_p)$ is the p-adic completion of $\pi/O^p(\pi)$ which is also an abelian group. It is an abelian finite p-group if $\pi_1(X)$ is finite (see [1, 5.7 (vi)]). □

t

7.7. **Lemma.** *Let X be a p-good space. Then, for any $G \leq \Sigma_n$ the diagram*

$$\begin{array}{ccc} X^\wedge_p & \xrightarrow{\Delta(X^\wedge_p)} & X^\wedge_p \wr G \\ {\scriptstyle \Delta(X)^\wedge_p}\downarrow & & \downarrow{\scriptstyle \eta} \\ (X \wr G)^\wedge_p & \xrightarrow[(\eta \wr G)^\wedge_p]{\simeq} & (X^\wedge_p \wr G)^\wedge_p \end{array}$$

is homotopy commutative where $(\eta \wr G)^\wedge_p$ is a homotopy equivalence.

Proof. The first statement follows from the naturality of η and of $- \wr G$. The map $\eta \wr G$ is a mod p homology equivalence by a Serre spectral sequence argument, hence $(\eta \wr G)^\wedge_p$ is a homotopy equivalence by [2, Lemma I.5.5]. □

7.8. **Proposition.** *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group, let P be a finite p-group and consider a map $f\colon BP \to |\mathcal{L}|^\wedge_p$. Then*

(a) *$\pi_i(|\mathcal{L}|^\wedge_p)$ are finite p-groups for all $i \geq 1$.*

(b) *$\pi_i(\mathrm{map}^{\eta\circ\Delta\circ f}(BP, (|\mathcal{L}|^\wedge_p \wr \Sigma_k)^\wedge_p))$ are finite p-groups for all $i \geq 1$ and $k \geq 0$. Moreover, if $k \geq 3$, $\pi_1(\mathrm{map}^{\eta\circ\Delta\circ f}(BP, (|\mathcal{L}|^\wedge_p \wr \Sigma_k)^\wedge_p))$ is abelian.*

Proof. (a) The fundamental group $\pi_1(|\mathcal{L}|^\wedge_p)$ is a finite p-group by [3, Theorem B]. Using a Serre class argument (see [25, Ch 9.6, Theorem 15]), we only need to show that the integral homology is finite at each degree. In [22], it is proven that the suspension spectrum $\Sigma^\infty|\mathcal{L}|^\wedge_p$ is a retract of $\Sigma^\infty BS$, hence its integral homology groups are finite abelian p-groups.

(b) If $S = 1$ then $|\mathcal{L}| = *$ hence $(|\mathcal{L}|^\wedge_p \wr \Sigma_k)^\wedge_p \simeq (B\Sigma_k)^\wedge_p$ and $\eta \circ \Delta \circ f$ is null-homotopic. Dwyer-Zabrodsky's result [10] shows that the space under study is homotopy equivalent to $(B\Sigma_k)^\wedge_p$ and the result follows from Proposition 7.5 together with [5, Proposition A.2] and part (a).

We now assume that $S \neq 1$. By [6, Theorem 4.4(a)] f is homotopic to

$$BP \xrightarrow{B\rho} BS \xrightarrow{\Theta} |\mathcal{L}| \xrightarrow{\eta} |\mathcal{L}|^\wedge_p$$

for some $\rho\colon P \to S$. Denote $f' = \Theta \circ B\rho$. We may, and will, assume that $f = \eta \circ f'$. By Theorem A there exists a p-local finite group $(S', \mathcal{F}', \mathcal{L}')$ together with a homotopy equivalence $\omega\colon |\mathcal{L}| \wr \Sigma_k \xrightarrow{\simeq} |\mathcal{L}'|$. Since $|\mathcal{L}|$ is p-good by [6, Proposition 1.12], Lemma 7.7 now implies that $(|\mathcal{L}| \wr \Sigma_k)^\wedge_p \simeq (|\mathcal{L}|^\wedge_p \wr \Sigma_k)^\wedge_p$. The following diagram

$$\begin{array}{ccc} \mathrm{map}^{\Delta\circ f'}(BP, |\mathcal{L}| \wr \Sigma_k) & \xrightarrow{(\eta_*)\circ(\eta\wr\Sigma_k)_*} & \mathrm{map}^{\eta\circ\Delta\circ f}(BP, (|\mathcal{L}|^\wedge_p \wr \Sigma_k)^\wedge_p) \\ {\scriptstyle \omega_*}\downarrow{\scriptstyle \simeq} & & {\scriptstyle \simeq}\downarrow{\scriptstyle (\omega^\wedge_p)_*} \\ \mathrm{map}^{\omega\circ\Delta\circ f'}(BP, |\mathcal{L}'|) & \xrightarrow{\eta_*} & \mathrm{map}^{\eta\circ\omega\circ\Delta\circ f'}(BP, |\mathcal{L}'|^\wedge_p) \end{array}$$

is homotopy commutative. By Proposition 2.11(b) both horizontal maps induce, after p-completion, split surjections on all homotopy groups. Moreover note that the spaces at the right hand side of the diagram are p-complete by Proposition 2.11(a). Therefore it suffices to prove that the homotopy groups of

