Integrable NLS equation with time-dependent nonlinear coefficient and self-similar attractive BEC

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Abstract. We investigate the Nonlinear Schrödinger Equation with a time-dependent nonlinear coefficient. By means of Painlevé analysis we establish the integrability of a particular case, when the nonlinear coefficient decays with t^{-1}. The corresponding soliton solution is shown to be of the self-similar kind. We discuss implications of the result to the dynamics of attractive Bose-Einstein condensates under Feshbach-managed nonlinearity and explore the possibility of a managed self-similar evolution in 1D condensates.

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1. Introduction

The Nonlinear Schrödinger (NLS) Equation

$$i\psi_t - \frac{\beta}{2}\psi_{xx} + \gamma|\psi|^2\psi = 0 \quad , \tag{1}$$

is one the ubiquitous equations in nonlinear dynamics of spatially extended systems. It is a (1+1)-dimensional completely integrable equation finding applications in optics, hydrodynamics, condensed matter physics, to site a few, [1]. Its wide applicability comes from the fact that it is a normal form in perturbative expansions connected to wave modulation [2]. Integrability grants the existence of particularly important solutions, the solitons, which dominate the long-time behavior of the system. Depending on the sign of the product $\gamma\beta$, the NLS equation has different kinds of solitonic solutions: if $\gamma\beta < 0$ it has localized, bright soliton, solutions and if $\gamma\beta > 0$ it has the so-called dark soliton solutions.

In recent years, a new area of applicability for the NLS equation has appeared. The dynamics of a single-species Bose-Einstein condensate (BEC) is indeed governed in the mean-field limit by the Gross-Pitaevskii equation (GP), [3], which has many similarities to NLS equation. GP is a (3+1)-dimensional equation of the NLS kind, supplemented with an extra term, describing the trapping potential that sustains the condensate. In GP the second derivative becomes a Laplacian. The coefficient β is in this case $\beta = -\frac{\hbar}{m}$, where m is the mass of the bosons. The parameter γ, is the scattering length and is connected to the self-interaction of the condensate. It can be both positive (attractive condensate) or negative (repulsive condensate). In the most frequent case, the potential is a single-well, harmonic, one.

BECs are very manageable objects in laboratory, allowing for manipulations of their form and interactions. It is thus possible to obtain effectively lower dimensional condensates by using an anisotropic trapping that strongly constrains the dynamics in one or two spatial directions. For instance, we will be interested here in one-dimensional condensates, that is, condensates where two directions have their dynamics frozen. It is possible to show that such condensates are governed by a one-dimensonal GP equation [4], with a rescaled scattering length. Furthermore, if we consider attractive condensates in one dimension, the dynamics in the "unfrozen" direction can be considered as free – not subject to the action of the potential – as long as we have the characteristic radius of the condensate smaller then the linear dimension of the trap. Summing up, an attractive BEC in a one-dimensional configuration will be described by the usual NLS equation. The expected solitons have been observed experimentally [5].

Part of the manageability of the BEC system is translated into the fact that γ can be tuned. The presence of spatially homogeneous magnetic fields acting upon the condensate modifies the interaction between atoms and affects γ. Besides, γ can be made time-dependent and can be used to manage the condensate, the so-called, Feshbach-resonance management [6]. This opens the path analogies with the dispersion managed systems of fiber optics [7]. Many interesting effects have been found in this respect, not only in (1+1)-dimensional case, as for example the stabilization of two-dimensional solitons [8].

Having thus the freedom to consider γ as a function of time a natural question arises. Given that Eq. (1) is integrable, are there other integrable cases when we consider a time-dependent $\gamma(t)$. For the case of optical fibers, where β, instead of γ is variable, it is know that one integrable case exist, with an exponentially decaying β, [9].

In this paper we will show that the NLS equation with a time-dependent γ admits a further integrable case which is decaying with t^{-1}. We will resort to the method of Painlevé analysis [10], which gives all possible integrable cases. Next we will show that the integrable case found can be mapped to a constant coefficient NLS by a gauge transformation. Finally, we will obtain explicit solutions and discuss their properties. In particular, we will show the possibility of existence Feshbach-managed expanding self-similar solutions of an attractive condensate.