$$(\mathrm{map}^{\Delta\circ f}(BP, |\mathcal{L}|^\wedge_p \wr \Sigma_k))^\wedge_p \tag{7.9}$$

are finite p-groups. It follows from Proposition 3.8(b) and Remark 3.6 that

$$\pi_1 \mathrm{map}^{\Delta\circ f}(BP, |\mathcal{L}|^\wedge_p \wr \Sigma_k) \cong \pi_1(\mathrm{map}^f(BP, |\mathcal{L}|^\wedge_p)) \wr \Sigma_k \qquad \text{and}$$
$$\pi_i \mathrm{map}^{\Delta\circ f}(BP, |\mathcal{L}|^\wedge_p \wr \Sigma_k) \cong \oplus_k\, \pi_i(\mathrm{map}^f(BP, |\mathcal{L}|^\wedge_p)) \qquad \text{for } i > 1.$$

Since $\mathrm{map}^f(BP, |\mathcal{L}|^\wedge_p)$ is the p-completed classifying space of a p-local finite group by Proposition 2.11(a), its homotopy groups are finite p-groups by (a). Now [2, Proposition VII.4.3] shows that the homotopy groups of (7.9) are finite p-groups. Finally, if $k \geq 3$, Proposition 3.8(b) and Corollary 7.6 show that the fundamental group is abelian. □

Proof of Theorem 7.3. First, we assume that $S \neq 1$, or else the result is a triviality. We begin by constructing a sequence of spaces and maps $Y_0 \xrightarrow{g_0} Y_1 \xrightarrow{g_1} \cdots$ where $Y_0 = |\mathcal{L}'|^\wedge_p$ with the following properties:

(i) For every $i \geq 0$ there exists some $m_{i+1} \geq 2$ such that $Y_{i+1} = (Y_i \wr \Sigma_{p^{m_{i+1}}})^\wedge_p$ and such that g_i is the composite $Y_i \xrightarrow{\Delta} Y_i \wr \Sigma_{p^{m_i}} \xrightarrow{\eta} Y_{i+1}$.

(ii) $\pi_{*>0}\mathrm{map}^{g_{i-1}\circ\cdots\circ g_0\circ f|_{BP}}(BP, Y_i)$ are finite abelian p-groups for all $i \geq 1$.

(iii) The homomorphism

$$\pi_i \mathrm{map}^{g_{i-1}\circ\cdots\circ g_0\circ f|_{BP}}(BP, Y_i) \xrightarrow{(g_i)_*} \pi_i \mathrm{map}^{g_i\circ\cdots\circ g_0\circ f|_{BP}}(BP, Y_{i+1})$$

is trivial for all $i \geq 1$ and all $P \leq S$ in $\mathcal{F}^c$.

Property (i) states explicitly how to construct the sequence from Y_0 and the m_i's. Since $Y_0 = |\mathcal{L}'|^\wedge_p$, property (i) with Lemma 7.7 and Theorem A shows that for every $i \geq 0$ there is a homotopy equivalence $Y_i \simeq |\mathcal{L}_i|^\wedge_p$ for some p-local finite group $(S_i, \mathcal{F}_i, \mathcal{L}_i)$ with $S_i \neq 1$. We assume this fact from now on.

To begin with, set $\mathcal{L}_0 = \mathcal{L}'$ and $Y_0 = |\mathcal{L}_0|^\wedge_p$ and $m_1 = 2$. Let $g_0\colon Y_0 \to Y_1$ be the composite $Y_0 \xrightarrow{\eta\circ\Delta} (Y_0 \wr \Sigma_{p^2})^\wedge_p$. Proposition 7.8 guarantees that (ii) holds for Y_1 since $p^{m_1} = p^2 > 3$.

Assume by induction that we have constructed $Y_0 \xrightarrow{g_0} \cdots \xrightarrow{g_{k-1}} Y_k$, where $k \geq 1$, for which conditions (i)–(iii) hold. By hypothesis (ii) on Y_k we can choose $m_{k+1} \geq 2$ such that $p^{m_{k+1}}$ annihilates every element in the abelian group

$$\bigoplus_{P\in\mathcal{F}^c} \pi_k \mathrm{map}^{g_{k-1}\circ\cdots\circ g_0\circ f|_{BP}}(BP, Y_k).$$

Define $Y_{k+1} = (Y_k \wr \Sigma_{p^{m_{k+1}}})^\wedge_p$ and $g_k = \eta \circ \Delta(Y_k)$. Thus, condition (i) holds for $Y_k \xrightarrow{g_k} Y_{k+1}$. Proposition 7.8 implies that condition (ii) holds for $i = k+1$ since $p^{m_{k+1}} \geq p^2 > 3$. It now follows from Proposition 2.11 that the mapping space $\mathrm{map}^{g_k\circ\cdots g_0\circ f|_{BP}}(BP, Y_{k+1})$ is p-complete and we are in position to apply Lemma 4.3 (with $Y = Y_k$ and $X = BP$) to deduce that condition (iii) holds for g_k. This completes the inductive step of the construction.

We now prove inductively that for every $k \geq 0$ there is a homotopy equivalence $Y_k \simeq (|\mathcal{L}'| \wr G_k)^\wedge_p$, where $G_k \leq \Sigma_{p^{m_1+\cdots+m_k}}$, such that

$$(1) \qquad |\mathcal{L}'|^\wedge_p \xrightarrow{g_{k-1}\circ\cdots\circ g_0} Y_k \simeq (|\mathcal{L}'| \wr G_k)^\wedge_p \qquad \text{is homotopic to}$$
$$|\mathcal{L}'|^\wedge_p \xrightarrow{\Delta(|\mathcal{L}'|)^\wedge_p} (|\mathcal{L}'| \wr G_k)^\wedge_p.$$

This is a triviality when $k = 0$. We assume inductively for $k \geq 1$ that the left triangle in the following diagram is homotopy commutative.

$$\begin{tikzcd}
 & Y_k \arrow[r, "\Delta(Y_k)"] \arrow[d, "\simeq"] & Y_k \wr \Sigma_{p^{m_{k+1}}} \arrow[r, "\eta"] \arrow[d, "\simeq"] & Y_{k+1} \arrow[d, "\simeq"] \\
|\mathcal{L}'|^\wedge_p \arrow[ur, "g_{k-1}\circ\cdots\circ g_0"] \arrow[r, "(\Delta(|\mathcal{L}'|)^\wedge_p"'] & (|\mathcal{L}'| \wr G_k)^\wedge_p \arrow[r, "\Delta"'] & (|\mathcal{L}'| \wr G_k)^\wedge_p \wr \Sigma_{p^{m_{k+1}}} \arrow[r, "\eta"'] & ((|\mathcal{L}'| \wr G_k)^\wedge_p \wr \Sigma_{p^{m_{k+1}}})^\wedge_p
\end{tikzcd}$$

The composite at the top is $g_k \circ \cdots \circ g_0$. By Theorem A and [6, Prop. 1.12], $|\mathcal{L}'| \wr G_k$ is p-good. The induction step now follows from Lemma 7.7 and Proposition 3.5.

Now consider the category $\mathcal{C} = \mathcal{O}(\mathcal{F}^c)^{\mathrm{op}}$ and the functor $\tilde{B}\colon \mathcal{C} \to \mathbf{Top}$ recalled in 2.8. Clearly $f\colon BS \to |\mathcal{L}'|^\wedge_p$ gives rise to a system of homotopy compatible maps $f_0\colon \tilde{B}(-) \to |\mathcal{L}'|^\wedge_p$ in the sense described in Section §6. Recall from [6, Corollary 3.4] that $\mathcal{C}$ is a finite category with p-height $d < \infty$ (see Defn. 6.1). Theorem 6.2 applied to f_0 and $Y_0 \xrightarrow{g_0} Y_1 \xrightarrow{g_1} \cdots$ shows that there is a map $\tilde{f}_0\colon |\mathcal{L}| \to Y_d$ such that $\tilde{f}_0 \circ \Theta \simeq g_{d-1} \circ \cdots \circ g_0 \circ f$. Part (a) of the Theorem now follows because the following diagram is homotopy commutative by (1).

$$\begin{tikzcd}
BS \arrow[r, "f"] \arrow[d, "\Theta"'] & |\mathcal{L}'|^\wedge_p \arrow[d, "\Delta(|\mathcal{L}'|)^\wedge_p"'] \arrow[dr, "\Delta(|\mathcal{L}'|)^\wedge_p"] & \\
|\mathcal{L}| \arrow[r, "\tilde{f}_0"'] & Y_d \simeq (|\mathcal{L}'| \wr G_d)^\wedge_p \arrow[r] & (|\mathcal{L}'| \wr \Sigma_{p^{m_1+\cdots+m_d}})^\wedge_p
\end{tikzcd}$$

Part (b) of the Theorem follows similarly: Given $\tilde{f}_1, \tilde{f}_2\colon |\mathcal{L}| \to Y_0$ such that $\tilde{f}_1 \circ \Theta \simeq \tilde{f}_2 \circ \Theta \simeq f$, we have $g_d \circ \cdots g_0 \circ \tilde{f}_1 \simeq g_d \cdots \circ g_0 \circ \tilde{f}_2$ which implies that the composites $|\mathcal{L}| \xrightarrow{\tilde{f}_1, \tilde{f}_2} |\mathcal{L}'|^\wedge_p \xrightarrow{\Delta^\wedge_p} Y_{d+1} \simeq (|\mathcal{L}'| \wr \Sigma_{p^{m_1+\cdots+m_{d+1}}})^\wedge_p$ are homotopic. □

Proof of Theorem B. The induced map $BS \xrightarrow{B\rho} BS' \xrightarrow{\eta\circ\Theta'} |\mathcal{L}'|^\wedge_p$ is clearly $\mathcal{F}$-invariant because $BS' \to |\mathcal{L}'|^\wedge_p$ is $\mathcal{F}'$-invariant by 7.2 and ρ is fusion preserving. The result is now a direct consequence of Theorem 7.3 and Theorem A. □

We say that $\rho\colon S \to \Sigma_n$ is $\mathcal{F}$-invariant if $\rho|_P$ and $\rho \circ \varphi$ are equivalent representations for every $P \leq S$ and $\varphi \in \mathcal{F}(P, S)$.