2. Painlevé Analysis

Consider a variable coefficient NLS equation in the form:

$$i\psi_t - \frac{\beta}{2}\psi_{xx} + \gamma(t)|\psi|^2\psi = 0. \tag{2}$$

We now apply the well established Painlevé analysis to Eq. (2) to derive the parametric condition on $\gamma(t)$ for which the NLS equation (2) is completely integrable [10]. To proceed further with the Painlevé analysis, we introduce a new set of variables $a(=\psi)$ and $b(=\psi^*)$. By Eq. (2), a and b can be written as

$$ia_t - \frac{\beta}{2}a_{xx} + \gamma(t)a^2b \quad = 0, \tag{3}$$

$$-ib_t - \frac{\beta}{2}b_{xx} + \gamma(t)b^2a = 0. \tag{4}$$

Next, expand a and b in generalized Laurent series as:

$$a = \sum_{r=0}^{\infty} a_r\varphi^{r+\mu}, \quad b = \sum_{r=0}^{\infty} b_r\varphi^{r+\delta}, \tag{5}$$

with $a_0, b_0 \neq 0$, where μ and δ are negative integers, a_r and b_r are a set of expansion coefficients which are analytic in the neighborhood of the non-characteristic singular manifold $\varphi(x,t) = 0$. Standard Painlevé analysis consists of looking at the leading order, when $a \approx a_0\varphi^{\mu}$ and $b \approx b_0\varphi^{\delta}$ are substituted in Eqs. (3). Upon balancing dominant terms, the following results are obtained:

$$\mu = \delta = -1 \quad \text{and} \quad a_0b_0 = \varphi_x^2\beta/\gamma(t) \quad . \tag{6}$$

Substituting the full Laurent series (5) in Eqs. (3) and considering the leading order terms, the resonances are found to be $r = -1, 0, 3, 4$. The resonance at $r = -1$ represents the arbitrariness of the singularity manifold and $r = 0$ corresponds to the fact that either a_0 or b_0 is arbitrary. Collecting and balancing the coefficients of the different powers of φ show that a sufficient number of arbitrary functions exists only for the following parametric condition on $\gamma(t)$:

$$\frac{d^2\gamma(t)}{dt^2}\gamma(z) - 2\left[\frac{d\gamma(t)}{dt}\right]^2 = 0. \tag{7}$$

On solving this equation, we have:

$$\gamma(t) = \frac{\gamma_0}{t + t_0}, \tag{8}$$

where γ_0 and t_0 are integration constants.

Thus the Painlevé analysis implies that the nonlinear coefficient must vary in the manner given by Eq. (8) for the system equation (2) to be completely integrable. Thus the integrable form of Eq. (2) can be written as

$$i\psi_t - \frac{\beta}{2}\psi_{xx} + \frac{\gamma_0}{t + t_0}|\psi|^2\psi = 0, \tag{9}$$

3. Soliton Solutions and Transformations

We would now like to point out a conjecture regarding the resonance values derived in the Painlevé analysis. The resonances $r = -1, 0, 3, 4$ obtained here for the variable coefficient NLS equation (2) are the same as those for the constant coefficient NLS equation. Past experience has shown that such coincidences usually imply that the newly derived integrable nonlinear evolution equation could be connected to existing systems of equations. This is in fact true and there is a connection between the variable coefficient NLS equation (9) and the conventional NLS equation. We consider the following dependent variable scaling

$$\psi = \sqrt{2(t + t_0)}\psi', \tag{10}$$

which maps the variable coefficient NLS equation (9) into the following variable coefficient NLS equation (with $\beta = -2$, for convenience):

$$i\psi'_t + \psi'_{xx} + 2\gamma_0|\psi'|^2\psi' + \frac{i}{2(t + t_0)}\psi' = 0. \tag{11}$$

This variable coefficient NLS equation (11) has been analyzed for its integrability through Painlevé analysis and possesses a transformation connecting it to constant coefficient NLS equation [11]. Thus the above mentioned conjecture about the resonance values of the Painlevé analysis holds good as there is a connection between the integrable nonlinearity varying NLS equation (9) and the conventional constant coefficient NLS equation.