7.6. **Proposition.** *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group and let $\rho\colon S \to \Sigma_n$ be a homomorphism. Then the following statements are equivalent:*

(1) *ρ is $\mathcal{F}$-invariant.*
(2) *$B\rho\colon BS \to B\Sigma_n$ is an $\mathcal{F}$-invariant map.*
(3) *$\eta \circ B\rho\colon BS \to (B\Sigma_n)^\wedge_p$ is an $\mathcal{F}$-invariant map.*

Proof. It follows immediately from a result of Mislin [16, Proof of the main theorem] which gives rise to bijections $\mathrm{Rep}(P, \Sigma_n) \approx [BP, B\Sigma_n] \xrightarrow[\approx]{\eta_*} [BP, (B\Sigma_n)^\wedge_p]$ for all $P \leq S$. □

7.7. **Proposition.** *The regular permutation representation of a finite p-group S induces an $\mathcal{F}$-invariant map $B\,\mathrm{reg}_S\colon BS \to B\Sigma_{|S|}$ for any fusion system $\mathcal{F}$ on S.*

Proof. By Proposition 7.6, it is enough to check that $\mathrm{reg}_S\colon S \to \Sigma_{|S|}$ is $\mathcal{F}$-invariant. Note that S acts freely on S via $\mathrm{reg}_S\colon S \to \Sigma_{|S|}$, that is all the isotropy subgroups are trivial. In particular, any group monomorphism $\varphi\colon P \to S$ where $P \le S$ renders S a free P-set via $\mathrm{reg}_S \circ\varphi$. Since any two free P-sets of the same cardinality are equivalent, it follows that $\mathrm{reg}_S|_P$ and $\mathrm{reg}_S \circ\varphi$ are conjugate in Σ_n. $\square$

By Example 7.2 and Proposition 7.6, every map $f\colon |\mathcal{L}| \to (B\Sigma_n)^\wedge_p$ gives rise to an $\mathcal{F}$-invariant representation ρ of S of rank n where $B\rho \simeq f|_{BS}$. Not every $\mathcal{F}$-invariant representation of S arises necessarily in this way. However, the next proposition gives a partial answer to that question.

7.8. **Proposition.** *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group.*

(a) *Given $\rho \in \mathrm{Rep}_n(\mathcal{F})$, there exists some $k \ge 0$ and an element $\tilde{f} \in \mathrm{Rep}_{p^k n}(\mathcal{L})$ such that $\tilde{f}|_{BS}$ is homotopic to $BS \xrightarrow{B(p^k\cdot\rho)} B\Sigma_{p^k n} \xrightarrow{\eta} (B\Sigma_{p^k n})^\wedge_p$.*

(b) *Consider $f_1, f_2 \in \mathrm{Rep}_n(\mathcal{L})$ such that $f_1|_{BS} \simeq f_2|_{BS}$. Then there exists some $e \ge 0$ such that $p^e \cdot f_1 = p^e \cdot f_2$ in $\mathrm{Rep}_{p^e n}(\mathcal{L})$.*

Proof. Let $(S, \mathcal{F}, \mathcal{L})$ be the p-local finite group associated with Σ_n. Since $|\mathcal{L}|$ is p-good by [6, Prop. 1.12], a standard Serre spectral sequence argument shows that

$$(1)\quad (B\Sigma_n)^\wedge_p \simeq |\mathcal{L}|^\wedge_p \xrightarrow{\Delta^\wedge_p} (|\mathcal{L}|^\wedge_p \wr \Sigma_k)^\wedge_p \simeq ((B\Sigma_n)^\wedge_p \wr \Sigma_k)^\wedge_p \xrightarrow{B\mathrm{incl}^\wedge_p} (B\Sigma_{nk})^\wedge_p \qquad \text{and}$$

$$(B\Sigma_n)^\wedge_p \xrightarrow{(B\Delta)^\wedge_p} (B\Sigma_{nk})^\wedge_p$$

where $\Delta\colon \Sigma_n \le \Sigma_{nk}$ is the diagonal inclusion, are homotopic. Both (a) and (b) follow directly from Proposition 7.6, Theorem 7.3 and (1) taking into account the definition of the operation $+$ in $\coprod_{n\ge0} \mathrm{Rep}_n(\mathcal{F})$ and $\coprod_{n\ge0} \mathrm{Rep}_n(\mathcal{L})$. $\square$

7.9. **Proposition.** *Every S-regular permutation representation $|\mathcal{L}| \xrightarrow{f} (B\Sigma_n)^\wedge_p$ is a homotopy monomorphism at p.*

Proof. By [5, Lemma 2.3], $H^*(S;\mathbb{F}_p)$ is a finitely generated module over the Noetherian $\mathbb{F}_p$-algebra $H^*(B\Sigma_{m\cdot|S|};\mathbb{F}_p)$ via the homomorphism $(m \cdot \mathrm{reg}_S)^*$. Finally, $H^*(|\mathcal{L}|;\mathbb{F}_p)$ is a submodule of $H^*(S;\mathbb{F}_p)$ by [6, Theorem B], whence it is finitely generated. $\square$

Proof of Theorem C. Apply Propositions 7.7 and 7.8(a) to obtain $f \in \mathrm{Rep}_{p^k\cdot|S|}(\mathcal{L})$ such that $f|_{BS}$ is homotopic to $\eta \circ B(p^k \cdot \mathrm{reg}_S)$, that is, $\Phi(f) = p^k \cdot \mathrm{reg}_S$. By Proposition 7.9, f is a homotopy monomorphism at p . $\square$

8. The p-local index of the Sylow subgroup

Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group and let $f\colon |\mathcal{L}| \to (B\Sigma_n)^\wedge_p$ be a map. The restriction $f|_{BS} = f \circ \Theta$ is $\mathcal{F}$-invariant by Example 7.2 and is homotopic to $(B\rho)^\wedge_p$ for a unique $\rho \in \mathrm{Rep}(S, \Sigma_n)$ which is $\mathcal{F}$-invariant by Proposition 7.6 and [10]. There results maps $\mathrm{Rep}_n(\mathcal{L}) \to \mathrm{Rep}_n(\mathcal{F})$ which are compatible with the operations $+$ and $\times$ defined in the introduction. They give rise to a ring homomorphism

$$\Phi\colon \mathrm{Rep}(\mathcal{L}) \to \mathrm{Rep}(\mathcal{F}).$$

8.1. **Proposition.** *Additively,* $\ker(\Phi)$ *and* $\mathrm{coker}(\Phi)$ *are p-torsion.*

Proof. An element in $\ker(\Phi)$ has the form $f_1 - f_2$ where $f_1, f_2 \in \mathrm{Rep}_n(\mathcal{L})$ for some n and $f_1|_{BS} \simeq f_2|_{BS}$. Proposition 7.8 implies that $p^e \cdot (f_1 - f_2) = 0$ in $\mathrm{Rep}(\mathcal{L})$ and it follows that $\ker(\Phi)$ is p-torsion.

An element of $\mathrm{Rep}(\mathcal{F})$ has the form $\rho_1 - \rho_2$ for some $\rho_1 \in \mathrm{Rep}_{n_1}(\mathcal{F})$ and $\rho_2 \in \mathrm{Rep}_{n_2}(\mathcal{F})$. By Proposition 7.8, the definition of Φ and the definition of the operations $+$ in $\mathrm{Rep}(\mathcal{F})$ and $\mathrm{Rep}(\mathcal{L})$, we see that there exist integers $k_1, k_2 \geq 0$ and representations $f_1 \in \mathrm{Rep}_{p^{k_1} n_1}(\mathcal{L})$ and $f_2 \in \mathrm{Rep}_{p^{k_2} n_2}(\mathcal{L})$ such that $\Phi(f_1) = p^{k_1} \cdot \rho_1$ and $\Phi(f_2) = p^{k_2} \cdot \rho_2$. Then $\omega = p^{k_2} \cdot f_1 - p^{k_1} \cdot f_2$ is an element of $\mathrm{Rep}(\mathcal{L})$ such that $\Phi(\omega) = p^{k_1+k_2}(\rho_1 - \rho_2)$. It follows that $\mathrm{coker}(\Phi)$ is p-torsion. □

By Proposition 7.7 the ring $\mathrm{Rep}(\mathcal{F})$ contains $\mathrm{reg}_S \colon S \to \Sigma_{|S|}$ which generates an (additive) infinite cyclic group $\mathrm{Rep}^{\mathrm{reg}}(\mathcal{F}) := \{n \cdot \mathrm{reg}_S\}_{n \in \mathbb{Z}}$ in $\mathrm{Rep}(\mathcal{F})$. Similarly let $\mathrm{Rep}^{\mathrm{reg}}(\mathcal{L})$ denote the additive subgroup of the ring $\mathrm{Rep}(\mathcal{L})$ generated by all the S-regular representations of $(S, \mathcal{F}, \mathcal{L})$ (see Definition 1.2).

It follows directly from the definitions that Φ restricts to a group homomorphism

$$\Phi^{\mathrm{reg}} : \mathrm{Rep}^{\mathrm{reg}}(\mathcal{L}) \to \mathrm{Rep}^{\mathrm{reg}}(\mathcal{F}).$$

8.2. **Corollary.** *The cokernel of* Φ^{reg} *is a cyclic* p*-group. The kernel of* Φ^{reg} *is an abelian torsion* p*-group and* $\mathrm{Rep}^{\mathrm{reg}}(\mathcal{L})$ *is isomorphic to the direct sum of* $\mathbb{Z}$ *with an abelian* p*-torsion group.*

Proof. This follows from Proposition 8.1 which in particular implies that the image of Φ^{reg} is isomorphic to $\mathbb{Z}$, whence it splits off from $\mathrm{Rep}^{\mathrm{reg}}(\mathcal{L})$. □

Given a finite group G there is a natural one-to-one correspondence between equivalence classes of permutation representations $G \to \Sigma_n$ and equivalence classes of G-sets of cardinality n. Sum and products of representations (as described in the introduction) correspond to disjoint unions and products of the associated G-sets. Note that reg_G corresponds to a free G-set with one orbit.