Using standard mathematical techniques the 1-soliton solution of the variable coefficient NLS equation (11) can be derived as

$$\psi'(x,t) = \sqrt{\frac{2}{\gamma_0}}\frac{\zeta}{t+t_0}\mathrm{sech}\left(\frac{\sqrt{2}\zeta x}{t+t_0}\right)\exp\left[\frac{i}{t+t_0}(x^2/4 - 2\zeta^2)\right]. \tag{12}$$

where ζ is an arbitrary constant.

The Hirota bilinear transform for 1- and 2-soliton solutions:-

The 1- and 2-soliton expressions can also be derived by the Hirota bilinear transformation, a method proven to be effective over the years in handling multi-soliton for nonlinear evolution equations (NEEs). To simplify the presentation, we consider

$$i\psi'_t + \psi'_{xx} + 2\gamma_0|\psi'|^2\psi' + \frac{i\sigma\psi'}{t} = 0, \tag{13}$$

where we have set $t_0 = 0$ as the time origin is immaterial. Remarkably, the bilinear map will dictate the constant σ to be $1/2$, consistent with earlier analysis. Recently, a modified bilinear method has been proposed to treat NEEs with variable coefficients. The main ideas are (a) to separate an appropriate chirp factor and (b) to employ time or space dependent wave numbers [12, 13, 14].

Our goal is to demonstrate that a similar algorithm will also succeed in the present case. To begin, one starts with transformation

$$\psi' = \exp\left[\frac{iB(x)}{t}\right]\frac{u}{t}. \tag{14}$$

A remark on the present choice of chirp factor is in order. For the GP / NLS equations with a quadratic potential in the spatial coordinate x, a quadratic factor in x with modifications / modulations in time t form the successful combination [12, 13, 14]. In the present case with a reciprocal in t being the inhomogeneous term in (13), it is plausible to seek this same format in t modified by a suitable phase factor in x.

The standard bilinear transform for envelope type equations, namely,

$$u = \frac{g}{f}, \qquad f \text{ real}, \tag{15}$$

is now implemented and the resulting, decoupled bilinear equations are (D is the usual Hirota differentiation operator [15])

$$\left[iD_t + D_x^2 + \frac{2iB_x}{t}D_x + \frac{B}{t^2} - \frac{i}{t} + \frac{iB_{xx}}{t} - \frac{B_x^2}{t^2} + \frac{i\sigma}{t}\right]g\cdot f = 0, \tag{16}$$

$$D_x^2 f\cdot f = \frac{2\gamma_0 g g^*}{t^2}. \tag{17}$$

To achieve a 1-soliton, an expansion with time dependent wave number is sought:

$$g = \exp[xh_1 + h_0], h_1 = h_1(t), h_0 = h_0(t), \tag{18}$$
$$f = 1 + m_{11}\exp[x(h_1 + h_1^*) + h_0 + h_0^*]. \tag{19}$$

The entity m_{11} may be a function of t in general, even though we shall prove shortly that it will be a constant in the present situation. Equation (17) now immediately yields

$$2m_{11}(h_1 + h_1^*)^2 = \frac{2\gamma_0}{t^2}. \tag{20}$$

The coefficient of $\exp(xh_1 + h_0)$ in (16) gives

$$i(xh_1' + h_0') + h_1^2 + \frac{2iB_x h_1}{t} + \frac{B}{t^2} - \frac{i}{t} + \frac{iB_{xx}}{t} - \frac{B_x^2}{t^2} + \frac{i\sigma}{t} = 0. \tag{21}$$