Let us return to discuss $\mathrm{Rep}(\mathcal{F})$. Since the product of a free S-set with any other S-set is again a free set, it follows that $\mathrm{Rep}^{\mathrm{reg}}(\mathcal{F})$ and $\mathrm{Rep}^{\mathrm{reg}}(\mathcal{L})$ are in fact ideals in $\mathrm{Rep}(\mathcal{F})$ and $\mathrm{Rep}(\mathcal{L})$ and that Φ^{reg} is a ring homomorphism.

8.3**. Example.** Let $(S, \mathcal{F}, \mathcal{L})$ be the p-local finite group of a finite group G. The restriction of $(B\,\mathrm{reg}_G)^\wedge_p \colon |\mathcal{L}|^\wedge_p \to (B\Sigma_{|G|})^\wedge_p$ to BS is homotopic to $n \cdot (B\,\mathrm{reg}_S)^\wedge_p$ where $n = |G \colon S|$ because $\mathrm{reg}_G|_S = n \cdot \mathrm{reg}_S$. In particular $(B\,\mathrm{reg}_G)^\wedge_p$ is an element in $\mathrm{Rep}^{\mathrm{reg}}(\mathcal{L})$ which is mapped by Φ to $n \cdot \mathrm{reg}_S$. It follows that $|G \colon S| \in \mathrm{Im}(\Phi^{\mathrm{reg}})$, whence $|\mathrm{coker}(\Phi^{\mathrm{reg}})|$ divides $|G \colon S|$.

8.4. **Definition.** Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group. Define the upper and lower *p-local index* of S in $\mathcal{L}$ by

$$\mathrm{Uind_p}(\mathcal{L} \colon S) = |\mathrm{coker}(\Phi^{\mathrm{reg}})|$$
$$\mathrm{Lind_p}(\mathcal{L} \colon S) = |\mathrm{Rep}^{\mathrm{reg}}(\mathcal{F}) : \mathrm{Rep}^{\mathrm{reg}}(\mathcal{F}) \cap \mathrm{Im}(\Phi)|.$$

Clearly $\mathrm{Lind_p}(\mathcal{L} \colon S)$ divides $\mathrm{Uind_p}(\mathcal{L} \colon S)$ because $\mathrm{Im}(\Phi^{\mathrm{reg}}) \leq \mathrm{Im}(\Phi) \cap \mathrm{Rep}^{\mathrm{reg}}(\mathcal{F})$.

8.5. **Lemma.** *Let* $(S, \mathcal{F}, \mathcal{L})$ *be a* p*-local finite group. Then* $\mathrm{Uind_p}(\mathcal{L} \colon S)$ *is a* p*-power. If there exists a permutation representation* $\rho \colon |\mathcal{L}| \to (B\Sigma_n)^\wedge_p$ *such that* $\rho|_{BS} \simeq B(n \cdot \mathrm{reg}_S)$ *with* $n \geq 1$ *prime to* p*, then* $\mathrm{Uind_p}(\mathcal{L} \colon S) = 1$*, and in particular also* $\mathrm{Lind_p}(\mathcal{L} \colon S) = 1$.

Proof. The first statement follows from Corollary 8.2. The existence of ρ shows that $n \in \mathrm{Im}(\Phi^{\mathrm{reg}})$ hence, $\mathrm{Uind_p}(\mathcal{L}\colon S) = 1$. □

The depth of a fusion system $\mathcal{F}$ on S is the largest number of elements in a chain of *proper* inclusions of $\mathcal{F}$-centric $\mathcal{F}$-radical subgroups of S. This includes chains ending in S. Following [23] we call these subgroups "$\mathcal{F}$-Alperin". Thus, if the depth of $\mathcal{F}$ is n then there exists no chain $P_1 \lneq \cdots \lneq P_{n+1}$ of proper inclusions of $\mathcal{F}$-Alperin subgroups.

8.6. **Proposition.** *If the depth of a p-local finite group $(S, \mathcal{F}, \mathcal{L})$ is equal to 2 then* $\mathrm{Uind_p}(\mathcal{L}\colon S) = 1$.

Proof. Let $\mathcal{R}$ denote the collection of the $\mathcal{F}$-Alperin subgroups of S. Fix representatives $S, P_1, \dots, P_n$ for the $\mathcal{F}$-conjugacy classes in $\mathcal{R}$ where P_i are fully $\mathcal{F}$-normalised. Consider the poset $\bar{\mathrm{s}}\mathrm{d}\mathcal{R}$ defined in [14, Defn. 1.3]. Its objects are the $\mathcal{F}$-conjugacy classes $[P]$ of elements $P \in \mathcal{R}$ and the $\mathcal{F}$-conjugacy classes $[P \lneq S]$ of proper inclusion $P \lneq S$ in $\mathcal{R}$. Here we use the fact that $\mathcal{F}$ has depth 2. The only relations in $\bar{\mathrm{s}}\mathrm{d}\mathcal{R}$ are $[P \lneq S] \prec [P]$ and $[P \lneq S] \prec [S]$. By Alperin's fusion theorem [6, Theorem A.10], if $[Q] = [P_i]$ then $[Q \lneq S] = [P_i \lneq S]$. It follows that $\bar{\mathrm{s}}\mathrm{d}\mathcal{R}$ is isomorphic to the poset $\mathcal{C}_n$ whose objects are $\{c_0, c_1^i, c_2^i \,|\, i = 1, \dots, n\}$ and whose only relations are $c_1^i \prec c_0$ and $c_1^i \prec c_2^i$ for all $i = 1, \dots, n$. Specifically, $c_0 = [S], c_2^i = [P_i]$ and $c_1^i = [P_i \lneq S]$. We view $\mathcal{C}_n$ as a small category with an arrow $x \to y$ if $x \prec y$.

In [14, Theorem A] a functor $F\colon \mathcal{C}_n \to \mathbf{Top}$ with the following properties is constructed. The values of F are the classifying spaces of finite groups G_0, G_1^i and G_2^i for $i = 1, \dots, n$ and the maps $F(c_1^i) \to F(c_0)$ and $F(c_1^i) \to F(c_2^i)$ are induced by inclusion of groups $G_1^i \le G_0$ and $G_1^i \le G_2^i$. In addition S is a subgroup of $G_0 = \mathrm{Aut}_{\mathcal{L}}(S)$ of index prime to p. Also, $k_i = |G_2^i \colon G_1^i|$ are prime to p by [14, Theorem A] and the fact that P_i is fully $\mathcal{F}$-normalised, whence $N_S(P_i)$ is a Sylow p-subgroup of $G_1^i = \mathrm{Aut}_{\mathcal{L}}(P \le S)$ which is a subgroup of $G_1^i = \mathrm{Aut}_{\mathcal{L}}(P_i)$ by [14, Prop. 1.5]. Finally, the map $\Theta\colon BS \to |\mathcal{L}|$ factors up to homotopy through $BG_0 \simeq F(c_0) \to \mathrm{hocolim}_{\mathcal{C}_n} F \simeq |\mathcal{L}|$.

Set $k = \prod_1^n k_i$ and $k_0 = |G_0| \cdot k$. Note that k_0 is divisible by $|G_1^i|$ and $|G_2^i|$ for all i because $k_0 = k \cdot |G_0| = k \cdot |G_1^i| \cdot |G_0 \colon G_1^i|$ and k_i divides k. Set $\ell_i = k_0/|G_1^i|$ and $m_i = k_0/|G_2^i|$. Consider the following permutation representations for $i = 1, \dots, n$

$$k \cdot \mathrm{reg}_{G_0} \colon G_0 \to \Sigma_{k_0}, \qquad \ell_i \cdot \mathrm{reg}_{G_1^i} \colon G_1^i \to \Sigma_{k_0}, \qquad m_i \cdot \mathrm{reg}_{G_2^i} \colon G_2^i \to \Sigma_{k_0}.$$

Note that $(k \cdot \mathrm{reg}_{G_0})|_{G_1^i}$ and $(m_i \cdot \mathrm{reg}_{G_2^i})|_{G_1^i}$ are equivalent to $\ell_i \cdot \mathrm{reg}_{G_1^i}$ because all of them give the set $\{1, \dots, k_0\}$ the structure of a free G_1^i-set with ℓ_i orbits. By taking classifying spaces there results a system of homotopy compatible maps $F \to B\Sigma_{k_0}$. It can be replaced by a system of compatible maps $F \to B\Sigma_{k_0}$ as follows. First, set the maps $F(c_1^i) \to B\Sigma_{k_0}$ to be the composite $F(c_1^i) \to F(c_0) \to B\Sigma_{k_0}$. Next, replace the maps $F(c_1^i) \to F(c_i^2)$ by cofibrations and change the maps $F(c_2^i) \to B\Sigma_n$ up to homotopy to obtain a system of compatible maps $F \to B\Sigma_{k_0}$.

There results a map $f\colon |\mathcal{L}| \simeq \mathrm{hocolim}\, F \to B\Sigma_{k_0}$ such that $f|_{BS} = f \circ B\iota_S^{G_0} \simeq k \cdot |G_0 \colon S| \cdot B\,\mathrm{reg}_S$ where $k \cdot |G_0 \colon S|$ is prime to p. Now Lemma 8.5 applies. □

We shall now prove Theorem D. In fact we prove the following stronger result.

8.7. **Theorem.** *Let $(S, \mathcal{F}, \mathcal{L})$ be a p-local finite group. Then* $\mathrm{Uind_p}(\mathcal{L}\colon S) = 1$ *if*

(1) *$(S, \mathcal{F}, \mathcal{L})$ is associated with a finite group, or*

(2) $(S, \mathcal{F}, \mathcal{L})$ *is one of the exotic examples in* [6, Examples 9.3 and 9.4] *or in* [23] *or in* [9] *or in* [7, Example 5.3].

Proof. (1) This follows from Lemma 8.5 and Example 8.3.
(2) We will apply Proposition 8.6. The p-local finite groups in [6, Examples 9.3-4] as well as the ones in [23] and in [9] were shown to have depth 2 in Examples 7.6, 7.7, 7.3 and 7.4 of [13] respectively. The information on the structure of the exotic p-local finite groups in [7, Example 5.3] implies quite directly that these fusion systems have depth 2. We leave the straightforward details to the reader. □

8.8. **Conjecture.** *For all p-local finite groups* $\mathrm{Uind_p}(\mathcal{L}\colon S) = 1$.