Both the real and imaginary parts of this expression must of course vanish independently. The real part is

$$h_1^2 + \frac{B}{t^2} - \frac{B_x^2}{t^2} = 0,$$

$$\text{or} \qquad h_1^2 t^2 = B_x^2 - B = \text{constant} \\ = r^2 \tag{22}$$

Elementary separation of variables arguments imply now both sides of (22) must be constant. Furthermore, h_1 is real (as B is real), and this separation constant is positive (r^2). The differential expression governing B is

$$\frac{dB}{dx} = \sqrt{B + r^2}. \tag{23}$$

This separable equation gives

$$B = \frac{x^2}{4} - r^2. \tag{24}$$

With

$$h_1 = \frac{r}{t}, \tag{25}$$

we deduce from expression (20) that m_{11} is not a function of t,

$$m_{11} = \frac{\gamma_0}{4r^2}. \tag{26}$$

The imaginary part of (21) now generates

$$h_0' = -\frac{\left(\sigma - \frac{1}{2}\right)}{t}. \tag{27}$$

The other term in the bilinear equation (4) now gives

$$h_0' = -\frac{\left(\sigma - \frac{1}{2}\right)}{t}. \tag{28}$$

Equations (27) and (28) now force us to conclude that σ must be 1/2, and h_0 must be zero (a constant that can be easily scaled out). This formulation can be readily shown to reduce to the 1-soliton solution (12). Extension to 2-soliton from (18) and (19) is straightforward.

With a minor modification, the appropriate expansions for a 2-soliton wave profile are then

$$\psi' = \exp\left(\frac{ix^2}{4t}\right)\frac{g_2}{f_2}, \tag{29}$$

$$\begin{aligned} g_2 = {} & \exp(\phi) + \exp(\chi) + n_1 \exp(\phi + \chi + \chi^*) + n_2 \exp(\chi + \phi + \phi^*), \\ f_2 = {} & 1 + m_{11} \exp(\phi + \phi^*) + m_{12} \exp(\phi + \chi^*) + m_{21} \exp(\chi + \phi^*) \\ & + m_{22} \exp(\chi + \chi^*) + M \exp(\phi + \phi^* + \chi + \chi^*), \end{aligned}$$

where

$$\phi = \frac{rx}{t} - \frac{ir^2}{t}, \qquad \chi = \frac{Rx}{t} - \frac{iR^2}{t}, \tag{30}$$

with r, R being real constants. The parameters m_{ij}, n_i, and M can be computed in manners very similar to earlier references [15]:

$$m_{11} = \frac{\gamma_0}{4r^2}, \qquad m_{22} = \frac{\gamma_0}{4R^2}, \qquad m_{12} = m_{21} = \frac{\gamma_0}{(r+R)^2}, \tag{31}$$

$$n_1 = \frac{\gamma_0 (R-r)^2}{4R^2 (r+R)^2}, \qquad n_2 = \frac{\gamma_0 (R-r)^2}{4r^2 (r+R)^2}, \qquad M = \frac{\gamma_0^2 (r-R)^4}{16 r^2 R^2 (r+R)^4}. \tag{32}$$

Gauge transformation:-

For completeness in the following we present the gauge transformation

$$\begin{aligned} \psi'(x,t) &= \left(\frac{t_0 - T}{t_0}\right) Q(X,T) \exp\left[\frac{iX^2}{4(t_0 - T)}\right], \\ t &= \left(\frac{t_0}{t_0 - T}\right) T, \\ x &= \left(\frac{t_0}{t_0 - T}\right) X, \end{aligned}$$

which connects the variable coefficient NLS equation (11) to the following constant coefficient NLS equation

$$iQ_T + Q_{XX} + 2\gamma_0 |Q|^2 Q = 0. \tag{33}$$

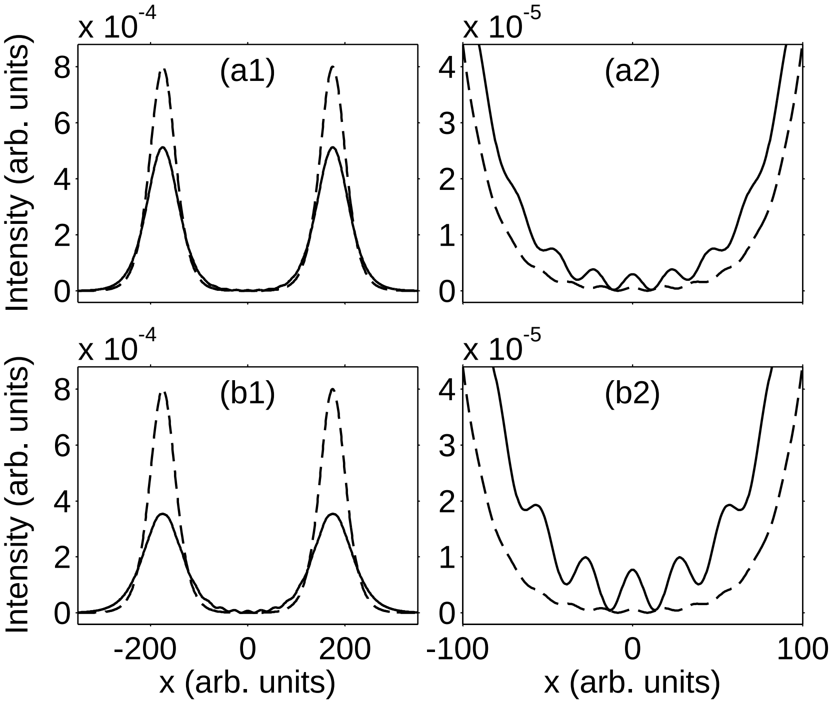

Figure 1. Plots showing the evolution of two single solitons with $\gamma_0 = 1$, $\zeta = 10$ and $t_0 = 500$. Dashed curves represent the initial solitons and solid curves represent the solitons after evolution of $t = 125$ ((a1) & (a2)) and $t = 250$ ((b1) & (b2)).

4. Numerical Simulations

Given the situation described above, the possibility arises of having two Feshbach-managed attractive BEC expanding and interfering. The two-soliton soliton is however much more general than this. Therefore to explore the possibility of merging two BEc in the conditions described, we numerically integrate the equations with the adequate initail conditions.

Figures 1 show the evolution $|\psi|^2$ for two single solitons initially separated by a distance of $x = 350$ and the other parameters $\gamma_0 = 1$, $\zeta = 10$ and $t_0 = 500$. Dashed curves represent the initial solitons and solid curves represent the solitons after evolution. The solid curve in Fig. 1 (a1) corresponds to the solitons evolved after $t = 125$. Whereas the solid curve in Fig. 1 (b1) corresponds to the solitons evolved after $t = 250$. For more insight in Figs. 1 (a2) and (b2) we have zoomed in the interacting tail part of the solitons corresponding to Figs. 1 (a1) and (b1), respectively.

As is clear from the above plots, an interference pattern can readily be observed.

5. Conclusions

The solution given by Eq. (12) implies a self-similar dynamics for the condensate density. Indeed, if we plug it into Eq. (10) and calculate the modulus squared of ψ we will get the following function:

$$|\psi|^2 = \frac{4\zeta^2}{\gamma_0(t+t_0)}\mathrm{sech}^2\left(\frac{\sqrt{2}\zeta x}{t+t_0}\right) \quad , \tag{34}$$

which has the form

$$|\psi|^2 = \frac{1}{t+t_0}F(x/(t+t_0)) \quad . \tag{35}$$

This represents an expanding condensate with typical width $\zeta \sim t+t_0$. Therefore, if we associate an expansion front to the condensate, it progresses with constant speed. The amplitude drops with t^{-1} and the evolution maintains the shape of the BEC.

We have thus an attractive condensate that expands under Feshbach-resonance control. Expansion is possible because the scattering length drops with time, allowing for larger condensates. The self-similarity of the solution is, however, the effect of the precise time-dependence of the scattering length with time. If, for instance, we had a different time-behavior of the scattering length we could still have an expanding condensate but not obeying a shape preserving evolution as the one found in the above results.

Self-similar evolution of BEC has been studied before [16] in different settings. In general, expansion is the effect of repulsion and self-similar evolution can occur for in small or large condensate limits [17]. On the the hand, self-similar evolution is also connected to collapsing attractive condensates [18]. In the present case, however, we would like to point out the possibility of a managed self-similar evolution, arising from the interplay of controlled nonlinearity and dispersion. Furthermore, we can have shown that a self-similar merging of two BEC is possible, leading to interference.

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