References

[1] Bousfield, A. K. On the p-adic completions of nonnilpotent spaces. Trans. Amer. Math. Soc., 331 (1992), 335–359.

[2] Bousfield, A. K.; Kan, D. M. Homotopy limits, completions and localizations. Lecture Notes in Mathematics, Vol. 304. Springer-Verlag, Berlin-New York, 1972.

[3] Broto,C.; Castellana, N.; Grodal, J.; Levi, R.; Oliver,B. Extensions of p-local finite groups. Trans. Amer. Math. Soc. 259 (2007), 3791–3858.

[4] Broto,C.; Castellana, N.; Grodal, J.; Levi, R.; Oliver,B. Subgroup families controlling p-local finite groups. Proc. of the London Math. Soc. 91 (2005), 325–354.

[5] Broto, C. Levi, R. Oliver, B. Homotopy equivalences of p-completed classifying spaces of finite groups. Invent. Math. 151 (2003), no. 3, 611–664.

[6] Broto, C.; Levi, R.; Oliver, B. The homotopy theory of fusion systems. J. Amer. Math. Soc. 16 (2003), no. 4, 779–856

[7] Broto, C.; Levi, R.; Oliver, B. A geometric construction of saturated fusion systems. (English summary) An alpine anthology of homotopy theory, 11–40, Contemp. Math., 399, Amer. Math. Soc., Providence, RI, 2006.

[8] Chermak A.;Oliver Bob; Shpectorov S. The simple connectivity of $B\mathrm{Sol}(q)$, Preprint.

[9] Díaz A.; Ruiz, A.; Viruel, A. All p-local finite groups of rank two for odd prime p. Trans. Amer. Math. Soc. 359 (2007), no. 4, 1725–1764 (electronic).

[10] Dwyer, W., Zabrodsky, A. Maps between classifying spaces. Algebraic topology, Barcelona, 1986, 106–119, Lecture Notes in Math., 1298, Springer, Berlin, 1987.

[11] Levi R.; Oliver, B. Construction of 2-local finite groups of a type studied by Solomon and Benson. Geom. Topol. 6 (2002), 917–990

[12] Libman, A. Minami-Webb splittings and some exotic p-local finite groups, To appear in Math. Zeit.

[13] Libman, A.; Viruel, A. On the homotopy type of the non-completed classifying space of a p-local finite group. To appear.

[14] Libman, A. The normaliser decomposition for p-local finite groups. Algebr. Geom. Topol. 6 (2006), 1267–1288

[15] Martino, J. and Priddy, S. Unstable homotopy classification of $BG^\wedge_p$. Math. Proc. Cambridge Philos. Soc. 119 (1996), no. 1, 119–137.

[16] Mislin, G. On group homomorphisms inducing mod-p cohomology isomorphisms. Comment. Math. Helv. 65 (1990), no. 3, 454–461.

[17] Oliver, B. Equivalences of classifying spaces completed at odd primes. Math. Proc. Cambridge Philos. Soc. 137 (2004), no. 2, 321–347.

[18] Oliver, B. Equivalences of classifying spaces completed at the prime two. Mem. Amer. Math. Soc. 180

[19] Oliver, B. Extensions Of Linking Systems and Fusion Systems. Preprint.

[20] Puig, Ll. Frobenius categories. J. Algebra 303 (2006), 309–357.

[21] Quillen, A. Higher algebraic K-theory. I. Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972). Lecture Notes in Math., Vol. 341, Springer, Berlin 1973. 85–147.

[22] Ragnarsson, K. Classifying spectra of saturated fusion systems. Algebr. Geom. Topol. 6 (2006), 195–252 (electronic).

[23] Ruiz, A.; Viruel, A. The classification of p-local finite groups over the extraspecial group of order p^3 and exponent p. Math. Z. 248 (2004), no. 1, 45–65.
[24] Solomon, R. Finite groups with Sylow 2-subgroups of type 3. J. Algebra 28 (1974), 182–198.
[25] Spanier, E. Algebraic topology. Springer-Verlag, New York-Berlin, 1981.
[26] Thomason, R. W. Homotopy colimits in the category of small categories. Math. Proc. Cambridge Philos. Soc. 85 (1979), no. 1, 91–109.
[27] Whitehead, G. W. Elements of homotopy theory. Graduate Texts in Mathematics, 61. Springer-Verlag, New York-Berlin, 1978.
[28] Wojtkowiak, Z. On maps from $\operatorname{ho}\varinjlim F$ to Z. Algebraic topology, Barcelona, 1986. Lecture Notes in Math. 1298, Springer, Berlin, 1987, 227–236.

Departament de Matemàtiques, Facultat de Ciències, Universitat Autònoma de Barcelona, 08193 Bellaterra, Catalonia, Spain.
E-mail address: `natalia@mat.uab.cat`

Department of Mathematical Sciences, King's College, University of Aberdeen, Aberdeen AB24 3UE , Scotland, U.K.
E-mail address: `assaf@maths.abdn.ac.uk`